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Power-Law between the Apparent Drainage Density and the Pruning Area

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Abstract. Self-similar structures of river networks have been quantified as diverse scaling laws. Among them we investigated a power functional relationship between the apparent drainage density ρ_a and the pruning area A_p with an exponent η. We analytically derived the relationship between η and other scaling exponents known for fractal river networks. The analysis of 14 real river networks covering diverse range of climate conditions and free-flow connectivity levels supports our derivation. We further linked η with non-integer fractal dimensions found for river networks. Synthesis of our findings through the lens of fractal dimensions provides an insight that the exponent η has fundamental roots in fractal dimension for the whole river network organization.

1 Introduction

Since first proposed by Horton (1945), the drainage density ρ has long been recognized as an important metric to describe geomorphological and hydrological characteristics of a catchment. Defined as $\rho = L_T / A_\Omega$ where A_Ω is the constant catchment area, ρ is a function of the total channel length L_T in a catchment. Alternatively, ρ is a function of the channel forming area A_o (also called the source area or the critical contributing area) (Band, 1986; Montgomery and Dietrich, 1988; Tarboton et al., 1988), which is directly related to L_T. The variation of ρ among catchments is associated with the climatic condition, which can be represented by measures such as the precipitation effectiveness index (Melton, 1957; Madduma Bandara, 1974). A_o reduces as the catchment becomes wetter, which leads to the expansion of the stream network (greater L_T) and vice versa (Godsey and Kirchner, 2014; Hooshyar et al., 2015; Durighetto et al., 2020). Therefore, L_T and ρ are inversely related to A_o (Tarboton et al., 1991).

On another note, the 'rate' at which L_T (and so ρ) varies with A_o is determined by the given topography. The close relationship between the main channel length L and the drainage area A is well known as a power function with a positive exponent h (Hack, 1957), i.e.,

$$L \propto A^h. \tag{1}$$

Although Eq. (1) provides a clue about the relationship between L_T and A_o, they differ in two senses: (1) L_T is the total length counting all tributaries, while L is the length of the main channel only; and (2) L is the length within the area A while L_T is the length of channels excluded from A_o. L_T reduces as A_o increases, while L grows with A (Eq. (1)).

The usage of the digital elevation models (DEMs) in the river network analysis introduced a constant called the pruning area A_p. In extracting a stream network from a DEM, cells of the upslope area A less than A_p are considered as hillslope and excluded from the network. For the ideal delineation of a river network, A_p is expected to be A_o. However, A_p can be any arbitrary value and differs from A_o by definition. If $A_p = 0$, every DEM cell is considered as channel while A_p can be as large as A_Ω for a completely dry landscape. As A_p increases, less channels are extracted, resulting in a smaller 'apparent' drainage density ρ_a.

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We distinguish ρ_a from the real drainage density ρ, accommodating the difference between A_p and A_o. It was found that ρ_a decreases as A_p grows following a power function (Moglen et al., 1998), i.e.,

$$\rho_a \propto {A_p}^{-\eta}. \qquad (2)$$

The background described above naturally leads us to the basic question about the physical origin of the power-law Eq. (2) and its scaling exponent η. Prancevic and Kirchner (2019) adopted the scaling relation of Eq. (2) in the derivation of the power function between L_T and the discharge at the catchment outlet Q (Godsey and Kirchner, 2014; Hooshyar et al., 2015; Jensen et al., 2017), i.e., $L_T \propto Q^\beta$, the relationship which quantifies the tendency of stream networks to expand and retreat. They expressed the exponent β as the combination of η and two other scaling exponents found in topographic attributes, i.e., $\beta = \eta / (\theta + \gamma + 1)$, where θ is the power-law exponent relating local channel slope to drainage area called the concavity (Montgomery and Foufoula-Georgiou, 1993; Mcnamara et al., 2006), and γ is the exponent of a hypothetical power function between valley transmissivity and A. Adopting this, we can reason $\eta = \beta(\theta + \gamma + 1)$. However, Prancevic and Kirchner (2019) acknowledged that the above expression of β does not hold confident generalization across a range of sizes and landscapes, suggesting the presence of diverse descriptions for η besides $\beta(\theta + \gamma + 1)$. Eq. (2) and the exponent η have awaited for deeper investigations.

Moglen et al. (1998) attempted direct DEM analyses to investigate the ρ_a–A_p relationships in real river networks. But, A_o and A_p were undistinguished and further discussion on η itself was missed in their study. To properly approach the given subject with terrain analyses, a greater resolution DEM for catchments of known A_o or blue-lines are needed. It is worth to realize that the power-law relationship of Eq. (2) implies fractal network formation. A river network is fractal, and many regular power-laws have been reported as characteristic signatures of a naturally evolved river network (Dodds and Rothman, 2000). As the power-law relationship between ρ_a and A_p can also serve as a signature reflecting the self-similarity, it is plausible to claim the linkage between ρ_a–A_p relationship and other power-laws known in natural river networks. In particular, $\eta = 0.5$ is anticipated to satisfy dimensional consistency in Eq. (2) (Tarboton et al., 1991). But the rough analysis of Moglen et al. (1998) raises a doubt whether η estimated from any real catchment meets this consistency. This issue is analogous to the question about the exponent h in Eq. (1), which should also be 0.5 to keep consistency in dimension (Hjelmfelt, 1988). In fact, h values reported for natural rivers are mostly greater than 0.5, i.e., between 0.5 and 0.7 (Hack, 1957; Gray, 1961; Robert and Roy, 1990; Crave and Davy, 1997). This has brought the introduction of the fractal dimension (Mandelbrot, 1977). Similarly, we can claim that the dimensional inconsistency in Eq. (2), if any, can be resolved by introducing the fractal dimension to express η.

Here, we aim to corroborate the aforementioned claims and hypothesis about the ρ_a–A_p relationship and its exponent η. To this end, in the next Sect. 2, we reviewed the scaling relationships known in a river network. Then, we presented analytical derivation of Eq. (2), and demonstrated how this is related with other power-laws known for a river network. To support our argument, many real catchments under the wide range of climate conditions and free-flow connectivity levels were analyzed with terrain analysis methods in a thorough manner using high resolution DEMs and trust-worthy blueline data. These are described in Sect. 3. With these results, we explored physical meanings embedded in the power-law relationship between ρ_a and A_p with the notion of fractal dimension in Sect. 4. Summary and conclusions are given in Sect. 5.

2 Cross-Relationships among Scaling Laws

2.1 Review on scaling laws of a river network

The river network has been perceived as an archetypal fractal network in nature (Mandelbrot, 1977; Rodríguez-Iturbe and Rinaldo, 2001), exhibiting scale-invariant organization. Systematic measures for characterizing structural hierarchy help manifest the self-similarity. Horton-Strahler ordering scheme (Horton, 1945; Strahler, 1957) has been popularly employed to investigate their structural characters. In this framework, the number, the mean length, and the mean drainage area of ω-order streams in a catchment, stated as $N_\omega, \bar{L}_\omega$, and $\bar{A}_\omega$, respectively, are defined for an order ω ranging from 1 to Ω, where Ω is the highest order in the network. There is only one Ω–order stream in a river network (i.e., N_Ω=1). Then, the total channel length L_T used for the definition of the drainage density ρ, is given as

$$L_T = \sum_{\omega=1}^{\Omega} N_\omega \bar{L}_\omega. \tag{3}$$

Following its definition, the length of any lower order stream is excluded in $\bar{L}_\omega$. By contrast, $\bar{A}_\omega$ includes the drainage area of all upstream branches (of $\omega - 1$ and lower orders). Therefore, $\bar{L}_\Omega$ is neither the upslope length L of a main channel, nor L_T, while $\bar{A}_\Omega$ is identical to the total drainage area of the catchment. To resolve the discrepant definitions of $\bar{L}_\omega$ and $\bar{A}_\omega$, the cumulative mean length was proposed to match the definition of area (Broscoe, 1959) as

$$\Xi_\omega = \sum_{k=1}^{\omega} \bar{L}_k \tag{4}$$

which is an order-discretized approximation of L. Alternatively, to match the definition of length, the eigenarea, also called the interbasin area (Strahler, 1964) or the contiguous area (Marani et al., 1991), was proposed as the area directly draining to the ω–order stream (Beer and Borgas, 1993). The mean eigenarea $\bar{E}_\omega$ of ω–order streams is

$$\bar{E}_\omega = \bar{A}_\omega - \bar{A}_{\omega-1}(N_{\omega-1}/\, N_\omega). \tag{5}$$

The self-similar structure of a river network has been captured through the linear scaling of above quantities ($N_\omega, \bar{L}_\omega, \bar{A}_\omega$, and $\bar{E}_\omega$) with ω on a semi-log paper (Horton, 1945; Schumm, 1956; Yang and Paik, 2017) as

$$N_\omega = R_B{}^{\Omega-\omega};\ \bar{L}_\omega = \bar{L}_\Omega R_L{}^{\omega-\Omega};\ \bar{A}_\omega = \bar{A}_\Omega R_A{}^{\omega-\Omega};\ \text{and}\ \bar{E}_\omega = \bar{E}_\Omega R_E{}^{\omega-\Omega} \tag{6}$$

where R_B, R_L, R_A, and R_E are the bifurcation, the length, the area, and the eigenarea ratios, respectively. They are dimensionless ratios of quantities between nearby orders, i.e., $R_B = N_\omega / N_{\omega+1}$, $R_L = \bar{L}_{\omega+1}/\bar{L}_\omega$, $R_A = \bar{A}_{\omega+1}/\bar{A}_\omega$, and $R_E = \bar{E}_{\omega+1}/\bar{E}_\omega$, and often called the Horton ratios as a group. They are dependent on each other (Morisawa, 1962; Rosso, 1984; Tarboton et al., 1990) and typically range as $3 < R_B < 5$, $1.5 < R_L < 3$, and $3 < R_A < 6$ (Smart, 1972), and $R_E \approx R_L$ (Yang and Paik, 2017).

In addition to Eq. (6), power functional relationships between geomorphologic variates have also been found and served as evidence of the scale-invariant river network structures. The Hack's law (Eq. (1)) is a classical principle in this line. Another interesting power-law relationship lies in the exceedance probability distributions of upstream area. Using a theoretical aggregation model, Takayasu et al. (1988) showed that the exceedance probability distribution of injected mass in a tree network always follows a power-law. In fact, their model is equivalent to the random-walk model of Scheidegger (1967) devised to mimic a river network (Takayasu and Nishikawa, 1986). Replacing the mass (flow) in the aforementioned study with the drainage area (which is rational if rainfall is spatially uniform), it leads to the power-law exceedance probability distribution of 'drainage area.' In a detail, the probability for a randomly designated point within a catchment to have A exceeding a reference value δ ($0 \le \delta \le \bar{A}_\Omega$) decreases with δ (Rodríguez-Iturbe et al., 1992a), following a power-law as

$$P(A \ge \delta) \propto \delta^{-\varepsilon} \tag{7}$$

where the exponent ε is reported as between 0.40 and 0.46 for most river networks (Rodríguez-Iturbe et al., 1992a; Crave and Davy, 1997). Above two power-laws (Eqs. (1) and (7)) are related as $h + \varepsilon = 1$ (Maritan et al., 1996), which suggests a trade-off between the two relationships to form the catchment boundary within a confined 2-d space.

Two classes of scaling relationships reviewed above, i.e., Horton's laws (Eq. (6)) and power-law relationships are linked as shown by La Barbera and Roth (1994), i.e.,

$$\varepsilon = 1 - h = \frac{\ln (R_B/R_L)}{\ln R_A}. \tag{8}$$

Two other expressions, comparable to Eq. (8), appear in literature. De Vries et al. (1994) derived $\varepsilon = 1 - \ln R_L / \ln R_B$, which is a special case of Eq. (8) where $R_B = R_A$. Empirical studies support that R_B is indeed close to R_A (Smart, 1972). For a 'topological'

Hortonian tree where no constraint on stream length in a finite area is given, Veitzer et al. (2003) and Paik and Kumar (2007) showed that $\varepsilon = \ln R_B/\ln R_A - 1$. This is another special case of Eq. (8) where $R_L = R_A$, the assumption used in the analysis of 'topological' self-similar trees where only connections among nodes matter with no spatial constraint (Paik and Kumar, 2007).

2.2 Linkage to ρ_a–A_p relationship

Below, we analytically derived the relationship between the pruning area A_p and the resulting apparent drainage density ρ_a (Eq. (2)), using the scaling relationships reviewed above. Through this investigation, we importantly revealed $\eta = \varepsilon$, i.e., the scaling exponents in Eqs. (2) and (7) are identical. We arrived at the same conclusion from two different approaches, described below.

2.2.1 Derivation 1

For the Hortonian tree, A_p can vary in a discrete manner (order-by-order), as we set $A_p = \bar{A}_\omega$. Given that up to ω-order streams are pruned in a river network, the total length after pruning is $\sum_{k=\omega+1}^{\Omega} N_k \bar{L}_k$, by revising Eq. (3). Replacing N_k and $\bar{L}_k$ in this equation with Eq. (6) leads to the expression of ρ_a as

$$\rho_a = \frac{L_\Omega}{A_\Omega} \sum_{k=\omega+1}^{\Omega} {R_B}^{\Omega-k} {R_L}^{k-\Omega}. \tag{9}$$

Above sum of the given geometric series is

$$\rho_a = \frac{L_\Omega}{A_\Omega (R_B/R_L - 1)} \left[\left(\frac{R_B}{R_L} \right)^{\Omega-\omega} - 1 \right]. \tag{10}$$

The logarithm of the term $(R_B/R_L)^{\Omega-\omega}$ in Eq. (10) can be written, using Eq. (6), as

$$\ln \left(\frac{R_B}{R_L} \right)^{\Omega-\omega} = (\Omega - \omega) \ln \frac{R_B}{R_L} = \frac{\ln (A_\Omega/\bar{A}_\omega)}{\ln R_A} \ln \frac{R_B}{R_L} = \frac{\ln (R_B/R_L)}{\ln R_A} \ln \frac{A_\Omega}{\bar{A}_\omega}. \tag{11}$$

Given that $\bar{A}_\omega = A_p$, from Eq. (11) we can state

$$(R_B/R_L)^{\Omega-\omega} = (A_\Omega/A_p)^{\frac{\ln (R_B/R_L)}{\ln R_A}}. \tag{12}$$

Substituting this into Eq. (10) yields an approximate power-law, i.e.,

$$\rho_a = \frac{L_\Omega}{A_\Omega (R_B/R_L - 1)} \left[\left(\frac{A_p}{A_\Omega} \right)^{-\frac{\ln (R_B/R_L)}{\ln R_A}} - 1 \right] \propto A_p^{-\frac{\ln (R_B/R_L)}{\ln R_A}}. \tag{13}$$

Given that $R_B \approx R_A > R_L$ (Smart, 1972) for a typical river network, $-1 < -\ln (R_B/R_L)/\ln R_A < 0$. With this range and for $A_p << A_\Omega$, $(A_p/A_\Omega)^{-\ln (R_B/R_L)/\ln R_A} = (A_\Omega/A_p)^{\ln (R_B/R_L)/\ln R_A} \gg 1$. This allows the approximation $[(A_\Omega/A_p)^{\ln (R_B/R_L)/\ln R_A} - 1] \approx (A_\Omega/A_p)^{\ln (R_B/R_L)/\ln R_A}$. Empirical studies suggested $A_o < 0.1A_\Omega$ to characterize fluvial channel networks (Mcnamara et al., 2006; Montgomery and Foufoula-Georgiou, 1993), implying the scope of this derivation, i.e., $A_p << A_\Omega$, of practical range. Comparing Eqs. (2) and (13), we can explicitly express

$$\eta = \frac{\ln (R_B/R_L)}{\ln R_A}. \tag{14}$$

This expression is identical to Eq. (8), which implies $\eta = \varepsilon$.

2.2.2 Derivation 2

The conclusion of $\eta = \varepsilon$ can also be derived by employing the eigenarea (Yang, 2016). Approximating an ω-order sub-catchment as a rectangle, $\bar{E}_\omega$ can be rewritten as $\bar{E}_\omega = W\bar{L}_\omega$ where W is the mean overland flow length. As W is regarded almost a constant (Yang and Paik, 2017; Hack, 1957), the apparent drainage density for the pruning area $A_p = \bar{A}_\omega$ becomes

$$\rho_a = \frac{1}{A_\Omega} \sum_{k=\omega+1}^{\Omega} N_k \bar{L}_k = \frac{1}{A_\Omega W} \sum_{k=\omega+1}^{\Omega} N_k \bar{E}_k. \tag{15}$$

On the other hand, $P(A \geq A_p)$ is defined from geometry as

$$P\left(A \geq A_p\right) = \frac{1}{A_\Omega} \sum_{k=\omega+1}^{\Omega} N_k \bar{E}_k \qquad (16)$$

which equals to $W\rho_a$ from Eq. (15). As $P\left(A \geq A_p\right) \propto A_p^{-\varepsilon}$ (Eq. (7)), we realize that $\rho_a \propto A_p^{-\varepsilon}$ and thereby $\eta = \varepsilon$. While equation (13) was derived for $A_p << A_\Omega$, this alternative derivation shows the power-law regardless of the range in A_p. Earlier, we discussed the reciprocal nature of two relationships; one between L_T and A_o, and the other between L and A. Combining above conclusion of $\eta = \varepsilon$ and $h + \varepsilon = 1$, we realize that $\eta = 1 - h$, indeed implying the compensating function between the two relationships.

3 Analyses of Real River Networks

3.1 Data and methods

To evaluate the power-law Eq. (2) and the derivation of $\eta = \varepsilon$, we analyzed real river networks in the contiguous United States. We have chosen 14 study networks (**Fig. 1**) from the pool investigated in previous studies of Tarboton et al. (1991), Rodríguez-Iturbe et al. (1992a), Botter et al. (2007), Hosen et al. (2021), and Carraro and Altermatt (2022). They are carefully selected to cover distinct hydro-climatic regions and a range of free-flowing capacity (**Table 1**). The climate feature is described by the Köppen-Geiger climate classification system (Beck et al., 2018). The free-flow characteristic is referred as an integrated connectivity status index (CSI) created at a global scale by Grill et al. (2019) for the first time. The CSI comprehensively and quantitatively describes the capacity of individual river reaches to freely flow based on the synthesis of observed and modelled datasets. The reported CSI values, ranging from 0 to 100 %, are the weighted average of estimated five pressure indicators - river fragmentation, flow regulation, sediment trapping, water consumption, and infrastructure development in riparian areas and floodplains - which represent natural and human inferences within longitudinal, lateral, vertical, and temporal dimensions. If a river reach loses connectivity due to any of aforementioned pressures, its CSI value decreases. We calculated a catchment-unit CSI by weighting the length of individual reaches in a given catchment. Our 14 study sites cover the CSI from 58 to 100 % which is irrelevant to each catchment size.

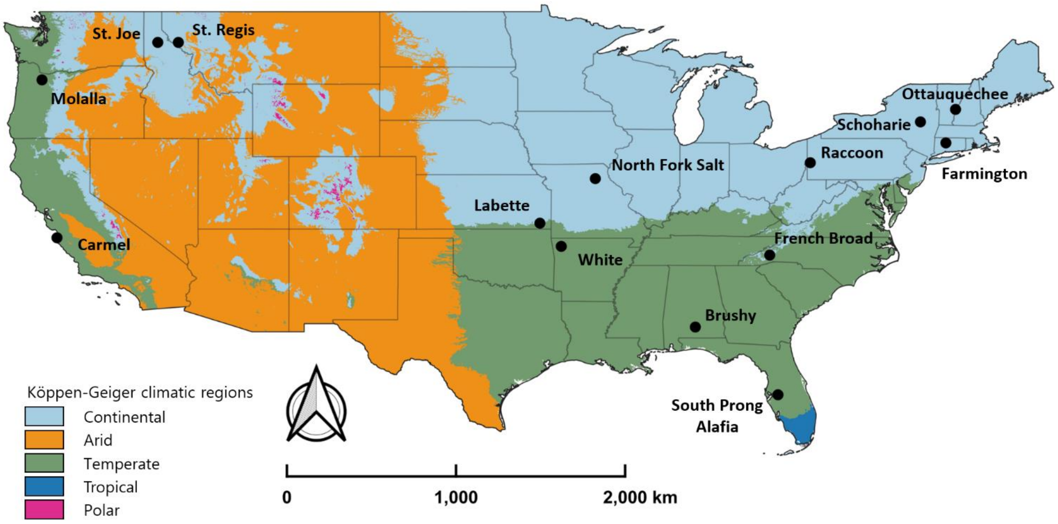

Figure 1. Studied 14 river networks across the contiguous United States and the Köppen-Geiger climate classifications.

To shape the structure of each river network at the grid domain, we used the 1 arc-second raster data of flow direction and upslope area provided in the National Hydrography Dataset Plus Version 2 (NHDPlusV2) (Mckay et al., 2012). Flow direction for each cell was assigned through the deterministic 8 method (O'callaghan and Mark, 1984) on the basis of post-processed DEM to discard depression or sink cell. Accordingly, upslope area was calculated for each cell. For detailed calculation steps and process, readers may refer to the user guide of NHDPlusV2. Detailed layouts of study networks are given in **Fig. S1** in the Supporting Information (SI). To extract river networks resembling individual blue-lines most, we referred to the source areas recorded in the NHDPlusV2. In NHDPlusV2, a channel forming area A_o^* is given for every reach in each network. This is very detailed information, while A_o in our notion is a single value that can represent the entire network. We draw probability distribution of A_o^* for each catchment (**Fig. S2** in SI) and A_o was determined as the median (**Table 1**). Horton-Strahler ordering was assigned on the pruned river networks.

Regarding the exceedance probability distribution of upstream area (Eq. (7)), three segments are often characterized: curved-head, straight-trunk, and truncated-tail. The head reflects hillslope (Moglen and Bras, 1995; Maritan et al., 1996) while the trunk indicates channels. As the upslope area becomes close to $\bar{A}_\Omega$, the probability rapidly drops because the size of a network is finite (Moglen et al., 1998; Perera and Willgoose, 1998; Rodríguez-Iturbe et al., 1992a). To accommodate such an effect in the distribution function, the exponentially tempered power function was adopted (Aban et al., 2006; Rinaldo et al., 2014) as

$$P(A \geq \delta) = c_d \delta^{-\varepsilon} \exp(-k_d \delta)\,, \text{for } \delta > A_o \tag{17}$$

where c_d is a constant and k_d is the tempering parameter. As k_d approaches zero, the function represents abrupt truncation. Similarly, we proposed an exponentially truncated power function for ρ_a, as a general form of Eq. (2), as

$$\rho_a = c_p {A_p}^{-\eta} \exp\left(-k_p A_p\right)\,, \text{for } A_p > A_o \tag{18}$$

where c_p is a constant and k_p is the tempering parameter. To estimate the best fitted parameters, we employed Matlab's *nlinfit* function of which the objective function is to minimize the sum of the squares of the residuals for the fitted model. The estimated range for a parameter was calculated with 95% confidence intervals.

3.2 Results and discussion

All studied networks well follow the power-law Eq. (1) (**Fig. S3 in SI**). The range of estimated Hack's exponent h is 0.55±0.03 (mean ± standard deviation) with $R^2 > 0.95$ (**Table 1**), which is within the typical range shown in earlier studies (Hack, 1957). The laws of stream number, length, drainage area, and eigenarea (Eq. (6)) are satisfied for all study networks with $R^2 > 0.85$ (**Figs. S4 – S5 in SI**). The resultant Horton ratios range as $R_B = 4.2 \pm 0.5$, $R_L = 2.3 \pm 0.3$, $R_A = 4.6 \pm 0.7$, and $R_E = 2.2 \pm 0.3$ (**Table 1**), which are within typical ranges (Horton, 1945; Schumm, 1956; Smart, 1972). These imply that our study networks hold statistically robust self-similar features.

In the exceedance probability distributions of upstream area, three segments of curved-head, straight-trunk, and truncated-tail are clearly characterized for all study catchments (**Fig. S6a in SI**). The visual interpretation is well demonstrated by the results of parameters fitted through Eq. (17) (mean squared error values < 2×10^{-8}). The tempering parameter k_d values are very small for all river networks, indicating an abrupt truncation in the tail part (**Table 1; Fig. S6b in SI**). The power-law exponent ε ranges as 0.45 ± 0.02 (**Table 1**), which agrees with the range reported in earlier studies (e.g., Rodríguez-Iturbe et al., 1992a). ε values estimated in our study networks satisfy the coupled relation with Hack's exponent h, resulting in $\varepsilon + h = 1.00 \pm 0.03$.

The ρ_a–A_p relationship is plotted over all possible value of A_p from the area of a single DEM cell (~900 m^2) to the drainage area at the direct upstream of the basin outlet. The plot greatly resembles the $P(A \geq \delta)$ distribution, exhibiting three segments of curved-head, straight-trunk, and truncated-tail (**Fig. 2a**). The representative source area A_o as the median of a given A_o^* distribution is clearly located in the upper part of straight-trunk section for all studied rivers, i.e., $A_o = 0.29 \pm 0.12$ km^2. For each study network, both minimum and maximum A_o^* are also laid within the trunk section, summarized as the minimum A_o^* = 0.04 ± 0.02 km^2 and the maximum A_o^* = 3.86 ± 3.40 km^2.

Indeed, Eq. (18) satisfies quantitative description of the ρ_a–A_p relationship for all study rivers (mean squared error values < 10^{-3}). The fitted tempering parameter k_p is nearly zero, corroborating the extremely sharp cut-off in the tail of a distribution (**Table 1**; **Fig. 2b**). The power-law exponent η ranges as 0.45 ± 0.04 (**Table 1**), which is close to but slightly smaller than the ranges of 0.48 ± 0.04 reported in Moglen et al. (1998) for 7 catchments with the median size of 30 km^2, and 0.47 ± 0.12 in Prancevic and Kirchner (2019) for 17 small mountainous catchments with the median size of 1.1 km^2. Integrating these earlier empirical outcomes and results from this study, it is clear that mostly $\eta < 0.5$. Further exploration linked to this dimensional inconsistency and fractal dimensions is given in the next section.

For every study network, the fitted η value is very close to its ε value (difference in % = 0.47 ± 0.30), which supports our theoretical derivation of $\varepsilon = \eta$ in Sect. 2.2. This means that the scaling exponent η also has intimate relation with h to be $\eta + h \sim 1$. In addition, the entire shapes of the two distributions are almost identical given $\varepsilon \approx \eta$ as well as $k_d \approx k_p$. The findings suggest that known physical meaning of ε can provide insights into what η physically stands for. By investigating the full range of binary trees from totally random to completely deterministic, Paik and Kumar (2007) highlighted that ε represents how compact the hierarchy of a given binary network is. Since they deal with tree topology, ε can be more explicitly expressed as ‘compactness of topological hierarchy.’ In the consistent context, ‘compactness of geometric hierarchy’ can be symbolized by η that is dependent on the concrete term of stream length.

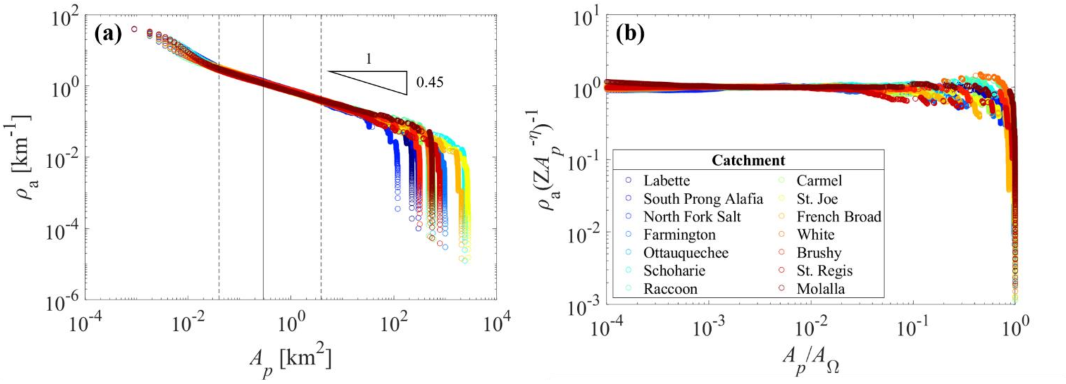

Figure 2. Relationship between the apparent drainage density ρ_a and the pruning area A_p for 14 studied river networks. (a) Distribution of ρ_a over varying A_p in a log-log scale. The averaged η is calculated as 0.45. Bold black line indicates the average of all A_o values reported in Table 1. Dashed lines depict the average of minimum and maximum A_o^* values shown in Fig. S2 in SI. (b) Normalized A_p–ρ_a distribution by individual power-law exponents. Color-codes for each catchment is consistent between (a) and (b).

Table 1. Topographic characteristics of the 14 river networks analyzed in this study

Catchment	**State**	**Climate [(1)] (%)**	**CSI [(2)]**	**Final stream-order Ω**	**Total area A_Ω (km²)**	**Source area A_o (km²)**	**Horton ratios**				**Hack's exponent**	**Area-exceedance probability distribution**		**Apparent drainage density-pruning area relationship**		**Fractal dimension**	
							R_B	R_L	R_A	R_E	h	ε	k_d (10^{-4})	η	k_p (10^{-4})	D_s	D_b
Labette Creek*	KS	Dfa (100)	58	5	222	0.21	4.2	2.3	4.8	2.1	0.60	0.46	20	0.46	22	1.0	1.8
South Prong Alafia River*	FL	Cfa (100)	65	5	350	0.32	4.1	2.3	4.4	2.2	0.52	0.47	27	0.46	28	1.1	1.7
North Fork Salt River*	MO	Dfa (100)	66	5	126	0.23	3.4	1.8	3.8	1.6	0.51	0.50	120	0.49	110	1.0	2.0
Farmington	CT	Cfa, Cfb, Dfb (42, 42, 16)	87	6	979	0.35	3.8	2.0	4.0	2.0	0.50	0.45	25	0.45	24	1.0	1.9
Ottauquechee	VT	Dfb (100)	94	6	572	0.55	3.1	1.8	3.0	1.8	0.53	0.45	11	0.45	11	1.1	1.9
Schoharie	NY	Dfb, Dfa (99, 1)	94	6	2,408	0.34	4.3	2.3	4.8	2.2	0.56	0.46	0.4	0.46	0.5	1.0	1.8
Raccoon	PA	Dfa (100)	96	5	476	0.20	5.0	3.0	5.4	2.8	0.58	0.43	3.3	0.43	3.6	1.3	1.4
Carmel	CA	Csb, Csa (99, 1)	96	6	593	0.13	4.1	2.2	4.5	2.2	0.53	0.45	24	0.45	26	1.0	1.8
St. Joe	ID	Dsb, Dsc (86, 14)	100	7	2,834	0.32	4.2	2.2	4.0	2.1	0.58	0.44	5.1	0.44	5.2	1.1	1.8
French Broad	NC	Cfa, Cfb, Dfb (42, 42, 16)	100	6	2,074	0.20	4.8	2.6	5.3	2.3	0.59	0.43	6.6	0.43	6.8	1.2	1.6
White River	AR	Cfa (100)	100	5	503	0.24	5.0	2.7	5.1	2.6	0.56	0.46	2.6	0.46	2.8	1.2	1.6
Brushy	AL	Cfa (100)	100	5	322	0.14	4.1	2.4	4.5	2.4	0.55	0.43	22	0.43	23	1.2	1.6
St. Regis River	MT	Dsb, Dsc, Dfb, Dfc (54, 39, 5, 2)	100	5	796	0.35	4.7	2.5	5.3	2.4	0.52	0.43	52	0.43	50	1.1	1.7
Molalla River	OR	Csb, Csc, Dsb (90, 2, 8)	100	5	569	0.47	4.1	2.5	4.8	2.2	0.58	0.40	7.1	0.40	7.4	1.2	1.6

Note: *Catchment name was referred from the Open Street Map as a creek or stream name at the outlet. (1) Climate zone was based on the Köppen climate classification scheme. (2) The reported Connectivity Status Index CSI was weighted by stream lengths for a given CSI.

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4 Interpretation of Dimensional Inconsistency in η

It is worthwhile to investigate η from dimensional perspective. Although $\eta = 0.5$ is anticipated for dimensional consistency (Tarboton et al., 1991), observed values are smaller than this in every network (see **Table 1**). As stated earlier, an analogous issue resides in Eq. (1): h is expected to be 0.5 but observed values are mostly greater. This inconsistency was relaxed by introducing the fractal dimension of a stream as $D_s=2h$ (Mandelbrot, 1977), which was based on the assumption that the shapes of catchments are self-similar in a downstream direction (Feder, 1988; Rigon et al., 1996). For a stream reach, the fractal nature stems from stream sinuosity. Considering the typical range of h, D_s is greater than unity, i.e., exceeding the dimension of a line, and mostly between 1 and 1.4 (Rosso et al., 1991). Motivated by this, we hypothesized that the deviation of the observed η values from 0.5 implies the presence of non-integer fractal dimension of the topography. We sought for a simple expression of η as a function of fractal dimension, like $h = D_s/2$. As $\eta = \varepsilon = 1 - h$, from $h = D_s/2$ it is clear that

$$\eta = 1 - D_s/2. \qquad (19)$$

We found that η values estimated from Eq. (19) well agrees with observed values.

However, above relationship is deceptive as Eq. (19) is identical to $\varepsilon + h = 1$ given $D_s=2h$. To resolve this issue, an independent relationship for D_s should be introduced. We can employ the expression of D_s from Horton ratios (Rosso et al., 1991) as

$$D_s = \max(1, 2\ln R_L/\ln R_A). \qquad (20)$$

Two extreme values of D_s, i.e., 1 (a line with no sinuosity) and 2 (full sinuosity of streams filling a plane), correspond to cases of $R_A = R_L^2$ and $R_A = R_L$, respectively. Our 14 study networks show the D_s range of 1.10 ± 0.10 (**Table 1**). Substituting Eq. (20) into Eq. (19) gives

$$\eta = 1 - \ln R_L/\ln R_A. \qquad (21)$$

While D_s represents the fractal dimension originated from the aforementioned fractal stream (single corridor), there is another fractal nature stemming from the network organization of stream branches. Denoting the fractal dimension covering the latter feature as D_b, La Barbera and Roth (1994) derived an expression of ε as a function of two fractal dimensions D_s and D_b. As $\eta = \varepsilon$, we can use their derivation as

$$\eta = \varepsilon = D_s(D_b - 1)/2. \qquad (22)$$

For D_b, we refer to the equation of La Barbera and Rosso (1989) as

$$D_b = \min(2, \ln R_B/\ln R_L). \qquad (23)$$

According to Eq. (23), the lower and upper limits in D_b (1 and 2) correspond to the cases of $R_B = R_L$ and $R_B = R_L^2$, respectively. Considering the typical ranges of R_B and R_L found in river networks, D_b is mostly between 1.5 and 2 (Rosso et al., 1991; La Barbera and Rosso, 1989), and our study networks present D_b ranging 1.73 ± 0.16 (**Table 1**). Substituting Eqs. (20) and (23) into (22) yields

$$\eta = \ln(R_B/R_L)/\ln R_A. \qquad (24)$$

In that both D_s and D_b are considered, Eq. (24) is regarded as a general form of Eq. (21). Indeed, one can notice that Eq. (24) becomes Eq. (21) if $R_B = R_A$. As stated, empirical findings suggest $R_B \approx R_A$, but calculated η can be sensitive to their differences. For $R_B < R_A$, which are found in most of our study networks (**Table 1**), Eq. (24) gives smaller value for η than Eq. (21).

Whilst Eq. (24) is regarded as a general expression of η as a function of Horton ratios, derived on the foundation of fractal dimension studies, we can suggest another relationship which is from a very different perspective. Examining analyzed results, we found $\eta=\alpha D_b$, the linear tendency. Further, the coefficient is fairly invariant as $\alpha = 0.26\pm0.01$, from our 14 networks, which is very close to 1/4. Interestingly, this is similar to the quarter-power scaling laws widely found in self-similar biological systems, such as the Kleiber's law (Ballesteros et al., 2018; Kleiber, 1932). Motivated by this finding and inspired by the simple expression of $h = D_s/2$, we suggest

$$\eta = D_b/4 = (\ln R_B/\ln R_L)/4. \tag{25}$$

For all studied river networks, η values estimated from Eqs. (24) and (25) were compared with those observed from the ρ_a–A_p relationship (**Fig. 3**). Between two, Eq. (24) yields much greater deviations from observations, and mostly under-estimates η values. It is interesting that the simple Eq. (25) is well supported by analysis results, with the estimated η mean of 0.44 under merely ~6 % difference from the observed η, which is around half of that calculated for Eq. (24). The inter-networks variability of the estimated η for each equation is fairly similar to that of the observed values (standard deviation = 0.06 and 0.04 for Eqs. (24) and (25), respectively).

We perceive the poor performance of Eq. (24) as the consequence of weak assumptions which form the basis of theoretical derivations of Eqs. (20) and (23), i.e., Horton's laws hold precisely at all scales of a unit length to measure (La Barbera and Rosso, 1989; Rosso et al., 1991). Indeed, this assumption is too ideal to be satisfied in real river networks, as corroborated in the non-perfect straight fits when estimating Horton's ratios of our studied networks (**Figs. S4- S5 in SI**). For D_s, the stream sinuosity cannot be directly analyzed with our DEM analysis due to limited resolution, and so large uncertainty is embedded. As a result, D_s values estimated from Eq. (20) (shown in **Table 1**) differ from $D_s=2h$ with h in **Table 1** (Mandelbrot, 1977). About D_b, the importance of fulfilling the assumption to employ Eq. (23) is also demonstrated by Phillips (1993) studying very small catchments in the Southern Appalachians in the USA.

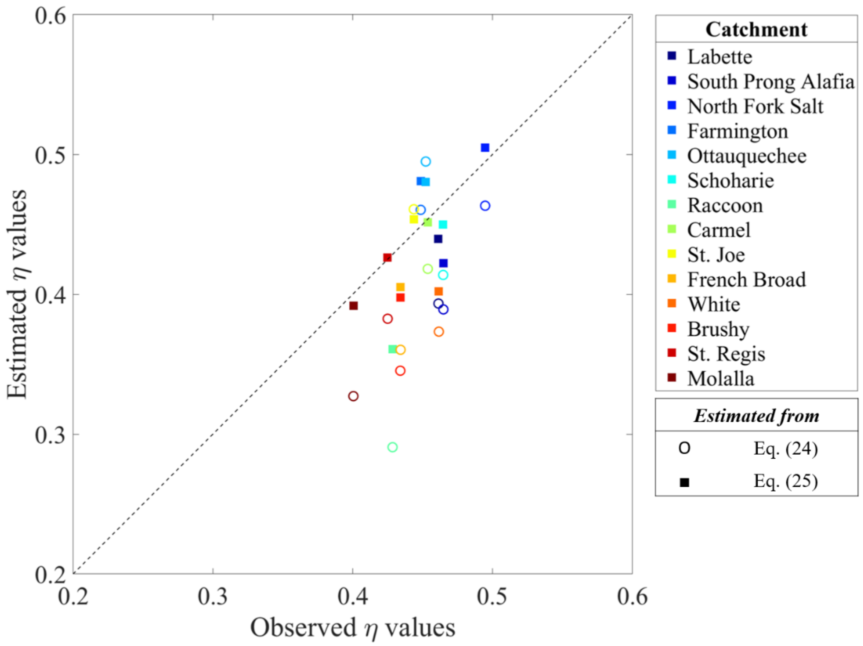

Figure 3. Comparison of η value observed from the ρ_a–A_p relationship (Eq. (18)), with η values estimated as the functions of the fractal dimensions expressed as the Horton ratios. Results of Eqs. (24) and (25) are presented as hollow-circle and filled-square markers, respectively. Color-codes for our studied river networks are the same as indicated in Fig. 2.

As shown in Fig. 3, estimated/observed η values are less than 0.5. This can be understood in three perspectives. First, taking Eq. (25), 0.5 becomes the upper limit of η, given the physical range of $1 \leq D_b \leq 2$. Second, the finding of $\eta < 0.5$ can also be understood from earlier studies on ε, given $\eta = \varepsilon$. In earlier studies about Eq. (7), $\varepsilon < 0.5$ is reported for most river networks (Rodríguez-Iturbe et al., 1992a; Crave and Davy, 1997). As Eq. (7) is about probability, no attention has been given to its dimensional consistency. Nevertheless, in theory, random critical trees should follow $\varepsilon \approx 0.5$ (Harris, 1963). Paik and Kumar (2007) investigated trees, ranging from purely deterministic to completely random, and according to observed ε values, river network organization is based on self-repetitive trees with some randomness in connectivity structure. In their follow-up study, Paik and Kumar (2011) dealt with more scaling laws of river networks to investigate the roles of the connectivity structures in tree organizations. Particularly for Hack's law analysis, they corroborated that partially random trees grounded on deterministic self-repetitive trees only exhibited the Hack's exponent h within the range found from river networks.

Lastly, $\eta < 0.5$ can be explored from plausible optimality in the network formation. To explain physical mechanisms resulting the connectivity pattern of treelike river structures, various optimality hypotheses have been proposed, such as minimizing total energy expenditure (Rodríguez-Iturbe et al., 1992b; Rinaldo et al., 2006), total stream power (Chang, 1979), and total energy dissipation rate (Yang and Song, 1979), as summarized in Paik and Kumar (2010). Although debates on the physical mechanisms are still ongoing (Paik, 2012), the typical hypotheses share the underlying principle: ***direct connectivity from individual elements to a common outlet is maximized while total length of flow paths is minimized, in turn efficient flow connection under a given space***. It is noteworthy that optimal channel networks, which were created towards achieving the minimum total energy expenditure, showed the satisfactions of Hack's law with $h \sim 0.6$ (Ijjasz-Vasquez et al., 1993) and the area-exceedance probability distribution with $\varepsilon \sim 0.44$ (Carraro et al., 2020). The results suggest that the minimization of total energy expenditure needs to be considered not as a necessary condition but a sufficient condition. The notion of optimality resides in the quarter-power scaling laws which is linked to Eq. (25). West et al. (1997) suggested a coarse-grained zeroth order theory to explain the emergence of the quarter-power scaling laws in biological systems, based on three essential and generic properties of networks in organisms: (1) space filling to serve sufficient resources to everywhere in a system, (2) invariant size and characteristics of terminal units, and (3) optimized designs to minimize energy loss. According to their theory (West et al., 1999; West, 2017), the ubiquitous number 'four' in the scaling law exponents indicates the total number of domains that all metabolic mechanisms are operated through optimized space-filling branching networks, thereby as a sum of the normal three domains representing three-dimensional appearance, and the additional one domain revealing fractal dimension feature. Indeed, it is broadly recognized that river network is an excellent analogue of biological networks in living organisms (Banavar et al., 1999). It implies that the interpretation for the number 'four' in the quarter-power scaling laws in biology may help to obtain a mechanism-based insight on the role of denominator 'four' in Eq. (25) for river networks of which fractal structures have been explained by optimality hypotheses.

5 Summary and Conclusions

Thorough investigations on the power-law relationship between the apparent drainage density ρ_a and the pruning area A_p with the exponent of η were conducted. We unraveled the meanings of η with dimensional inconsistency in diverse aspects. We analytically demonstrated that η is equivalent to the fractal scaling exponent ε in the area-exceedance probability distribution, based on a hypothetical network following the Hortonian tree framework. This pinpointed the coupled relationship between η and Hack's exponent h that is also deviated from the dimensional consistency, i.e., $(\eta = \varepsilon) + h = 1$.

Our arguments are well supported by evidence from many real river networks, analyzed with NHDPlusV2 dataset. Analyzed networks in this study cover wide ranges of climatic and free-flow connectivity conditions over the contiguous United States.

The ρ_a–A_p relationships for all studied catchments were obviously distinct into curved-head, straight-trunk and truncated-tail parts, which is identical shape as the area-exceedance probability distributions. We showed that the range of extracted source areas was clearly overlapped with the upper part of the straight-trunk section in each ρ_a–A_p distribution. Our findings highlighted that the empirical analyses results are in good agreement with the analytically found ones. It suggested that two scaling exponents η and ε are fundamentally identical but conceptually distinguishable, since geometric and topological attributes are inherent in the calculation procedure for η and ε, respectively. Hence, we enabled to define physical meaning of η as 'compactness of geometric hierarchy.'

We further examined the physical implication of η based on non-integer fractal dimensions. Such effort was elaborated as expressing η as the functions of fractal dimensions on a single stream and the entire river organization, including the quarter-power scaling relationship. Despite the presence of inevitable uncertainty in quantifying fractal dimensions, the estimated η values were likely aligned with the observed ones for all studied rivers. Given that, this study contributed to deeper understanding of the ρ_a–A_p relationship. Our findings, further, lay the foundation of future studies on the interlinkage between fractal dimensions and indicators characterizing self-similar structures of river networks.

Overall, our study sites followed representative scaling laws of river networks, despite the differences in climate condition and connectivity level. In particular, our findings suggest that the interplay between ε and h for rivers is insensitive to the diverse conditions. It leads to a natural curiosity whether the diversity scope of the conditions was not sufficient or critical anthropogenic stressors were missing to uncover exceptional real river networks exhibiting the deviation from the well-known scaling properties. A follow-up study may need to resolve such curiosity with extended study sites at a global scale and additional descriptors for anthropogenic effects on river network structures and functions.

Data availability

This study did not use any new data to conduct the presented analyses. The National Hydrography Dataset Plus Version 2 for the contiguous U.S. are publicly available.

Author contribution

SY conceptualized this study and conducted initial analyses through her Master's thesis under KP's supervision. SY and KC performed the topographic analyses for the study networks, and interpreted them. SY and KP wrote the paper, and all co-authors reviewed and edited it. Funding was acquired by KP.

Competing interests

The author declare that they have no conflict of interest.

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