Manuscript Number: hess-2022-153

Title: A robust Upwind Mixed Hybrid Finite Element method for transport in variably saturated porous media

RC2: 'Comment on hess-2022-153', Anonymous Referee #2, 11 Jul 2022 reply

The paper under review presents a new method to obtain an upwind discretization of the advection--diffusion equation, using mixed hybrid finite element. This is a non-trivial problem, as one would like to obtain a stable and monotone method, and because the requirement for a hybrid discretization means that the balance equation needs to be local to each element. This is difficult to reconcile with the need for upwinding so as to obtain a stable method. The author review an existing method, and propose a new variation. The proposed method is validated on two numerical examples.

In this reviewer's opinion, the main advantage of the proposed method would that it avoids the need to discretize in time to derive the hybrid discretization. As argued by the authors, this enables the use of a higher order, or variable time step, discretization.

However, the paper as it is currently written suffers from several deficiencies, some major, some minor, that I would like to see addressed before publication.

> **Answer:** We thank the Reviewer for the constructive comments which helped us to improve the quality of our manuscript. As detailed below, all suggestions are accounted for in the revised version.

Specific comments

- I do not understand the "derivation" of the method that is given is Section 3 (starting on page 12). Why is it allowed, or useful, or even correct, to use a steady state form of the basic mass conservation equation (12) ? This approximation is used to obtain equation (24), and then again in equation (29).

  Answer: We agree and improve the presentation of the method in the revised version. More details are provided to explain how and why it is useful to use the steady state form of the mass conservation equation.

  The main idea of the new method is to combine the upwind edge centered finite volume method with the lumped hybrid MFE method. A new section is added to better explain the principles of the new method. Indeed, the standard hybrid MFE method is based on two stages: (*i*) the discretization of the transient mass conservation equation over the element E and (*ii*) the continuity of fluxes across the edge i sharing the elements E and E'. With the lumped formulation, the previous two stages are transformed to new ones as following: (*i*) the transient mass conservation over E is transformed to a steady state one and (*ii*) the continuity equation, which is seen as a steady state mass conservation equation at the edge level, is transformed to a transient equation. The new scheme keeps the time derivative continuous which allows the use of efficient high order temporal discretization methods via the MOL.

These points are better specified in the revised version

- As a consequence of the previous point, why would the proposed method be consistent ? And why do the authors expect that the method is stable (or satisfies a maximum principle) ? I understand that the paper, or even the journal, is not a numerical analysis paper, but any discussion shedding some heuristic light in the two issues of consistency and stability would be welcome.

  Answer: In the new method, the term of mass (with time derivative) has a contribution only on the diagonal terms of the final system matrix. This improves the monotonous character of the hybrid-MFE solution. For dispersive transport, in the case of an acute triangulation, the maximum principle is respected whatever the heterogeneity of the porous medium (Younes *et al*., 2006). In the case of transport by only advection, the system matrix with the new upwind hybrid MFE scheme is always an *M*-matrix (a non singular matrix with $m_{ii} > 0, m_{ij} \le 0$). The *M*-matrix property insures the stability of the scheme since it guaranties the respect of the discrete maximum principle *i.e.* local maxima or minima will not appear in the C solution in a domain without local sources or sinks.

  Note that in the case of 2D triangular elements, the lumped formulation is algebraically equivalent to the nonconforming Crouzeix-Raviart (Crouzeix and Raviart, 1973) finite element method (see Younes et al, 2008). The nonconforming Crouzeix-Raviart method uses the chapeau functions as basis functions to approximate the concentration, like the standard finite element method, but seed nodes are the midpoints of the edges.

These points will be specified in the revised version

- If no analysis is possible, then the authors should at least include a numerical convergence study, say for example 1, to quantify the accuracy of the proposed method.

  Answer: We agree and will present results of simulations for the first test case (with the analytical solution) using different mesh refinements in order to investigate the order of convergence in space and time of the new upwind-hybrid MFE method.

- For the two examples discussed in Section 4, the proposed method is compared to the basic hybrid MFE method (that is equation (18) of the paper). But this method is already known to be non-monotone. A more significant comparison would be with the previous upwind variant from the Radu et al. (2011) reference quoted (that is the method described on page 11). This would allow the authors to discuss the possible pros and cons of their proposed method, in terms of accuracy, robustness and versatility. Even though I do not ask lightly that the authors do more numerical experiments, I feel this is really needed before the paper can be published.

  Answer: In the revised version, we add numerical experiments which show that the new scheme has optimal first order convergence in time and space as the upwind scheme of Radu *et al*. (2011) and the non-hybrid upwind mixed method of Dawson (1978).

  Note that the upwind scheme of Radu *et al*. (2011) has good convergence properties but uses the approximation that the concentration at the edge i is the average of the mean concentrations of the two adjacent elements sharing i. This can be a rough approximation since with the hybrid MFE method, the trace of concentration can be significantly different from the average of mean concentrations of adjacent cells, especially in the case of a heterogeneous domain where dispersion can vary with several orders of magnitude from element to element. Further, with this scheme one cannot ensure continuity of both advective and dispersive fluxes at the interface since the continuity is prescribed only for the total flux. These two drawbacks are avoided with the new developed scheme.

These points will be specified in the revised version

- The paper under review has a significant overlap with the recently published paper (see below) by the same authors. The authors should clarify the relationship between both papers.

  Answer: The aim of the present paper is (i) to develop a new upwind hybrid MFE scheme for advection-dispersion transport, (ii) to detail the main principles of the new scheme and show its benefits as compared to other upwind MFE schemes, (iii) to validate the method by comparison against an analytical solution, (iv) to investigate its order of convergence in time and space (in the revised version) and (v) to show its robustness when combined with the MOL for transport in unsaturated zones.

  The developed upwind scheme for transport has been recently successfully combined with the MFE method for fluid flow to simulate coupled flow and transport in unsaturated fractured porous media (Younes et al., 2022). The main objective of the paper of Younes *et al.* (2022) was to investigate the efficiency and robustness of a 1D-2D discrete fracture matrix (DFM) approach to model nonlinear flow and transport problems in fractured porous media with matrix-fracture and fracture-fracture fluid and mass exchanges.

These points will be specified in the revised version

A robust fully mixed finite element model for flow and transport in unsaturated fractured porous media, Anis Younes, Hussein Hoteit, Rainer Helmig, Marwan Fahs, Advances in Water Resources, Volume 166, 2022, https://doi.org/10.1016/j.advwatres.2022.104259.

Technical corrections

• The $\{\ \}'$ used first in equation (27) and several times after that is very confusing. Does it mean “additional terms that the authors do not want to detail”, or rather “the same as before, but for element E0 instead of E"? I know the answer is the latter, but I wasn't sure the first time. This really should be clarified, preferably by writing the equation in full, at least by giving an explanation.

Answer: We agree and change the text accordingly

• The exposition of the model in Section 2 should follow a more logical order: I would suggest starting with equation (3) (with (4) and (5)), then give equation (1) to say where $\boldsymbol{q}$ comes from (and explain what θ is, then conclude with equation (6). The discretized equations such as (2) and (7) should only appear later.

Answer: We agree and change the order of equations in the revised version

• I think there is a confusion between B and B^{-1} on line 162. And the correct notation should be $\tilde{B}_{i,j}^{E,-1}$

Answer: Corrected in the revised manuscript.

• In principle, the flux terms in equations (16) and later should have a superscript n + 1 attached.

Answer: We agree and specify in the text that the subscript n+1 is omitted to alleviate the notation

• The physical units in Table 1 should be written in an upright font (as in the text before).

Answer: Corrected in the revised manuscript.

• Example 1 has been used as a test case in the same context by (at least) two papers: [2] and [3], that should be cited.

Answer: The two papers are referenced in the revised version.