Report on the paper
“A robust upwind mixed hybrid finite element method for transport in variably saturated porous media “
by A. Younes, H. Hoteit, R. Helmig and M. Fahs

The paper under review presents a new method to obtain an upwind discretization of the advection–diffusion equation, using mixed hybrid finite element. This is a non–trivial problem, as one would like to obtain a stable ans monotone method, and because the requirement for a hybrid discretization means that the balance equation needs to be local to each element. This is difficult to reconcile with the need for upwinding so as to obtain a stable method. The author review an existing method, and propose a new variation. The proposed method is validated on two numerical examples.

In this reviewer’s opinion, the main advantage of the proposed method would that it avoids the need to discretize in time to derive the hybrid discretization. As argued by the authors, this enables the use of a higher order, or variable time step, discretization.

However, the paper as it is currently written suffers from several deficiencies, some major, some minor, that I would like to see addressed before publication.

Specific comments

- I do not understand the “derivation” of the method that is given is Section 3 (starting on page 12). Why is it allowed, or useful, or even correct, to use a steady state form of the basic mass conservation equation (12) ? This approximation is used to obtain equation (24), and then again in equation (29).

- As a consequence of the previous point, why would the proposed method be consistent ? And why do the authors expect that the method is stable (or satisfies a maximum principle) ? I understand that the paper, or even the journal, is not a numerical analysis paper, but any discussion shedding some heuristic light in the two issues of consistency and stability would be welcome.

- If no analysis is possible, then the authors should at least include a numerical convergence study, say for example 1, to quantify the accuracy of the proposed method.

- For the two examples discussed in Section 4, the proposed method is compared to the basic hybrid MFE method (that is equation (18) of the paper). But this method is already known to be non-monotone. A more significant comparison would be with the previous upwind variant from the Radu et al. (2011) reference quoted (that is the method described on page 11). This would allow the authors to discuss the possible pros and cons of their proposed method, in terms of accuracy, robustness and versatility. Even though I do not ask lightly that the authors do more numerical experiments, I feel this is really needed before the paper can be published.

- The paper under review has a significant overlap with the recently published paper [1] by the same authors. The authors should clarify the relationship between both papers.

Technical corrections

- The shortcut " $+\{\}'$ " used first in equation (27) and several times after that is very confusing. Does it mean "additional terms that the authors do not want to detail", or rather "the same as before, but for element E' instead of E" ? I know the answer is the latter, but I wasn't sure the first time. This really should be clarified, preferably by writing the equation in full, at least by giving an explanation.
- The exposition of the model in Section 2 should follow a more logical order: I would suggest starting with equation (3) (with (4) and (5)), then give equation (1) to say where $\mathbf{q}$ comes from (and explain what θ is, then conclude with equation (6). The discretized equations such as (2) and (7) should only appear later.
- I think there is a confusion between B and B^{-1} on line 162. And the correct notation should be $\tilde{B}^{E,-1}_{i,j}$.
- In principle, the flux terms in equations (16) and later should have a superscript $n+1$ attached.
- The physical units in Table 1 should be written in an upright font (as in the text before).
- Example 1 has been used as a test case in the same context by (at least) two papers: [2] and [3], that should be cited.

References

[1] *A robust fully mixed finite element model for flow and transport in unsaturated fractured porous media*, Anis Younes, Hussein Hoteit, Rainer Helmig, Marwan Fahs, Advances in Water Resources, Volume 166, 2022, `https://doi.org/10.1016/j.advwatres.2022.104259`.

[2] *A triangular finite volume approach with high-resolution upwind terms for the solution of groundwater transport equations*, M. Putti, W. W.-G. Yeh and W. A. Mulder, Water Resources Res., 26, 2865–2880 (1990).

[3] *Solution of the Advection–Diffusion Equation Using a Combination of Discontinuous and Mixed Finite Elements*, Siegel, P., Mosé, R., Ackerer, P. and Jaffre, J., Int. J. Numer. Meth. Fluids, 24: 595-613 (1997). `https://doi.org/10.1002/(SICI)1097-0363(19970330)24:6<595::AID-FLD512>3.0.CO;2-I`.