# A mixed distribution approach for low-flow frequency analysis – Part 1: concept, performance and effect of seasonality

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**Abstract.** In seasonal climates with a warm and a cold season, low-flows are generated by different processes so that the annual extreme series will be a mixture of summer and winter low-flow events. This leads to a violation of the homogeneity assumption for all statistics derived from the annual series and give rise to inaccurate conclusions. In this paper we propose a mixed distribution approach to perform frequency analysis in catchments with mixed low-flow regime. We formulate the theoretical basis of the mixed distribution approach for the lower extremes based on annual minima series. The main strength of the model is that it allows the user to estimate return periods of summer low-flows, winter low-flows and annual return periods in a theoretically sound and consistent way. Using archetypal examples, we show how the model behaves for a range of low-flow regimes, from distinct winter and summer regimes to mixed regimes where seasonal occurrence in summer and winter is equally likely. The examples show in a qualitative way the errors we have to expect with conventional extreme value statistics performed with the annual extremes series. The model is then applied to a comprehensive Austrian data set to quantify the expected gain of using the mixed distribution approach compared to conventional frequency analysis. Results indicate that the gain of using a mixed distribution approach is indeed large. On average, the error is reduced by 21, 39 and 63 % when estimating the low-flow with a 20-, 50- and 100-year return period. For the 100-year event, 75 % of stations show a performance gain of > 10 %, 41 % of stations > 50 % and 25 % of stations > 80.6. This suggests a broad relevance of the approach that goes beyond highly mixed seasonal regimes to include the strongly seasonal ones. We finally correlate the performance gain with seasonality indices in order to show the expected gain conditional to the strength of seasonality expressed by the ratio of average summer and winter low flow $SR$. For the 100-year event, the expected gain is about 70 % for $SR = 1.0$, 20 % for $SR = 1.5$, and 10 % for $SR = 2.0$. The errors are further allocated to the spatial patterns of $SR$ in the study area. The results suggest that the mixed estimator is relevant not only for mountain forelands but for a broad range of low flow regimes. The mixed distribution approach provides one consistent approach for summer, winter, and annual probabilities and should be used by default in seasonal climates with a cold winter season where summer and winter low flows can occur.

## 1 Introduction

Extreme value statistics are of particular importance to hydrology. The methodology was originally developed for floods, but has recently received increasing attention for low-flows. It allows the user to characterize the severity of an event by a statistical

25  probability or return period, i.e., the expected time interval until an event of the same magnitude occurs. This is a very natural method of characterizing extremes that is well understood in the scientific, policy, and public arenas. The underlying methods of general frequency analysis are well established and simple to use. However, they rely on the basic assumption of data homogeneity, meaning that the data belong to the same population, which is often not the case for the data or problems under study.

30  This is a particular challenge in low-flow hydrology. In seasonal climates, with a warm and a cold season, low-flows are generated by different processes, so that the annual extreme value series are a mixture of summer recessions and winter freeze events. This leads to a violation of the homogeneity assumption for all statistics derived from the annual series and gives rise to inaccurate conclusions. In such a case, the data set can be stratified into a summer and a winter part to make the extreme value series more homogeneous (Laaha and Blöschl, 2006b). Temporal stratifications require the beginning and end of low-flow

35  seasons to be well defined depending on climate and hydrological conditions. For European conditions, various studies have suggested a division into a summer season from about April or May to November, and a winter season for the remaining part of the year (Laaha et al., 2017). The classification has been shown to apply to other parts of the Northern Hemisphere as well, and can be generalized to the Southern Hemisphere when the seasons are shifted by six months (Hodgkins and et al., 2022).

Based on the seasonal extreme value series two types of analysis can be conducted. In the first case, summer and winter series

40  are analyzed separately, which yields distinct design values for each season. The seasonal characteristics are directly relevant for a number of water management tasks, including environmental flows and hydropower production. For other problems annual characteristics that describe the absolute minimum regardless of the season are more relevant. In this case, the seasonal distributions of summer and winter can be combined into a valid annual distribution using a mixed distribution approach.

Mixed distribution approaches for maxima have a longer tradition in hydrology, but have received increasing attention in

45  recent years. The methods date back to the original extreme value theory of Gumbel (1960), who developed concepts for bivariate distributions to capture the co-occurrence of events in different variables. Yue et al. (1999) used a bivariate Gumbel mixed model, that is, a bivariate extreme value distribution model with Gumbel marginal distributions, to perform a joint frequency analysis of flood peak and magnitude. However, they did not stratify the data by season because they focused their study on a specific regime where all events are uniformly triggered by rain over snowmelt and thus coincide in the

50  spring. Stedinger et al. (1993) presented a mixed distribution approach for the case of seasonally separable events. Assuming independence of summer and winter events, event probabilities are combined using the multiplication rule of statistics. A similar approach was used by Schumann (2005) to assess the 2002 extreme flood event in the Mulde river basin in Germany. The concept is straightforward and has gained increasing attention in recent years as the accuracy of flood statistics has come to the fore. As a result, the mixed probability estimator became the recommended method of the German flood estimation

55  guideline (Deutsche Vereinigung für Wasserwirtschaft, 2012) for the use of causal information augmentation as it allows the user to consider seasonal floods as well.

While mixed distribution approaches have been used for floods, we are not aware of any study that assesses the value of using mixed distribution approaches for low-flows. Since the methods were originally developed for maxima, some adjustments will be needed to make the methods suitable for low-flow statistics.

60    In this paper we propose a mixed distribution approach for low-flows to perform frequency analysis in catchments with a mixed summer/winter regime. The aim of the paper is threefold:

– To formulate the theoretical concept of the mixed distribution approach for low-flows;

– To illustrate the characteristics of the model for archetypal low-flow regimes, from pronounced summer and winter regimes to flow mixtures with weak seasonality;

65    – To evaluate the gain in performance from the mixed distribution model.

The analysis will be performed based of a comprehensive Austrian data set that covers a broad range of low-flow regimes.

## 2    Mixed distribution approach for low-flows

### 2.1    Basic concept

The common approach of low-flow frequency analysis is based on classical extreme value theory. It involves (a) the selection of
70    extreme vents from the flow record, (b) the choice of an appropriate distribution model, (c) the estimation of model parameters, and finally (d) the prediction of event probabilities $p$ and return periods $T = 1/p$. Low-flows are characterised by a time scale that is much larger than for other hydrological extremes such as floods or heavy precipitation events. It is therefore common to use the annual minima series

$$AM = min\{X_1, ..., X_n\} \tag{1}$$

75    for low-flow frequency analysis, where $X_1, ..., X_n$ is a sequence of independent random variables that have a common distribution function $F$ and are sampled on an annual time scale. This is termed the classical AMS-approach for minima in extreme value statistics (Coles, 2001). Tallaksen and Van Lanen (2004) found the procedure straightforward, but regarding its uncertainty strongly depending on the basic assumptions of the model adopted. In particular in snow- and ice-affected regions low-flow can occur in different seasons and are then not homogeneous. In such cases the elements of the AMS are not identi-
80    cally distributed, so that the general extreme value theory is likely less suitable. Adaptations are required on both the data and model side of the approach.

As for the data side, the annual minima series $AM$ can be viewed as the minima of the annual summer minima $AM_S$ and winter minima $AM_W$:

$$AM = min\{AM_S, AM_W\} \tag{2}$$

85    Under the assumption that summer and winter events are independent from each other, the occurrence probability of an event with magnitude $q$ is obtained from a multiplication of its respective non-occurrence probabilities in the summer and winter seasons (Stedinger et al., 1993). This yields to the definition of the mixed probability $p_{mix}$ for minima:

$$p_{mix} = 1 - (1 - p_S)(1 - p_W), \tag{3}$$

where $p_S$ and $p_W$ are the respective occurrence probabilities of the flow characterising the event in the summer and winter season.

## 2.2 Theoretical probability estimator

Equation 3 applies to single event probabilities and can be generalized for the entire distribution. Let $F_S(q)$ and $F_W(q)$ denote the marginal extreme value distributions of low-flow occurrence in summer and winter season, respectively. By the notation $F.(q)$ we make explicit that the same flow (i.e. the event magnitude of interest) is inserted into both marginal distributions. Then, if summer and winter low-flows are statistically independent, the cumulative distribution function (cdf) of the annual low-flow is

$$F(q) = F_{mix}(q) = F_S(q)F_W(q). \tag{4}$$

Following the Fisher-Tippett-Gnedenko theorem of extreme value theory, the annual low-flow follows a Weibull-distribution, with marginal summer distribution

$$G_S(q) = 1 - exp\left[-\left(\frac{q-\zeta_S}{\beta_S}\right)^{\delta_S}\right], \tag{5}$$

and marginal winter distribution

$$G_W(q) = 1 - exp\left[-\left(\frac{q-\zeta_W}{\beta_W}\right)^{\delta_W}\right]. \tag{6}$$

The $\zeta.$, $\beta.$ and $\delta.$ are the location, scale and shape parameters of summer (index $S$) and winter (index $W$) distributions, respectively. Inserting the marginal Weibull-distributions into Eq. 4 leads to the mixed Weibull-model for minima, which can be written as

$$G_{mix}(q) = 1 - \left\{1 - exp\left[-\left(\frac{q-\zeta_S}{\beta_S}\right)^{\delta_S}\right]\right\}\left\{1 - exp\left[-\left(\frac{q-\zeta_W}{\beta_W}\right)^{\delta_W}\right]\right\}. \tag{7}$$

By the model, the event probability is calculated from the product of the marginal (counter) Weibull-distributions with different, summer and winter parametrizations. Both are evaluated at the discharge value $q$, the event magnitude of interest.

## 2.3 Empirical probability estimator

Process heterogeneity leads to a violation of the homogeneity assumption for all statistics derived from the annual series. The empirical probability estimator, used for plotting the annual low-flow series, is no exception. Let $q$ be an element of an annual minima series with $n$ observations, and $m$ its rank in increasing order. Then, the common empirical probability estimator from a (homogeneous) sample is

$$p_m(q) = \frac{m}{n+1}. \tag{8}$$

115  For mixed low-flow regimes the annual minima are heterogeneous and the calculation of empirical probabilities should be based on the summer- and winter series instead. When defining $m_S$ as the rank of $q$ in the $AM_S$ and $m_W$ its rank in the $AM_W$ the mixed empirical probability estimator can be written as

$$p_{mix}(q) = 1 - \left[ (1 - \frac{m_S}{n_S + 1})(1 - \frac{m_W}{n_W + 1}) \right]. \tag{9}$$

Unlike for the common empirical probability estimator, the value $q$ of the annual series may be missing in the marginal series
120  and its empirical probability need to be approximated in this case. Such approximation can be obtained by taking the average rank of next smaller and bigger element to $q$ in the considered marginal series.

## 2.4  Demonstration of model behaviour

Using archetypal examples, we demonstrate how the model behaves for a range of low-flow regimes: from weakly seasonal ones where summer and winter occurrences are equally likely, to strongly seasonal ones where low-flows in one season predominate.
125      Our first example is gauge Ebensee at river Langbathbach situated in the northern foothills of the Alps in the federal country Upper Austria. It represents the case of a fully mixed low-flow regime resulting from tributaries draining alpine parts and lowland parts of the catchment equally. It has a mixture rate of $MR = 0.57$, which means that about half (57 % to be exact) of the events in the annual minima series are summer events and the other half are winter events. This is also reflected by a very weak seasonality, such as indicated by a low variability measure of the circular seasonality index ($r = 0.49$), and a seasonality
130  ratio close to one ($SR = 1.09$) (for the definition of indices see Laaha and Blöschl, 2006b). In such case we can expect that there is a large difference between the annual distribution fitted to the heterogeneous annual series, and the mixture distribution resulting from the distributions fitted to the homogeneous seasonal series. Figure 1a shows that this is indeed the case for the considered catchment. Three interesting areas can be distinguished in the graph. At the lower tail the mixed distribution coincides with the summer distribution, as the lowest values occur in summer and make the summer distribution fall much
135  below the winter distribution. In the central part the mixture distribution follows the annual distribution, as summer and winter distributions coincide in this part. At the upper tail the mixed distribution deviates from the annual distribution which reflects the divergence of summer and winter distributions in the upper part. In low-flow frequency analysis, we are most interested in the lower tail, at return periods of 20 years and more. Figure 1a shows that there is a large difference between the distributions in this area, so we can expect a large gain in accuracy from using the mixed distribution approach for strongly mixed low-flow
140  regimes. In the case of gauge Ebensee the mixed distribution falls below the annual distribution so the mixed distribution approach will yield lower discharge quantiles.
     Figure 1b gives another example of a highly mixed low-flow regime, albeit with a slightly different behavior. The example is gauge Weg at river Isen situated in low-hilly terrain in Bavaria, Germany. Now 67 % ($MR$) of annual low-flows occur in summer, and the magnitude of summer low-flow is somewhat lower (-10 % on average) than the winter low-flow. This is
145  indicated by its $SR$ of 0.9. As can be seen from the graph, the summer distribution is always lower than the winter distribution, and only summer low-flows occur at the lowest part. The mixed distribution reflects this behavior by following the summer distribution at the lower tail and combining the probabilities in the range of discharges where low-flows in both seasons can
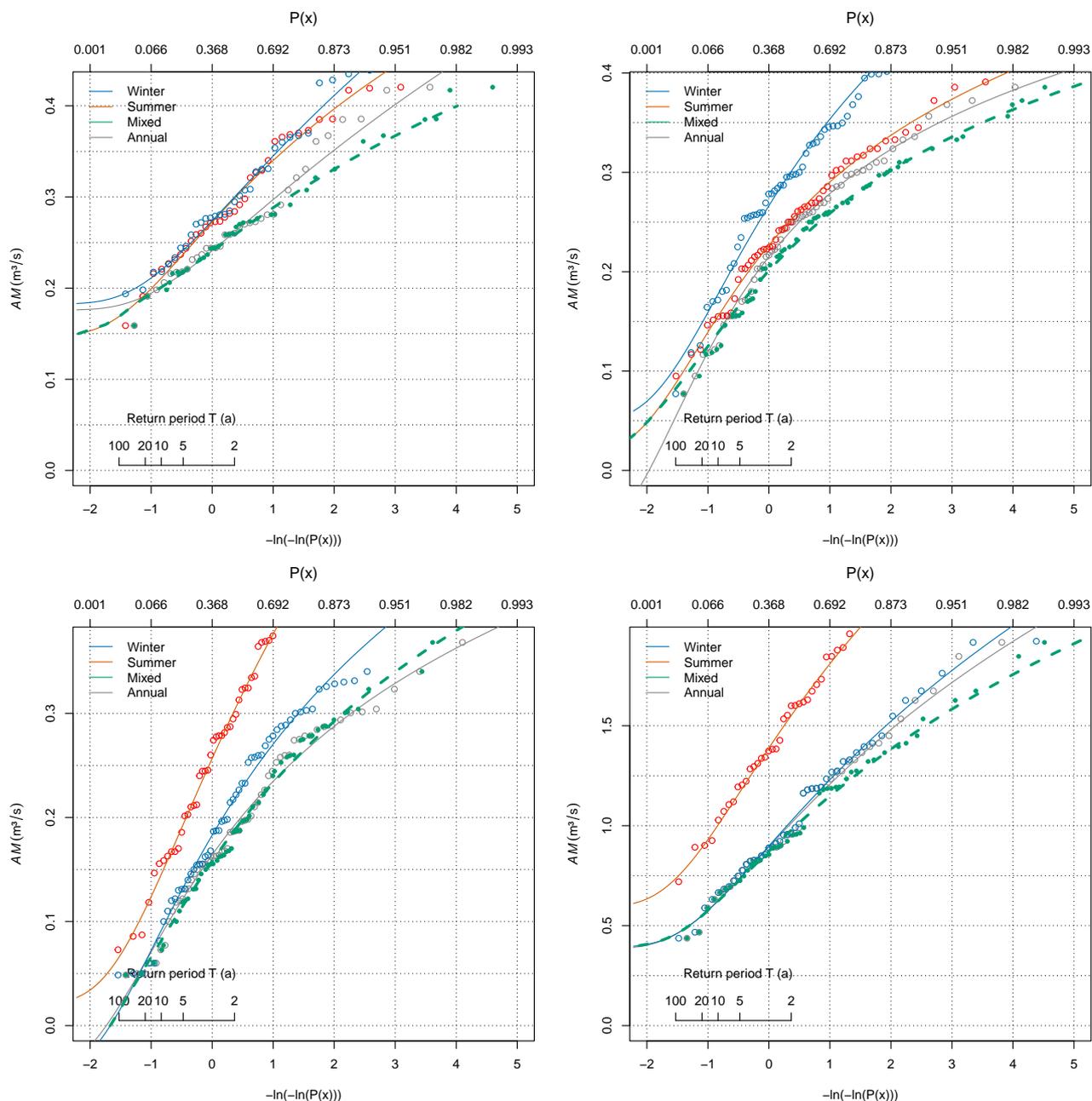
**Figure 1.** Probability plots of summer (red), winter (red), annual (black) and mixed (green) distributions for archetypal cases: (a) gauge Ebensee at river Langbathbach situated in the northern foothills of the Alps of Upper Austria, (b) gauge Weg at river Isen situated in low-hilly terrain in Bavaria, Germany, (c) gauge Schönenbach at river Subersach representing an alpine catchment of the Bregenzerwald in Vorarlberg, Austria, and (d) gauge St. Peter-Freienstein at river Vordernberger Bach representing an inner-alpine catchment in Styria, Austria.

occur. In such a case the mixed distribution is always different to the conventional annual distribution and is expected to lead to different results at all probabilities. Notably at high return periods (T > 20 years), the mixed probability estimator yields much

150 higher discharge quantiles than the annual probability estimator.

The third example (Fig. 1c) is an alpine catchment in the Bregenzerwald in Vorarlberg, Austria, which again has a lower mixture rate (28 %) and a stronger seasonality ($SR = 2.41$). This is also reflected in a larger gap between the summer and winter distributions, indicating a relatively low probability of low-flow events in summer compared to winter. Consequently, summer low-flows should have little effect on mixing probabilities (Eq. 4), and so should be little difference between the annual and

155 mixed probability estimators. This assumption is also supported by the data example (Fig. 1c). The annual, mixed and winter distributions coincide in the lower part of the distribution reflecting moderate to severe low-flow events. For the upper part (high magnitudes) summer low-flows can occur but are less likely than winter events. This leads to a slight divergence between the mixed and the annual distribution, although with little significance for the extreme value analysis.

The fourth, and final example (Fig 1d) is an inner-alpine catchment in Styria, Austria, which has the lowest mixture rate

160 ($MR = 0.11$) and again a pronounced seasonality ($SR = 1.72$). Here five summer low-flow events (between 0.90 $m^3s^{-1}$ for the year 2003 and 1.53 $m^3s^{-1}$ fort the year 1976) mix in the annual minima series and should have little effect on the distribution. Figure 1d shows that the mixed distribution in the data example does indeed completely match the annual distribution for the most interesting area for analysis. Slight divergence occurs only above 0.90 $m^3s^{-1}$, where the summer low-flows are mixed in.

165 The examples suggest that the mixed model is a valid generalization of the annual probability, since both models coincide for cases when no mixture occurs. In case of mixed series the annual probability model suffers from a violation of the basic homogeneity assumption and the mixed model should be used. The examples shed light on the errors we need to expect from the conventional extreme value statistics in a qualitative way. Below, we assess these errors quantitatively.

## 3   Evaluation using streamflow data

170 ### 3.1   Dataset

The mixture model is evaluated based on a comprehensive Austrian data set of natural-like catchments with no or little antropogenic disturbance to the low-flow regime. The data set was already used in a number of low-flow studies, addressing seasonality indices (Laaha and Blöschl, 2006b), catchment classification and regional regression (Laaha and Blöschl, 2006a), geostatistical methods (Laaha et al., 2014), estimation methods from short records and spot-gauging (Laaha and Blöschl,

175 2005), the link between meteorological drought indices and streamflow (Haslinger et al., 2014), and climate change (Laaha et al., 2016; Parajka et al., 2016; Karanitsch-Ackerl et al., 2019). Most recently, an updated version of the data set was used for evaluating statistical learning methods (Laimighofer et al., 2022b) and statistical space-time models for low-flow (Laimighofer et al., 2022a). In this study we use the data set of Laaha et al. (2016) consisting of 329 Austrian stream gauges with measurements from the 1950 to 2010 period (Fig. 2). The data cover catchments from 7 to 7000 $km^2$ with elevations from 159 to 3770

180 $m.a.s.l.$. The mean annual low-flow discharges of these catchments range from 0.03 to 1013 $m^3s^{-1}$.
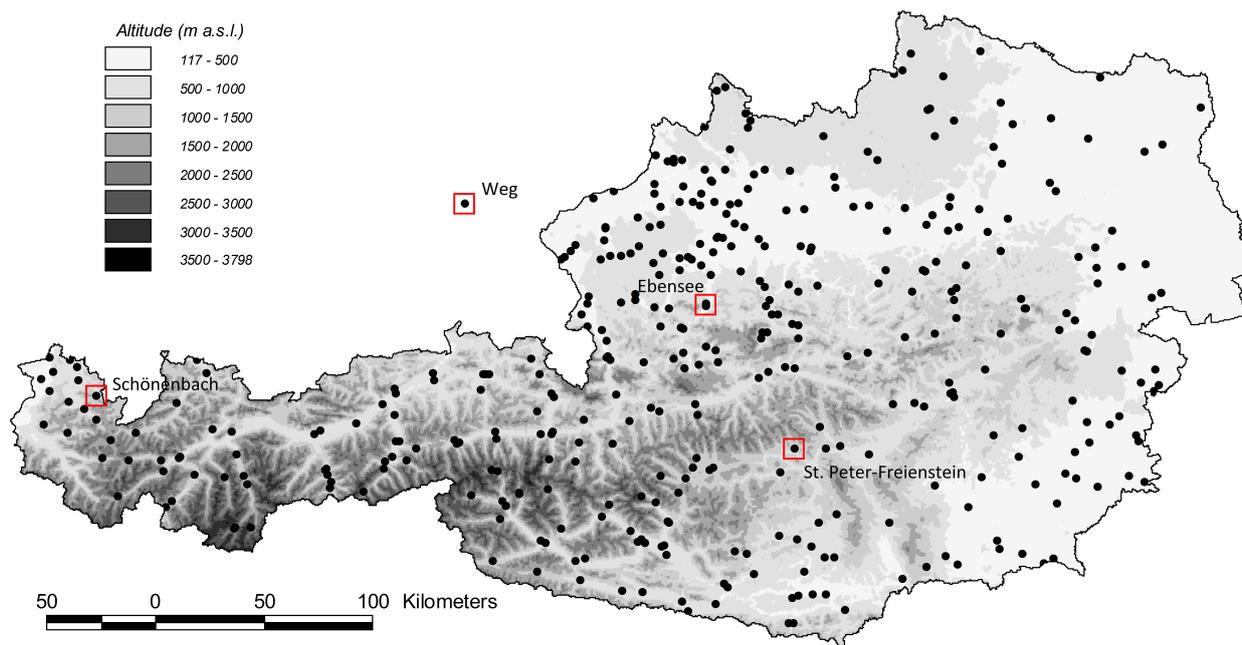
**Figure 2.** Topography and stream gauging network in Austria. Points indicate location of gauges used in this study. Example gauges of Fig. 1 are indicated by red squares. Redrawn from Laaha and Blöschl (2006b).

Austria covers an area of 84 000 $km^2$ and is climatically, physiographically and hydrologically, highly diverse. In the lowland east, low-flows mainly occur in the summer and are a result of a seasonal water balance deficit triggering stream flow recessions. In the Alpine areas in the central and western part of Austria, low-flows mainly occur in the winter and are a result of snow storage and frost processes. Both areas are characterised either by a pronounced summer or winter seasonality, with
185   mixture zones of weak seasonality in between. The study area has a very high gauging density and also a high data quality, therefore providing an ideal test bed for the the performance of the mixed seasonality estimator.

## 3.2   Assessment method

### 3.2.1   Performance measures

The very goal of frequency analysis is to characterise the severity of an event $q$ by its occurrence probability $p_q$ or, equivalently,
190   its return period $T_q = 1/(1-p_q)$. When we use a different probability model the return period of an event will change, so we are interested in the change of characteristic return periods between the two models. The difference between one model estimate and the other model estimate is referred to in statistics as *deviation*. When we can assume that one model is superior to the

other, the change in performance of the superior model compared to the inferior model can be termed the *gain* in accuracy. When we place the emphasis on the inferior model, the same quantity can be called an error, or *loss* in accuracy.

195    Based on this concept, the evaluation of the performance gain at a station of using the mixed probability estimator over the annual probability estimator proceeds in four steps:

–  Define the return period of interest (e.g. $T = 20, 50, 100$ years);

–  Calculate the associated flow quantile $q_T$ of the annual distribution model (using the inverse of its distribution, known as quantile function);

200    –  Insert the value so obtained into the mixed distribution model (Eq. 9) to calculate its probability $p_{mix}$;

–  Transform the obtained probability into the return period $T_{mix}$, which constitutes the *improved* estimate.

From the assumed input and the obtained output return periods, common deviation statistics are computed. The change of the return period ($d$) at a station is simply calculated as

$$d_T = T - Tmix \tag{10}$$

205 It can be expressed as a relative deviation ($rd$), what makes the measure comparable for different return periods:

$$rd_T = \frac{T - Tmix}{Tmix}. \tag{11}$$

The performance gain of a model is finally assessed based on the absolutes of the relative deviation ($rad$) which is calculated as:

$$rad_T = |rd_T|. \tag{12}$$

210 **3.2.2    Effect of seasonality**

The purpose of the mixed probability estimator is to enhance frequency analysis for mixed summer/winter regimes. Its use is based on the assumption that the annual probability estimator suffers from heterogeneity in mixed low-flow regimes and the mixed seasonality estimator will improve the calculations. Thus, to prove the concept, we test whether, and to what extent, the performance gain of the mixed seasonality estimator depends on seasonality.

215    In a first step, we assess the strength of the relationship with common low-flow seasonality indices (Laaha and Blöschl, 2006b). This is done by correlation analysis using Pearason and Spearman correlation coefficients. Since the percentage gains have a skewed distribution, we consider both their untransformed and log-transformed values for the analysis. The considered seasonality indices include the variability measure of the circular seasonality index ($r$) and the seasonality ratio ($SR$). The $SR$ shows a particular distribution that is symmetric around 1 (no seasonality) and has decreasing frequency towards the tails

220 of the distribution. We therefore examine the untransformed values and the absolutes of the log-transformed values of $SR$ to test possible linearizations. The circular seasonality measure $r$ has no particular distribution and enters the correlation analysis with untransformed values. The correlation coefficients are tested using standard t-tests with a significance level of $\alpha = 0.05$.

In a second step, we model the dependency of the performance gain from seasonality using linear regression. Since we expect log-linear relationships, we linearize the equations to estimate linear models that fit the particular variables. The models considered are of the following forms:

$$log(y)\~x \tag{13}$$

$$log(y)\~abs(log(x)) \tag{14}$$

where $y$ is the relative deviation ($rd_T$) and $x$ the seasonality measure under consideration ($r$ or $SR$). The symbol ~ in our notation corresponds to the formula syntax of the statistical software R and is used to specify the relationship between predictand and predictor.

In a final assessment, we evaluate the spatial cross-correlation of the gain in performance with seasonality patterns in Austria from Laaha and Blöschl (2006b). This will shed light under what hydrological conditions the mixed estimator is expected to improve the estimates.

## 4   Results

### 4.1   Performance gain of the mixed probability estimator

The error distribution of the annual probability estimator as compared to the mixed probability estimator is presented in Fig. 3, and summary statistics are presented in Table 1 and Table 2. The results show a very wide dispersion indicating that the differences between models are large. This applies to all return periods considered, but some variation can be seen. The deviations are most pronounced for the 100-year return period and decrease for 50- and 20-year events, as one would expect.

A surprising aspect of the broad distribution is that there must be a large portion of catchments where the mixed distribution approach improves performance. In fact, there are only 32 to 34 cases (about 10 % of the catchments) where the deviation between the models is zero and therefore the use of the improved estimator has no effect on the estimate. For the 100-year event, 75 % of stations show an (absolute) performance gain of > 10 %, 41 % of stations > 50 % and 25 % of stations > 80.6 (Table 2). This suggests that the mixed distribution approach is not only relevant in cases with a highly mixed low-flow regime, but also applies to catchments with a rather strong seasonal low-flow regime.

More details can be seen from Table 1. The results show a range of accuracy gains from -100 to + 1000 %, with a lower quartile of -35 % and an upper quartile of +44 % for the 100-year event. For the lower return periods, the lower and upper quartiles are -100 and +461 % for $T$=50, and -99 and +140 % for $T$=20 years. The expected value of the deviations is close to zero for the 20 and 50-year events, indicating that the annual estimator has no systematic bias compared to the mixed estimator for the lower and intermediate return periods. However, for the 100-year events, the expected deviation is +23 %, indicating a substantial overestimation of the return period by the annual estimator.

The effects of using the mixed estimator are most apparent when considering the absolute magnitude of the errors (Table 2). These make evident that the mixed estimator results in average performance gains of 21, 39, and 63 % in estimating low-flow

**Table 1.** Relative deviation between the annual and mixed probability estimator (%)

| T | Min. | 1st Qu. | Median | Mean | 3rd Qu. | Max. |
|---|------|---------|--------|------|---------|------|
| 100 | -100.00 | -34.70 | 0.00 | 23.20 | 44.20 | 1004.40 |
| 50 | -100.00 | -25.70 | 0.00 | 4.00 | 22.40 | 460.90 |
| 20 | -99.20 | -15.10 | 0.00 | -2.00 | 11.10 | 139.60 |

with T=20, 50, and 100-year return periods, respectively. Overall, the performance gain is much higher and affects a much
255 larger number of cases than we have expected.
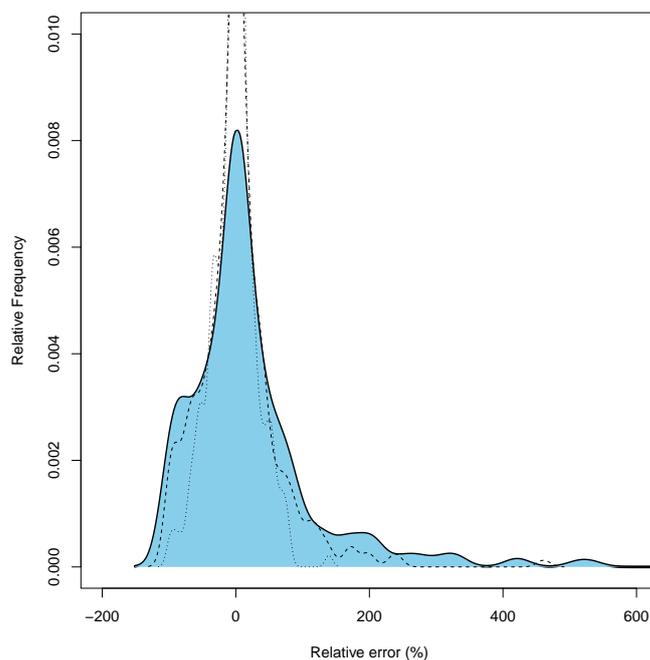
**Figure 3.** Uncertainty of the annual probability estimator as compared to the mixed probability estimator. Full line with blue shaded area
refers to the 100-year event. For comparison the 50-years event (dashed line) and the 20-year event (dotted line) are shown.

**Table 2.** Relative absolute deviation ($rad$) between the annual and mixed probability estimator (%).

| T | Min. | 1st Qu. | Median | Mean | 3rd Qu. | Max. |
|---|---|---|---|---|---|---|
| 100 | 0.00 | 10.00 | 38.30 | 63.00 | 80.60 | 1004.40 |
| 50 | 0.00 | 6.50 | 24.30 | 38.90 | 58.30 | 460.90 |
| 20 | 0.00 | 3.30 | 12.30 | 21.20 | 33.10 | 139.60 |

## 4.2 Effect of seasonality on performance gain

### 4.2.1 Correlation with seasonality indices

We first assess the strength of the relationship of the accuracy gain with three seasonality indices: The variability measure of the circular seasonality index (mean resultant $r$), the seasonality ratio $SR$ and its absolute logarithm $|log(SR)|$. The Spearman and Pearson correlation coefficients are shown in Table 3. All correlations are significant at the $\alpha = 0.05$ level. The strongest relationships can be found for the log-transformed relative absolute deviation. It is, overall, best correlated with the transformed vales of the seasonality ratio, $|log(SR)|$ according to both the Spearman and Pearson coefficients. For the Pearson coefficients the relationships are much stronger for the log-transformed than for the untransformed relative absolute deviation. This points to log-linear relationships that can be linearized using a log-transformation. Here, the untransformed $abs(SR)$ reaches almost the same correlation as $|log(SR)|$, and the correlation of $r$ is only slightly lower. For the Spearman coefficients the results are insensitive for a log-transformation of the relative absolute error. Overall the performance gain of the mixed estimator is strongly correlated with seasonality, and a log-transformation can help linearize the relationship.

**Table 3.** Correlation of relative absolute deviation ($rad$) and its logarithm ($log(rad)$) with seasonality indices $r$, $SR$, $|log(SR)|$. All correlations are significant at the $\alpha = 0.05$ level.

| | Spearman coefficient | | | Pearson coefficient | | |
|---|---|---|---|---|---|---|
| | $r$ | $SR$ | $|log(SR)|$ | $r$ | $SR$ | $|log(SR)|$ |
| rad (T=100) | -0.43 | -0.35 | -0.45 | -0.18 | -0.12 | -0.12 |
| rad (T=50) | -0.45 | -0.32 | -0.45 | -0.23 | -0.15 | -0.19 |
| rad (T=20) | -0.50 | -0.29 | -0.45 | -0.33 | -0.18 | -0.25 |
| log(rad) (T=100) | -0.43 | -0.35 | -0.45 | -0.43 | -0.47 | -0.48 |
| log(rad) (T=50) | -0.45 | -0.32 | -0.45 | -0.45 | -0.45 | -0.48 |
| log(rad) (T=20) | -0.50 | -0.29 | -0.45 | -0.49 | -0.41 | -0.47 |
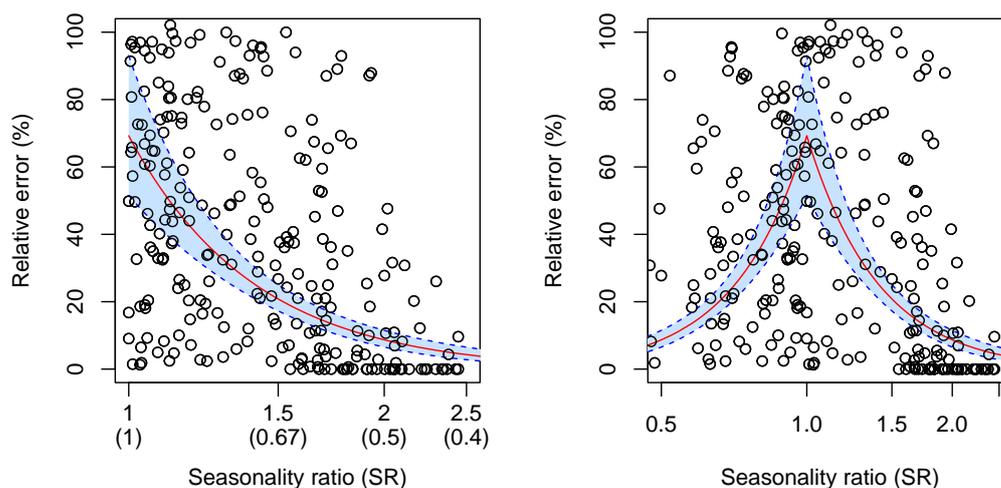
**Figure 4.** Performance gain by seasonality ratio ($SR$). Shown is the log-linear model between relative error and absolute logs of $SR$ (red lines refer to the expected values, blue to their 95 % confidence bounds). Black points are the observations. In panel (a) $SR < 1$ are plotted as inverted values ($1/SR$) and labeled by the numbers in parentheses. Panel (b) shows the relationship for the original values of $SR$.

### 4.2.2 Expected performance gain by seasonality

As a second step of the analysis we model the dependency of the performance gain from seasonality using log-linear regression.

270 Despite there is a large variability in the data the regression model appears well suited as it is able to represent the decline of variability with increasing seasonality. Table 4 shows the expected gains from the regression model for various return periods. For the 100-year event, the expected gain is about 70 % for $SR = 1.0$, 20 % for $SR = 1.5$, and 10 % for $SR = 2.0$. For the 50-year event the expected gain is about two-thirds as large, and for the 20-year event the gain is almost half as large. The results show that the gain of using the mixed probability estimator is not only limited to highly mixed regimes but also applies

275 to strongly seasonal ones. The gain is particular large for high return periods where results are expected to differ strongly for a wide range of regimes. For low return periods such as T=20 years strong effects can be found for highly mixed regimes (say, $SR < 1.5$) whereas there is little gain for highly seasonal regimes.

### 4.2.3 Spatial cross-correlation with seasonality patterns in Austria

In the final step of the analysis, the errors are compared with the spatial patterns of $SR$ in the study area to shed light on the

280 hydrological and topographic relevance of the mixed probability estimator. Figure 5 from Laaha and Blöschl (2006b) presents a map of seasonality index $SR$. The patterns correlate strongly with the topography of the study area, with strong winter

**Table 4.** Expected performance gains from the regression model for various return periods

| | Seasonality ratio (SR) | | | |
|---|---|---|---|---|
| T | 1 | 1.5 | 2 | 2.5 |
| 100 | 69.30 | 20.80 | 8.70 | 4.30 |
| 50 | 44.70 | 14.30 | 6.20 | 3.10 |
| 20 | 24.30 | 8.50 | 3.90 | 2.00 |
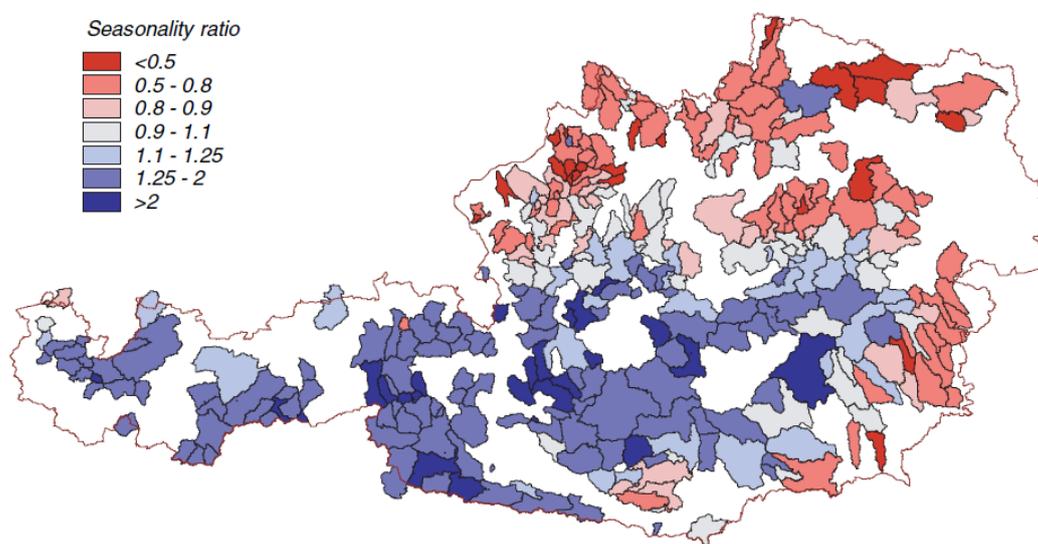
**Figure 5.** Ratio of summer and winter low flow discharges (SR) for 325 subcatchments in Austria. SR > 1 indicates a winter low flow regime; SR < 1 indicates a summer low flow regime. From Laaha and Blöschl (2006b)

seasonality in the high Alps, strong summer seasonality in the lowlands, and mixed regimes in between. Figure 6 shows the relative gain by seasonality ratio using the same classes as used in the map. The lowest gains can be observed for strong winter seasonality ($SR > 2$) where the mixed estimator is almost identical with the annual estimator. For all other classes the

285 differences are considerable as the have an expected value of at least 25 %. Importantly, this also includes the highly summer dominated catchments where one would expect the annual estimator well suited. This is because in cold temperate climate, low-flow relevant frost events can also occur in the lower catchments in some years, which are then mixed in the annual low-flow distribution. The highest gain is observed for mild summer low-flow regimes ($SR$ between 1.1 and 1.25) which correspond to typical mid-mountain catchments where the frost events are enhanced by the slightly higher altitude and the mixture between

290 summer and winter events is particularly strong. The results suggest that the mixed estimator is relevant not only for catchments in mountainous regions. It is relevant for most catchments, except for the high alpine ones.
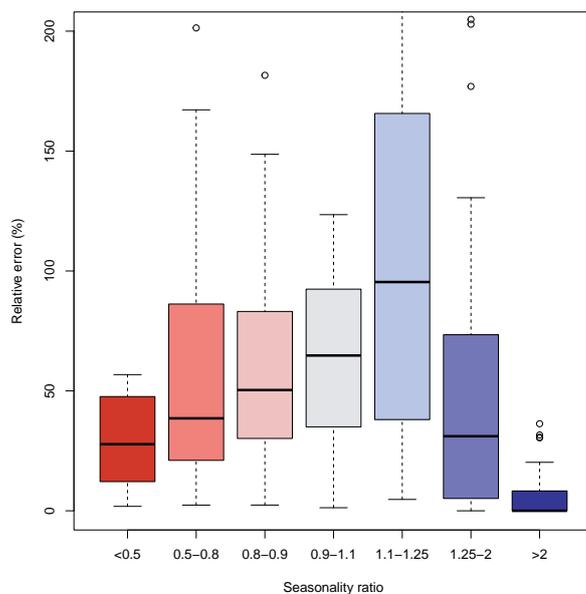
**Figure 6.** Performance gain by seasonality ratio classes of Laaha and Blöschl (2006b). The colour coding is those of Fig. 5.

## 5   Discussion and conclusions

This paper has proposed a mixed distribution approach to perform frequency analysis in catchments with mixed summer/winter low-flow regime. We formulated the theoretical basis of the mixed distribution approach for the lower extremes based on annual
295   minima series. The performance of the model was evaluated based on a comprehensive Austrian data set. Results indicate that the model performs well for a range of low-flow regimes, from distinct winter and summer regimes to mixed regimes where seasonal occurrence in summer and winter is equally likely. This finding goes far beyond our expectation that the mixed estimator would be relevant only for highly mixed regimes, and that its application would be limited mainly to mountain forelands. On average, for the study area, the error is reduced by 21, 39 and 63 % when the low-flow with a return period of
300   20, 50 and 100 years is estimated, respectively.

The performance gain depends on the degree of mixture, or the seasonality ratio, as on would expect. For the 100-year event, the expected gain is about 70 % for $SR = 1.0$, 20 % for $SR = 1.5$, and 10 % for $SR = 2.0$. For the 50-year event the expected gain is about two-thirds as large, and for the 20-year event the gain is almost half as large. The gain is substantial for high return periods where results are expected to differ strongly for nearly all seasonal regimes. For low return periods such as T=20
305   years strong effects can be found for highly mixed regimes (say, $SR < 1.5$) whereas the gain for strongly seasonal regimes is quite low.

As illustrated by archetypal examples, the mixed estimator has very favorable properties: If no mixture occurs, both approaches provide the same results. This means that the mixed estimator is a valid generalisation of the common annual estimator. If mixture occurs, the seasonally differentiated statistics can better take it into account. The mixed distribution approach is

310    therefore a more efficient estimator for seasonally mixed low flow series. An additional strength of the model is that it allows the user to estimate return periods of summer low-flows, winter low-flows and annual return periods in a theoretically sound and consistent manner, resulting in coherent seasonal and annual statistics. This supports the estimates of seasonal statistics that are increasingly needed, such as for assessing environmental flows, or hydropower production capacity.

Similar advantages of the mixed seasonality estimator have been found for high flows, such as by Schumann (2005) when

315    assessing the extreme 2005 event at the Mulde river basin in Germany. The study also found the seasonal statistics more robust with respect to single extreme values than statistics based on annual values. The same effect can be seen from our example in Fig. 1b, where the annual statistic largely underestimates the magnitude of extreme events. The mixed distribution, however, coincides with the seasonal distributions in an intuitive way, and thus offers the plausible estimator in such case.

Despite there is a high degree of similarity between flood and low-flow applications, some important differences exist. Fis-

320    cher et al. (2016) noted that distinguishing between summer and winter floods is not always as easy when seasonal series contain events with different genesis. Indeed, the published examples of mixed flood distributions (e.g. Deutsche Vereinigung für Wasserwirtschaft, 2012) are less clear, as they do not show a similar clear separation between the seasonal distributions as the examples for low-flow in our paper (Fig. 1). Floods are generated by a wide range of process combinations (Merz and Blöschl, 2003) that are not confined to a season to the same extent as low-flow events. This leads to some limitation of seasonal flood

325    statistics, or in other words, makes seasonal low-flow statistics even more relevant. As a remedy, Fischer et al. (2016) proposed to further subdivide summer floods into groups of long and short events based on their time scale to improve the separation of processes. However, they had to consider overlapping events because the events they identified were not independent anymore.

Although such statistical dependence exists to a much lesser extent for summer and winter low-flow events, there may be cases where the assumption of independence is not entirely fulfilled. This may for example occur in very large watersheds with

330    a high storage capacity and, therefore, a particularly long time of recession and recovery at the end of the drought. In such cases, the estimate could be further improved by considering the correlation structure of events. Such an approach appears to be a natural extension of the mixed probability estimator and will be explored in the companion paper to this study.

Overall, the mixed distribution approach provides one consistent approach for summer, winter, and annual probabilities. Because of all its beneficial properties, it should be used in the analysis of low-flow frequencies in climates with warm summer

335    and cold winter seasons, under conditions where mixed seasonal low flows regimes occur.

*Code availability.* The code will be made available in an updated version of R software package 'lfstat' via the CRAN repository.

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# References

Coles, S.: An introduction to statistical modeling of extreme values, Springer series in statistics, Springer, London ; New York, 2001.

Deutsche Vereinigung für Wasserwirtschaft, A. u. A., ed.: Ermittlung von Hochwasserwahrscheinlichkeiten, no. M 552 in DWA-Regelwerk,
DWA, Hennef, august 2012 edn., oCLC: 809196700, 2012.

Fischer, S., Schumann, A., and Schulte, M.: Characterisation of seasonal flood types according to timescales in mixed probability distributions, Journal of Hydrology, 539, 38–56, https://doi.org/10.1016/j.jhydrol.2016.05.005, 2016.

Gumbel, E. J.: Distributions des valeurs extremes en plusiers dimensions, Publ. Inst. Statist. Univ. Paris, 9, 171–173, 1960.

Haslinger, K., Koffler, D., Schöner, W., and Laaha, G.: Exploring the link between meteorological drought and streamflow: Effects of climate-catchment interaction, Water Resources Research, 50, 2468–2487, https://doi.org/10.1002/2013WR015051, 2014.

Hodgkins, G. A. and et al.: Low flow trends (in preparation), 2022.

Karanitsch-Ackerl, S., Mayer, K., Gauster, T., Laaha, G., Holawe, F., Wimmer, R., and Grabner, M.: A 400-year reconstruction of spring–summer precipitation and summer low flow from regional tree-ring chronologies in North-Eastern Austria, Journal of Hydrology, 577, 123 986, https://doi.org/10.1016/j.jhydrol.2019.123986, 2019.

Laaha, G. and Blöschl, G.: Low flow estimates from short stream flow records–a comparison of methods, Journal of Hydrology, 306, 264–286, https://doi.org/doi:10.1016/j.jhydrol.2004.09.012, 2005.

Laaha, G. and Blöschl, G.: A comparison of low flow regionalisation methods—catchment grouping, Journal of Hydrology, 323, 193–214, https://doi.org/10.1016/j.jhydrol.2005.09.001, 2006a.

Laaha, G. and Blöschl, G.: Seasonality indices for regionalizing low flows, Hydrological Processes, 20, 3851–3878, https://doi.org/10.1002/hyp.6161, 2006b.

Laaha, G., Skøien, J., and Blöschl, G.: Spatial prediction on river networks: comparison of top-kriging with regional regression, Hydrological Processes, 28, 315–324, https://doi.org/10.1002/hyp.9578, 00000, 2014.

Laaha, G., Parajka, J., Viglione, A., Koffler, D., Haslinger, K., Schöner, W., Zehetgruber, J., and Blöschl, G.: A three-pillar approach to assessing climate impacts on low flows, Hydrol. Earth Syst. Sci., 20, 3967–3985, https://doi.org/10.5194/hess-20-3967-2016, 2016.

Laaha, G., Gauster, T., Tallaksen, L. M., Vidal, J.-P., Stahl, K., Prudhomme, C., Heudorfer, B., Vlnas, R., Ionita, M., Van Lanen, H. A. J., Adler, M.-J., Caillouet, L., Delus, C., Fendekova, M., Gailliez, S., Hannaford, J., Kingston, D., Van Loon, A. F., Mediero, L., Osuch, M., Romanowicz, R., Sauquet, E., Stagge, J. H., and Wong, W. K.: The European 2015 drought from a hydrological perspective, Hydrology and Earth System Sciences, 21, 3001–3024, https://doi.org/10.5194/hess-21-3001-2017, 2017.

Laimighofer, J., Melcher, M., and Laaha, G.: Low flow estimation beyond the mean–expectile loss and extreme gradient boosting for spatio-temporal low flow prediction in Austria, Hydrology and Earth System Sciences Discussions, pp. 1–33, publisher: Copernicus GmbH, 2022a.

Laimighofer, J., Melcher, M., and Laaha, G.: Parsimonious statistical learning models for low-flow estimation, Hydrology and Earth System Sciences, 26, 129–148, https://doi.org/10.5194/hess-26-129-2022, 2022b.

Merz, R. and Blöschl, G.: A process typology of regional floods, Water Resources Research, 39, https://doi.org/10.1029/2002WR001952, 2003.

Parajka, J., Blaschke, A. P., Blöschl, G., Haslinger, K., Hepp, G., Laaha, G., Schöner, W., Trautvetter, H., Viglione, A., and Zessner, M.: Uncertainty contributions to low-flow projections in Austria, Hydrology and Earth System Sciences, 20, 2085–2101, https://doi.org/10.5194/hess-20-2085-2016, 2016.

Schumann, A.: Hochwasserstatistische Bewertung des Augusthochwassers 2002 im Einzugsgebiet der Mulde unter Anwendung der
380    saisonalen Statistik, Hydrologie und Wasserbewirtschaftung, 49, 200–206, 2005.

Stedinger, J. R., Vogel, R. M., and Foufoula-Georgiou, E.: Frequency analysis of extreme events, Chapter 18 in Handbook of Hydrology,
       edited by DR Maidment, McGraw-Hill, 1993.

Tallaksen, L. M. and Van Lanen, H. A. J.: Hydrological drought: processes and estimation methods for streamflow and groundwater, no. 48
       in Developments in water science, Elsevier, 2004.

385    Yue, S., Ouarda, T., Bobée, B., Legendre, P., and Bruneau, P.: The Gumbel mixed model for flood frequency analysis, Journal of Hydrology,
       226, 88–100, https://doi.org/10.1016/S0022-1694(99)00168-7, 1999.