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Power-Law between the Apparent Drainage Density and the Pruning Area

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Abstract. Self-similar structures of river networks have been quantified as diverse scaling laws. Among them we investigated a power functional relationship between the pruning area A_p and the associated apparent drainage density ρ_a with an exponent η. We analytically derived the relationship between η and other scaling exponents known for fractal river networks. The derivation is supported by analysis of four real river networks. The relationship between η and non-integer fractal dimensions found for natural river networks is suggested. Synthesis of our findings through the lens of fractal dimensions provides an insight that the exponent η has fundamental roots in fractal dimension for the whole river network organization.

1 Introduction

Since first proposed by Horton (1945), the drainage density ρ has long been recognized as an important metric to describe geomorphological and hydrological characteristics of a catchment. Defined as $\rho = L_T / A_\Omega$ where A_Ω is the constant catchment area, ρ is a function of the total channel length L_T in a catchment. Alternatively, ρ is a function of the channel forming area A_o (also called the source area or the critical contributing area) (Band, 1986; Montgomery and Dietrich, 1988; Tarboton et al., 1988), which is directly related to L_T. The variation of ρ among catchments is associated with the climatic condition, which can be represented by various measures such as the precipitation effectiveness index (Melton, 1957; Madduma Bandara, 1974). A_o reduces as the catchment becomes wetter, which leads to the expansion of the stream network (greater L_T) and vice versa (Godsey and Kirchner, 2014). Therefore, L_T and ρ are inversely related to A_o (Tarboton et al., 1991).

On another note, the rate at which L_T (and so ρ) varies with A_o is determined by the given topography. The close relationship between the main channel length L and the drainage area A is well known as a power function with a positive exponent h (Hack, 1957), i.e.,

$$L \propto A^h. \quad (1)$$

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Although Eq. (1) provides a clue about the relationship between L_T and A_o, they differ in two senses: (1) L_T is the total length counting all tributaries, while L is the length of the main channel only; and (2) L is the length within the drainage area A while L_T is the length of channels excluded from the coverage A_o. Hence, L_T reduces as A_o increases, while L grows with A (Eq. (1)).

The usage of the digital elevation models (DEMs) in the river network analysis introduced a constant called the pruning area A_p. In extracting a stream network from a DEM, cells of the upslope area A less than A_p are considered as hillslope and excluded from the network. If $A_p = 0$, every DEM cell is considered as channel while A_p can be as large as A_Ω for an extremely dry landscape. As A_p increases, less channels are extracted. This results in a smaller 'apparent' drainage density ρ_a. We distinguish ρ_a from the real drainage density ρ as A_p is an arbitrary value, and can be different from A_o. It was found that ρ_a decreases as A_p grows following a power function (Moglen et al., 1998), i.e.,

$$\rho_a \propto A_p{}^{-\eta}. \tag{2}$$

On the basis of the notion described above, we raise a fundamental question about the range of the scaling exponent η in Eq. (2). To satisfy dimensional consistency in Eq. (2), $\eta = 0.5$ is anticipated (Tarboton et al., 1991). This issue is analogous to the question about the exponent h in Eq. (1), which should also be 0.5 to meet the dimensional consistency (Hjelmfelt, 1988). In fact, h values reported for natural rivers are mostly greater than 0.5, i.e., between 0.5 and 0.7 (Hack, 1957; Gray, 1961; Robert and Roy, 1990; Crave and Davy, 1997). This has brought the introduction of the fractal dimension (Mandelbrot, 1977).

In this study, we aimed to address the raised question in three approaches. First, we investigated analytically the linkage between Eq. (2) and other power-laws known in natural river networks. A river network is fractal, and many regular power-laws have been reported as characteristic signatures of a naturally evolved river network (Dodds and Rothman, 2000). The power-law relationship between ρ_a and A_p can also serve as a signature reflecting the self-similarity of a river network. Then, it is plausible to claim the linkage between ρ_a–A_p relationship and other power-laws. Second, we investigated ρ_a–A_p relationship in real river networks, through DEM analysis. The relationship was analyzed for seven catchments in an earlier study (Moglen et al., 1998) but, A_o and A_p were undistinguished in their study. To properly estimate η, detailed analyses with a greater resolution DEM for catchments of known A_o or blue-lines are required. Lastly, we hypothesized that not only the power-law itself but also the likely dimensional inconsistency in Eq. (2) implies the fractal nature of a river network. On this background, we searched for a linkage between η and fractal dimension.

In the next Sect. 2, we reviewed the scaling relationships known in a river network. Then, we presented analytical derivation of Eq. (2), and demonstrated how this is related with other power-laws known for a river network. We implemented terrain analyses in a thorough manner as described in Sect. 3. With these results, we explored physical meanings embedded in the power-law relationship between ρ_a and A_p with the notion of fractal dimension in Sect. 4. Summary and conclusions are given in Sect. 5.

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2 Cross-Relationships among Scaling Laws

2.1 Review on scaling laws of a river network

The river network has been perceived as an archetypal fractal network in nature (Mandelbrot, 1977; Rodríguez-Iturbe and Rinaldo, 2001), exhibiting scale-invariant organization. Systematic measures for characterizing structural hierarchy help manifest the self-similarity. Horton-Strahler ordering scheme (Horton, 1945; Strahler, 1957) has been popularly employed to investigate their structural characters. In this framework, the number, the mean length, and the mean drainage area of ω-order streams in a catchment, stated as $N_\omega, \bar{L}_\omega$, and $\bar{A}_\omega$, respectively, are defined for an order ω ranging from 1 to Ω, where Ω is the highest order in the network. There is only one Ω–order stream in a river network (i.e., N_Ω=1) of which length and the drainage area are L_Ω and A_Ω, respectively. Then, the total channel length L_T used for the definition of the drainage density ρ, is given as

$$L_T = \sum_{\omega=1}^{\Omega} N_\omega \bar{L}_\omega. \quad (3)$$

Following its definition, $\bar{A}_\omega$ includes the drainage area of all upstream branches (of $\omega - 1$ and lower orders). By contrast, the length of any lower order stream is excluded in $\bar{L}_\omega$. Therefore, L_Ω is neither the upslope length L of a main channel, nor L_T, while A_Ω is identical to the total drainage area of the catchment. To resolve inconsistent definitions of $\bar{A}_\omega$ and $\bar{L}_\omega$, two metrics have been proposed. First, the cumulative mean length was proposed to match the definition of area (Broscoe, 1959) as

$$\Xi_\omega = \sum_{k=1}^{\omega} \bar{L}_k. \quad (4)$$

Ξ_ω is essentially an order-discretized version of L. Alternatively, to match the definition of length, the eigenarea, also called the interbasin area (Strahler, 1964) or the contiguous area (Marani et al., 1991), was proposed. It indicates the area directly draining to the ω–order stream (Beer and Borgas, 1993). The mean eigenarea $\bar{E}_\omega$ of ω–order streams is

$$\bar{E}_\omega = \bar{A}_\omega - \bar{A}_{\omega-1}(N_{\omega-1}/\ N_\omega). \quad (5)$$

The self-similar structure of a river network has been captured through the linear scaling of above quantities as ($N_\omega, \bar{L}_\omega$, $\bar{A}_\omega$, and $\bar{E}_\omega$) with ω on a semi-log paper (Horton, 1945; Schumm, 1956; Yang and Paik, 2017) as

$$N_\omega = {R_B}^{\Omega-\omega};\ \bar{L}_\omega = L_\Omega {R_L}^{\omega-\Omega};\ \bar{A}_\omega = A_\Omega {R_A}^{\omega-\Omega};\ \bar{E}_\omega = E_\Omega {R_E}^{\omega-\Omega} \quad (6)$$

where R_B, R_L, R_A, and R_E are the bifurcation, the length, the area, and the eigenarea ratios, respectively. As a group, they are often called the Horton ratios and dependent on each other (Morisawa, 1962). They are dimensionless ratios of quantities between nearby orders, i.e., $R_B = N_\omega/N_{\omega+1}$, $R_L = \bar{L}_{\omega+1}/\bar{L}_\omega$, $R_A = \bar{A}_{\omega+1}/\bar{A}_\omega$, and $R_E = \bar{E}_{\omega+1}/\bar{E}_\omega$. In natural river networks, they typically range as $3 < R_B < 5$, $1.5 < R_L < 3$, and $3 < R_A < 6$ (Smart, 1972). Yang and Paik (2017) suggested $R_E \approx R_L$.

In addition to Eq. (6), power functional relationships between geomorphologic variates have also been found and served as evidences of the scale-invariant river network structures. The Hack's law (Eq. (1)) is a classical principle in this line. Another power-law relationship of our interest lies in the exceedance probability distributions of upstream area. Using a theoretical aggregation model, Takayasu et al. (1988) showed that the exceedance probability distribution of injected mass in a tree

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network always follows a power-law. In fact, their model is equivalent to the random-walk model of Scheidegger (1967) devised to mimic a river network (Takayasu and Nishikawa, 1986). Replacing the mass (flow) in the aforementioned study with the drainage area (which is reasonable if rainfall is spatially uniform), it leads to the power-law exceedance probability distribution of 'drainage area.' In a detail, the probability for a randomly designated point within a catchment to have A exceeding a reference value δ ($0 \leq \delta \leq A_\Omega$) decreases with δ (Rodríguez-Iturbe et al., 1992), following a power-law as

$$P(A \geq \delta) \propto \delta^{-\varepsilon} \tag{7}$$

where the exponent ε is reported as between 0.40 and 0.46 for most river networks (Rodríguez-Iturbe et al., 1992; Crave and Davy, 1997). Aforementioned two power-laws (Eqs. (1) and (7)) are related to each other as $h + \varepsilon = 1$ (Maritan et al., 1996), which suggests a trade-off between the two exponents to form the catchment boundary within a confined 2-d space.

Two classes of scaling relationships reviewed above, i.e., Horton's laws (Eq. (6)) and power-law relationships are linked as shown by La Barbera and Roth (1994), i.e.,

$$\varepsilon = 1 - h = \frac{\ln (R_B/R_L)}{\ln R_A}. \tag{8}$$

Two other expressions, comparable to Eq. (8), appear in literature. de Vries et al. (1994) derived $\varepsilon = 1 - \ln R_L/\ln R_B$. For a 'topological' Hortonian tree where no constraint on stream length in a finite area is given, Veitzer et al. (2003) and Paik and Kumar (2007) showed that $\varepsilon = \ln R_B/\ln R_A - 1$. These expressions are special cases of Eq. (8) where $R_B = R_A$ and $R_L = R_A$, respectively. Many empirical studies support that R_B is indeed close to R_A (e.g., Smart, 1972). The assumption of $R_L = R_A$ was used in the analysis of 'topological' self-similar trees where only connections among nodes matter with no spatial constraint (Paik and Kumar, 2007).

2.2 Linkage to A_p-ρ_a relationship

Below, we analytically derive the relationship between the pruning area A_p and the resulting apparent drainage density ρ_a (Eq. (2)), using the aforementioned scaling relationships. Through this investigation, we importantly conclude that $\eta = \varepsilon$, i.e., the scaling exponents in Eqs. (2) and (7) are identical. We arrive at the same conclusion from two different approaches, described below.

2.2.1 Derivation 1

For the Hortonian tree, we vary A_p in a discrete manner (order-by-order), i.e., we set $A_p = \bar{A}_\omega$. Given that up to ω-order streams are pruned in a river network, the total length after pruning is expressed as $\sum_{k=\omega+1}^{\Omega} N_k \bar{L}_k$, by revising Eq. (3). Replacing N_k and $\bar{L}_k$ in this equation with Eq. (6) leads to the expression of ρ_a as

$$\rho_a = \frac{L_\Omega}{A_\Omega} \sum_{k=\omega+1}^{\Omega} R_B{}^{\Omega-k} R_L{}^{k-\Omega}. \tag{9}$$

Above sum of the given geometric series is

$$\rho_a = \frac{L_\Omega}{A_\Omega(R_B/R_L-1)}\left[\left(\frac{R_B}{R_L}\right)^{\Omega-\omega} - 1\right]. \quad (10)$$

The term $(R_B/R_L)^{\Omega-\omega}$ in Eq. (10) is rewritten as follows. From Eq. (6), we can state $\Omega - \omega = \ln(A_\Omega/\bar{A}_\omega)/\ln R_A$. The logarithm is taken for the term $(R_B/R_L)^{\Omega-\omega}$ as

$$\ln\left(\frac{R_B}{R_L}\right)^{\Omega-\omega} = (\Omega-\omega)\ln\frac{R_B}{R_L} = \frac{\ln(A_\Omega/\bar{A}_\omega)}{\ln R_A}\ln\frac{R_B}{R_L} = \frac{\ln(R_B/R_L)}{\ln R_A}\ln\frac{A_\Omega}{\bar{A}_\omega}. \quad (11)$$

Given that $\bar{A}_\omega = A_p$, from Eq. (11) we can state

$$(R_B/R_L)^{\Omega-\omega} = (A_\Omega/A_p)^{\frac{\ln(R_B/R_L)}{\ln R_A}}. \quad (12)$$

Substituting this into Eq. (10) yields an approximate power-law, i.e.,

$$\rho_a = \frac{L_\Omega}{A_\Omega(R_B/R_L-1)}\left[\left(\frac{A_p}{A_\Omega}\right)^{-\frac{\ln(R_B/R_L)}{\ln R_A}} - 1\right] \propto A_p^{-\frac{\ln(R_B/R_L)}{\ln R_A}}. \quad (13)$$

Given that $R_B \approx R_A > R_L$ (Smart, 1972) for a typical river network, the exponent of Eq. (13) ranges as $-1 < -\ln(R_B/R_L)/\ln R_A < 0$. With this range and for $A_p \ll A_\Omega$, $(A_p/A_\Omega)^{-\ln(R_B/R_L)/\ln R_A} = (A_\Omega/A_p)^{\ln(R_B/R_L)/\ln R_A} \gg 1$. This allows the approximation $[(A_\Omega/A_p)^{\ln(R_B/R_L)/\ln R_A} - 1] \approx (A_\Omega/A_p)^{\ln(R_B/R_L)/\ln R_A}$. Comparing Eqs. (2) and (13), we can explicitly express

$$\eta = \frac{\ln(R_B/R_L)}{\ln R_A}. \quad (14)$$

This expression is identical to Eq. (8), which implies $\eta = \varepsilon$.

2.2.2 Derivation 2

Our conclusion of $\eta = \varepsilon$ can also be derived by employing the eigenarea (Yang, 2016). For ω-order streams, it was claimed that $\bar{E}_\omega = W\bar{L}_\omega$ where the overland flow length W is regarded almost constant (Hack, 1957; Yang and Paik, 2017). Therefore, the apparent drainage density for the pruning area $A_p = \bar{A}_\omega$ becomes

$$\rho_a = \frac{1}{A_\Omega}\sum_{k=\omega+1}^{\Omega} N_k\bar{L}_k = \frac{1}{A_\Omega W}\sum_{k=\omega+1}^{\Omega} N_k\bar{E}_k. \quad (15)$$

On the other hand, $P(A \geq A_p)$ is defined from geometry as

$$P(A \geq A_p) = \frac{1}{A_\Omega}\sum_{k=\omega+1}^{\Omega} N_k\bar{E}_k \quad (16)$$

which equals to $W\rho_a$ from Eq. (15). As $P(A \geq A_p) \propto A_p^{-\varepsilon}$ (Eq. (7)), we realize that $\rho_a \propto A_p^{-\varepsilon}$ and thereby $\eta = \varepsilon$.

Earlier, we discussed the reciprocal nature of two relationships of $L_T - A_o$ and $L - A$. Combining above conclusion of $\eta = \varepsilon$ and $h + \varepsilon = 1$, we realize that $\eta = 1 - h$, indeed implying the compensating function between the two relationships.

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3 Analyses of Real River Networks

3.1 Data and methods

To evaluate the proposed power-law Eq. (2) and the derivation of $\eta = \varepsilon$, we analyzed four river networks in the contiguous United States under distinct hydro-climatic conditions: the Molalla River, the Racoon Creek, the St. Regis River, and the White River (Fig. 1, Table 1). The Molalla and the White Rivers were studied in the hydrological context by Botter et al. (2007), and the other two were examined in the geomorphological context by Tarboton et al. (1991).

We used 1 arc-second raster DEM, generated by Shuttle Radar Topography Mission and provided by United States Geological Survey (USGS), that has 1 m vertical resolution. Every depression or sink cell in the raw DEM was filled by raising its elevation to the lowest elevation among its neighboring eight cells, employing the algorithm of Planchon and Darboux (2002). Depression filling might yield flat surfaces over which flow directions are difficult to be specified. The imposed gradients method (Garbrecht and Martz, 1997) was applied to form the micro-differences of elevation especially around flat areas. Then, flow direction for each cell was assigned through the improved Global Deterministic 8 method (Shin and Paik, 2017). Accordingly, upslope area was calculated for each cell. To extract river networks resembling individual blue-lines most, we referred to the National Hydrography Dataset Plus Version 2 (NHDPlusV2) that includes river blue-lines and the corresponding source areas for the contiguous U.S. (McKay et al., 2012). In each study catchment, a channel forming area is given for every channel head in NHDPlusV2. They exhibit a range of distribution in each catchment (Fig. S1 in the Supporting Information, SI) and a single value A_o was determined for each catchment as the median (Table 1). Horton-Strahler ordering was assigned on the pruned river networks.

Regarding the exceedance probability distribution of upstream area (Eq. (7)), three segments are often characterized: curved-head, straight-trunk, and truncated-tail. The head reflects hillslope (Moglen and Bras, 1995; Maritan et al., 1996) while the trunk part indicates channels. As the upslope area becomes close to A_Ω, the distribution rapidly falls because of the finite size of a network (Rodríguez-Iturbe et al., 1992; Moglen et al., 1998; Perera and Willgoose, 1998). To accommodate such an effect in the distribution function, the exponentially tempered power function was adopted (Aban et al., 2006; Rinaldo et al., 2014; Yang et al., 2017) as

$$P(A \geq \delta) = K\delta^{-\varepsilon} \exp(-M\delta)\,, \text{for } \delta > A_o \quad (17)$$

where K and M are constants. As M approaches zero, the function represents abrupt truncation like as the finite size effect of a network (Rodríguez-Iturbe et al., 1992; Moglen et al., 1998; Perera and Willgoose, 1998). Similarly, we proposed an exponentially truncated power function for ρ_a, as a general form of Eq. (2), as

$$\rho_a = Z{A_p}^{-\eta} \exp\left(-NA_p\right), \text{for } A_p > A_o \quad (18)$$

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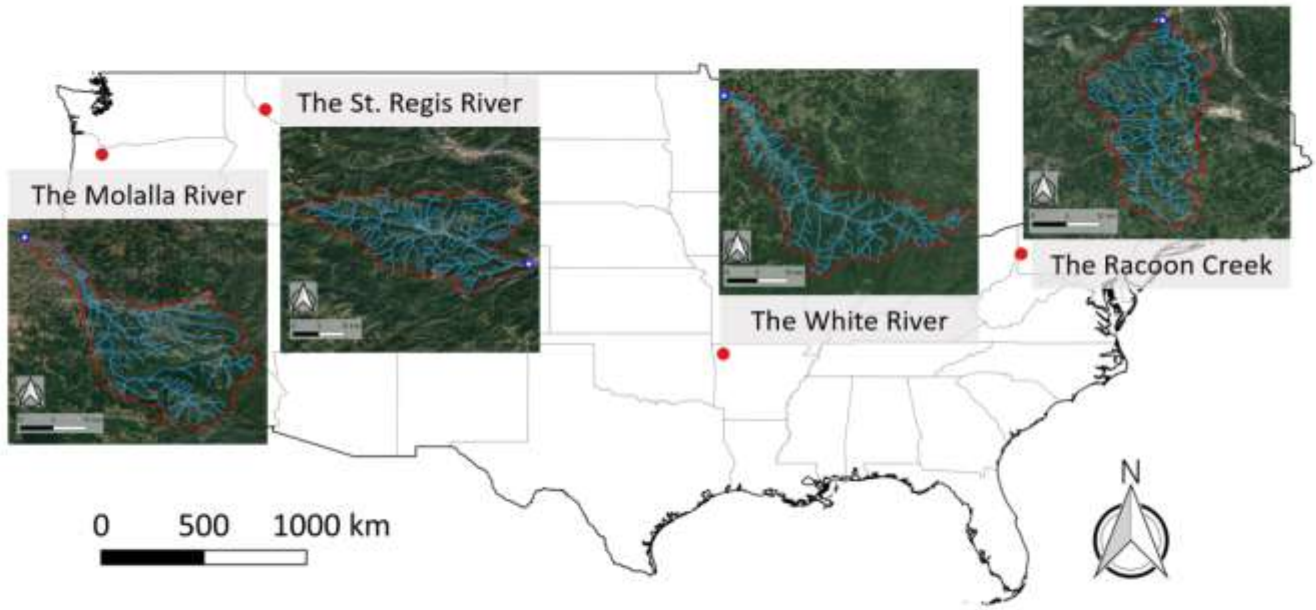
where Z and N are constants. To estimate the best fitted parameters, we employed Matlab's *nlinfit* function of which the objective function is to minimize the sum of the squares of the residuals for the fitted model. The estimated range for a parameter was calculated with 95% confidence intervals.



175 **Figure 1.** Locations and structures of four river networks investigated in this study. A circle mark in each nested figure represents the river basin outlet. River network layouts (light blue color lines) are originated from NHDPlusV2. Satellite images on the background of the study areas are obtained from ©Google Earth.

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Table 1. Topographic characteristics of study river networks

		River name in USA (state)			
		Molalla (OR)	Racoon (PA)	St. Regis (MT)	White (AR)
Final stream-order Ω		5	5	5	5
Total area A_Ω (km^2)		536	448	739	477
Source area A_o (km^2)		0.47	0.20	0.34	0.24
Horton ratios	R_B	4.1	4.9	4.5	5.0
	R_L	2.5	2.8	2.5	2.7
	R_A	4.5	5.4	5.1	5.1
	R_E	2.3	2.6	2.4	2.4
Hack's exponent	h	0.55	0.55	0.52	0.56
Area-exceedance probability distribution	ε	0.41	0.46	0.42	0.47
	M (10^{-4})	9.7	8.5	40	2.8
Apparent drainage density-pruning area relationship	η	0.42	0.46	0.43	0.47
	N (10^{-4})	7.2	6.3	50	3.1
Fractal dimension	D_s	1.1	1.1	1.0	1.1
	D_b	1.5	1.5	1.7	1.6

3.2 Results and discussion

All studied river networks follow the laws of stream number, length, drainage area, and eigenarea (Eq. (6)) with $R^2 > 0.9$ (Fig. S2 in SI). They show narrow ranges of Horton ratios ($R_B = 4.7 \pm 0.4$, $R_L = 2.6 \pm 0.2$, $R_A = 5.0 \pm 0.4$, $R_E = 2.4 \pm 0.1$; mean ± standard deviation) (Table 1), which are within the typical ranges reported in earlier studies. Further, all study networks well satisfy the power-law Eq. (1) (Fig. S3 in SI). The range of Hack's exponent h value is estimated as 0.55±0.02 with $R^2 > 0.95$ (Table 1), which is within the typical range known in earlier studies (Hack, 1957). These features imply that our study networks are self-similar and of typical natural river networks.

In the exceedance probability distributions of upstream area, three segments of curved-head, straight-trunk, and truncated-tail are clearly characterized (Fig. 2a). The visual interpretation is well demonstrated by the results of parameters fitted through Eq. (17) for all studied catchments (mean squared error values < 6×10^{-8}). The power-law exponent ε of the area-exceedance probability distribution ranges as 0.44±0.03 for the study areas (Table 1), which is close to the range reported in earlier studies

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(Rodríguez-Iturbe et al., 1992). The tempering parameter M values are very small for all river networks, indicating an abrupt truncation in the tail part (Table 1; inset figure in Fig. 2a).

The ρ_a–A_p relationship is plotted over all possible value of A_p from the area of a single DEM cell (~900 m^2) to the drainage area at the direct upstream of the basin outlet. The plot greatly resembles the $P(A \geq \delta)$ distribution, exhibiting three parts of head, trunk, and tail (Fig. 2b). We found that Eq. (18) satisfies quantitative description of the ρ_a–A_p relationship for all study rivers (mean squared error values < 10^{-4}). The power-law exponent η is estimated as 0.45 ± 0.02 (Table 1), which is close to the η range of 0.48 ± 0.04 reported in Moglen et al. (1998). For every river network, an η value is very close to its ε value (difference in % = 1.2 ± 1.4), which supports our theoretical argument of $\varepsilon = \eta$ derived in Sect. 2.2. Fitted tempering parameter N is also very small, corroborating the extremely sharp cut-off in the tail part (Table 1; inset figure in Fig. 2b). The $\eta(= \varepsilon) + h$ values for the four rivers are close to unity (0.99 ± 0.04; Table 1), which empirically verifies the interdependent relationship between the two scaling indicators.

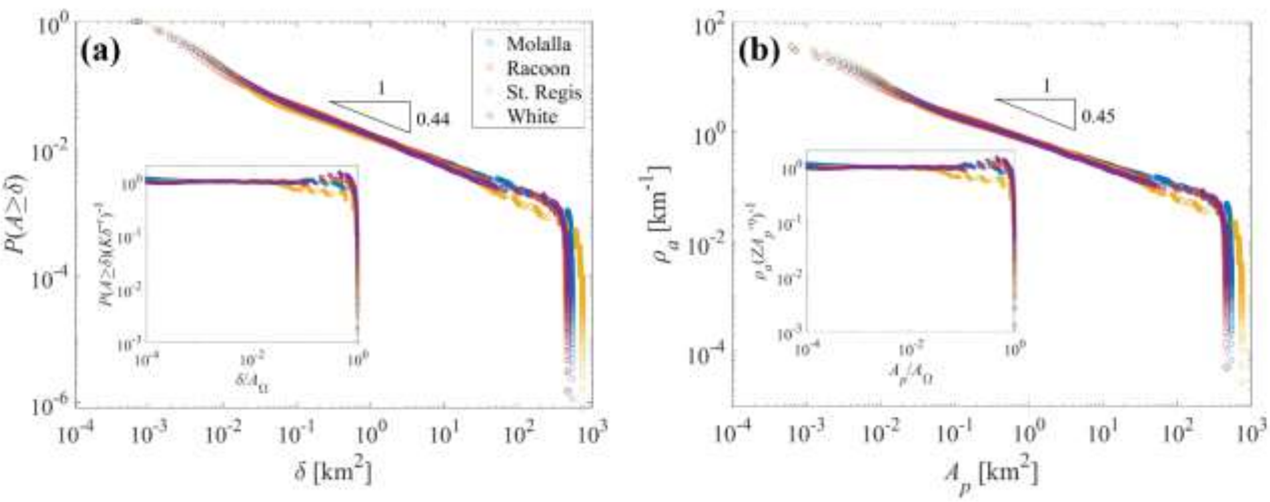

Figure 2. Power-law analyses for four studied river networks. (a) Exceedance probability distribution of upstream area δ, where the averaged ε is estimated at 0.44. (b) Relationship between the pruning area A_p and the apparent drainage density ρ_a, where the averaged η is calculated at 0.45. Inset figures in (a) and (b) represent individual normalized distributions with each power-law fit.

4 Fractal Dimension

It is worthwhile to investigate the physical implication of Eq. (2) from dimensional perspective. For dimensional consistency, η = 0.5 is anticipated (Tarboton et al., 1991). But observed values are slightly smaller than this (see Table 1). As stated earlier, this issue is similar to the dimensional inconsistency in Eq. (1): h is expected to be 0.5 but observed values are mostly greater.

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This inconsistency was relaxed by introducing the fractal dimension of a stream as D_s=2h (Mandelbrot, 1977), which was based on the assumption that the shapes of river basins are self-similar in a downstream direction (Feder, 1988). For a stream reach, the fractal nature stems from stream sinuosity. Considering the typical range of h, D_s is greater than unity, i.e., exceeding the dimension of a line, and mostly between 1 and 1.4 (Rosso et al., 1991). Motivated by this, we hypothesized that the deviation of the observed η values from 0.5 implies the fractal dimension of the topography. We sought for a simple expression of η as a function of fractal dimension, similarly to $h = D_s/2$. Three expressions can be claimed for η below.

First, as we learned $\eta = \varepsilon = 1 - h$, from $h = D_s/2$ we can conclude that

$$\eta = 1 - D_s/2. \quad (19)$$

We found that η values estimated from Eq. (19) well agrees with observed values. However, this is deceptive as Eq. (19) is identical to $\varepsilon + h = 1$ given D_s=2h. Accordingly, in this section, we employ another expression of D_s estimated from Horton ratios (Rosso et al., 1991) as

$$D_s = \max(1, 2\ \ln R_L / \ln R_A). \quad (20)$$

Two extreme values of D_s, i.e., 1 (a line with no sinuosity) and 2 (full sinuosity of streams filling a plane), correspond to cases of $R_A = R_L^2$ and $R_A = R_L$, respectively. Substituting Eq. (20) into Eq. (19) gives

$$\eta = 1 - \ln R_L / \ln R_A, \quad (21)$$

which is the first expression for η.

The fractal characteristic of a river network is originated from two features: (1) the aforementioned fractal stream (single corridor) and (2) the fractal network organization of such streams. While D_s is the fractal dimension to represent the former, the fractal dimension stemming from the latter feature is denoted as D_b. La Barbera and Roth (1994) derived an expression of ε as a function of two fractal dimensions D_s and D_b. As $\eta = \varepsilon$, we can use their derivation as

$$\eta = D_s(D_b - 1)/2. \quad (22)$$

For D_b, we refer to the equation of La Barbera and Rosso (1989) as

$$D_b = \min(2,\ \ln R_B / \ln R_L). \quad (23)$$

According to Eq. (23), the lower and upper limits in D_b (1 and 2) correspond to the cases of $R_B = R_L$ and $R_B = R_L^2$, respectively. Considering the typical ranges of R_B and R_L found in river networks, D_b is mostly between 1.5 and 2 (La Barbera and Rosso, 1989; Rosso et al., 1991). We can substitute Eqs. (20) and (23) into (22) as

$$\eta = \ln(R_B/R_L)/\ln R_A. \quad (24)$$

In addition to above two expressions (Eqs. (21) and (24)), we suggest a simple relationship, on the basis of examining our analysis of real river networks, as

$$\eta = D_b/4 = (\ln R_B / \ln R_L)/4. \quad (25)$$

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This functional form is similar to the quarter-power scaling laws widely found in biological systems. The metabolic rates of mammals scale with their respective mass spanning more than eight orders of magnitude with approximately 3/4 power-law exponent, known as Kleiber's law (Kleiber, 1932; Ballesteros et al., 2018). Almost all quantities characterizing physiological phenomena and biological networks in organisms systematically follow power-law over diverse spectrum of size (or mass), with scaling exponents very much approximating to quarter-multiples, such as -1/4 for heart rates (Dawson, 2001), and 3/4 for number of capillaries (Savage et al., 2004). For the emergence of the quarter-power scaling laws, West et al. (1997) suggested a coarse-grained zeroth order 'theory' based on three essential and generic properties of networks in organisms: (1) space filling to serve sufficient resources to everywhere in a system, (2) invariant size and characteristics of terminal units, and (3) optimized designs to minimize energy loss. According to their theory (West et al., 1999; West, 2017), the ubiquitous number 'four' in the scaling law exponents indicates the total number of domains that all metabolic mechanisms are operated through optimized space-filling branching networks, thereby as a sum of the normal three domains representing three-dimensional appearance, and the additional one domain revealing fractal dimension feature. Broadly recognized that river network is an excellent analogue of biological networks in living organisms (Banavar et al., 1999), the interpretation for the number 'four' in the quarter-power scaling laws in biology can be fully transferrable to obtain a mechanism-based insight on the role of denominator 'four' in Eq. (25). In case of the river network organization formed in three dimensions, the total number of domains characterizing the object and its fractality becomes 4 (as 3+1). Hence, $D_b/4$ ($=\eta=\varepsilon$) indicates the 'standardization' of fractal dimension D_b defined for the entire river network.

η values estimated from three Eqs. (21), (24), and (25) show notable differences among them and from the observed η values in the ρ_a–A_p relationship (Fig. 3). This result manifests that each expression holds an uncertainty forbidding exact correspondence with the observed. We suppose that the uncertainty may be mostly attributable to non-perfect straight fits when estimating Horton's ratios (as shown in Fig. S2 in SI). Despite the accumulated uncertainties, a remarkable finding is that the new Eq. (25) proposed from an off-the-wall perspective yields satisfactory result for η estimate.

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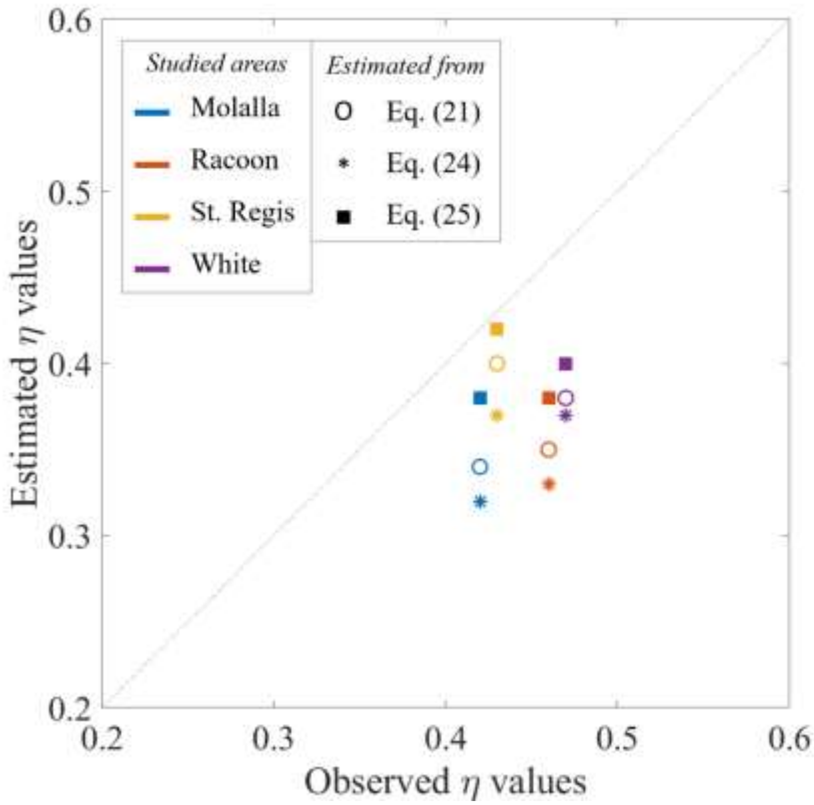

Figure 3. Comparison of η value observed from the ρ_a–A_p relationship (Eq. (18)), with η values estimated as the functions of the fractal dimensions expressed as the Horton ratios. Results of Eqs. (21), (24), and (25) are presented as circle, asterisk, and filled-square markers, respectively. Color-codes for our studied four river networks are the same as indicated in Fig. 2.

5 Summary and Conclusions

In this study, we thoroughly investigated the power-law relationship between the apparent drainage density ρ_a and the pruning area A_p, and revealed its meaning for characterizing river network organization. We analytically found that η is equivalent to the fractal scaling exponent ε in the area-exceedance probability distribution. Analyses for four river networks in the USA support our conclusion, suggesting the convergence of diverse descriptors for fractal river network at the most fundamental level. We further presented how η is associated with fractal dimensions, based on inspiration from fractality embedded in the dimensional inconsistency between the ideal and observed η values. We expect that the novel interpretation sheds a new light on the subject of the standardized fractal dimension.

Data availability

This study did not use any new data to conduct the presented analyses. Dataset of the 1 arc-second raster DEM and the National Hydrography Dataset Plus Version 2 for the contiguous U.S. are publicly available.

Author contribution

SY conceptualized this study and conducted initial analyses through her Master's thesis under KP's supervision. SY and KC performed the topographic analyses for the study areas, and interpreted them. SY and KP wrote the paper, and all co-authors reviewed and edited it. Funding was acquired by KP.

Competing interests

The author declare that they have no conflict of interest.

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