B8 The special case of standardized data

During the review process, reviewer Hoshin Gupta inspired us to think about what would happen to the NSE if the available data would always be standardized (i.e., both the observations and simulations have zero mean and unit variance for all partitions and the overall data). This section shows that in this special setting the NSE and the Kling-Gupta Efficiency (KGE just measures the Pearson's correlation coefficient r, and the correlation becomes the same as the cosine similarity.

Proposition B.8. *In a setting where we standardize the observations and model outputs for a given set of observations and simulations, we get* $\mathrm{NSE} = 2 * r - 1$.

Proof. As per Gupta et al. (2009) the NSE can be decomposed into

$$\mathrm{NSE} = 2 * \alpha * r - \alpha^2 - \beta, \tag{B14}$$

where α is the ratio of the standard deviations, i.e.: $\alpha = \frac{\sigma_s}{\sigma_o}$, r is Pearson's correlation coefficient, and $\beta = \frac{\mu_s - \mu_o}{\sigma_o}$ (here independent of the partition).

Since the means of the observations and simulations are zero, it always holds that $\beta = 0$ and $\alpha = 1$, which simplifies Eq. (B14) to

$$\mathrm{NSE} = 2 * r - 1.$$

In other words, in this special setting the NSE only measures the correlation. □

There exists a similar simplification for the KGE:

Proposition B.9. *In a setting where we standardize the observations and model outputs, it holds that* $\mathrm{KGE} = r$.

Proof. The KGE is defined as

$$\mathrm{KGE} = 1 - \sqrt{(r-1)^2 + (\sigma_s/\sigma_o - 1)^2 + (\mu_s/\mu_o - 1)^2}.$$

In the current setting $\frac{\mu_s}{\mu_o}$ is actually undefined because of the division by zero, but we might also interpret it as one because $\mu_s = \mu_o$. Similarly, $\frac{\sigma_s}{\sigma_o} = 1$. Thus, the only part within the square root that remains is $(r-1)^2$, which gives us:

$$\begin{aligned}\mathrm{KGE} &= 1 - \sqrt{(r-1)^2},\\ &= 1 - |r-1|,\\ &= r.\end{aligned}$$

Thus, we showed that in the special setting where observations and simulations are standardized the KGE measures the correlation only. $\square$

Next, we show that within the standardization setting the correlation becomes the cosine similarity.

Proposition B.10. *In a setting where we standardize the observations and model outputs for all data and all partitions, the correlation is the same as the cosine similarity.*

Proof. The cosine similarity between two N dimensional vectors a and b is defined as

$$s_c = \cos\theta = \frac{\sum_{i=1}^{N} a_i b_i}{\sqrt{\sum_{i=1}^{N} a_i^2} * \sqrt{\sum_{i=1}^{N} b_i^2}}, \tag{B15}$$

where θ is the angle between the two vectors, and equivalence to r is given because Eq. (B15) is the same as Eq. (B13) if the means are set to zero. $\square$

In other words, in this special setting the correlation just measures the difference in rotation given by the two centered vectors that contain the observations and the simulations respectively.