Comment on hess-2021-50

This study applied both analytical and numerical approaches to solve hillslope hydrological dynamics equation, and tested (as well as compared) the results in some idealized situations. However, the manuscript was written more like a mathematic article though dealing with a practical problem in hydrology. Thus I think some major revisions are needed to meet the criteria of HESS. Please see my detailed comments as following.

Major comments:

Just as I mentioned, simulating the outlet discharge water of a hillslope is a practical hydrological problem. While many mathematic tools can be employed to solve the problem, the only metric to evaluate them is to compare their outcomes with some real observed measurements. However, this study stopped by testing its framework in some idealized conditions without checking with the real situation and data. On the other side, the topic of explicitly solving hillslope hydrology is not new. In fact, based on my knowledge, some land models have already employed the conception of hillslope and solve its hydrology dynamics explicitly using numerical solutions. These models have been tested and applied at different scales, and the observations are also available at different scales. So at such stage, conducting a similar research but only in idealized conditions is not decent for publication in HESS (maybe more suitable for a journal for applied mathematics). To overcome this shortage, the authors may consider using some real data to configure and evaluate their model, even at a local scale. Thus it can let us see more clearly the ability of each (analytical or numerical) methods and benefit future research. Please note that all required real data must be available as hydrological modelers have already depicted and validated the hillslope from local to global scales. So I see no excuse to refuse this suggestion.

Response: Thank you for your precious comments, and I quite respect your opinion. Analytical approaches and numerical methods are commonly used two ways to do research. Analytical solutions are focused on their logic processes and systematic derivation. The limitation of analytical solutions is that the geometry (or shape) of simulated domain must be specific and regular. Conversely, numerical models can be applied to irregular geometry in a wide range. In the past, the reviewer usually asks us to compare our new analytical solutions with numerical solutions to validate the correctness of our analytical solutions. Nowadays, the analytical solutions have been compared with numerical solutions developed by ourselves as well as other analytical solutions. The reviewer presented that the present analytical solutions as well as numerical methods need to test by the real situation and data. We totally agree that an analytical solution could be verified by real data, but this is not the only way. As you know, both analytical and numerical solutions cannot conform with the real observation data because of the high uncertainty and irregular properties of soils, which influence the input parameters, in a real aquifer. Some input parameters in real situation are hard to be determined or acquired, for example, real hydraulic conductivity, real porosity, and real inclined angle of slope in a watershed. These parameters vary with space in a real watershed. There are no available data in the published papers listed in the references section. We searched for available real data and input parameters after the reviewer gave the comments. The following papers were found:

Matonse A. H. and Kroll C. (2009). "Simulating low streamflows with hillslope storage models". water resources research, 45, W01407, doi:10.1029/2007WR006529, 2009.

In the above paper, the parameters *a* and *c* of the hillslope equation in Eq. (2), $w(x) = ce^{ax}$, are lacked because they didn't give *w*(*x*). Moreover, the hydraulic conductivity and soil porosity only give their lower limits and upper limits.

Kong, J, Shen C., Luo Z., Hua G., and Zhao H. (2016) "Improvement of the hillslope-storage Boussinesq model by considering lateral flow in the unsaturated zone". water resources research, 52, 2965–2984, doi:10.1002/ 2015WR018054.

In the above paper, the authors studied saturated and unsaturated soil layers including sand, loam, and clay. The shape and wide are not the same with ours, i.e. not $w(x) = ce^{ax}$. Porosity varies with space and time. No real data is given for verification.

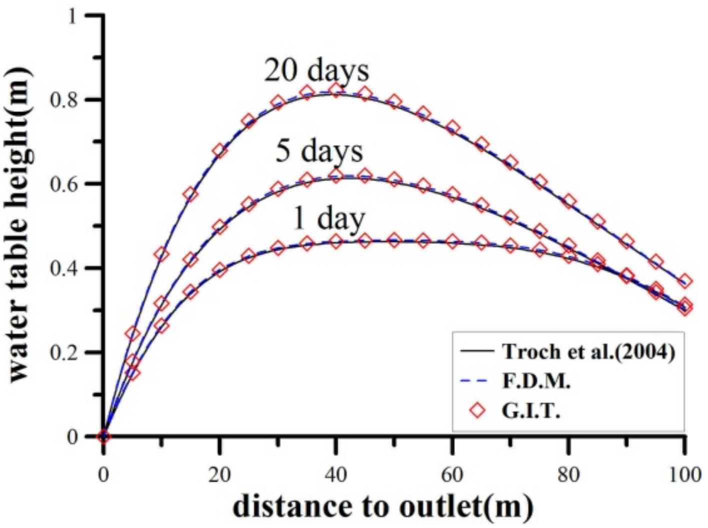
Norbiato D, Borga M. (2008) "Analysis of hysteretic behaviour of a hillslope-storage kinematic wave model for subsurface flow". Advance in Water Resources 31, 118-131.
In the above paper, the thickness and porosity of the aquifer are not given. No real data is given for verification.

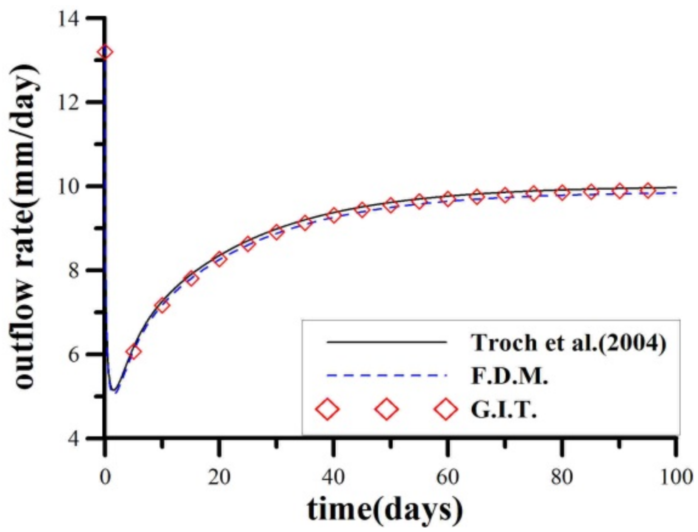
Based on the above literature surveying, we can find there is no sufficient information for us to compare the present solutions with real data/situation.

If we conduct the observation and measurement of the real data and parameters, we shall need a huge financial support and spent a lot of time. Therefore, we used different ways to verify our analytical solutions and numerical solutions in this study.

The analytical solutions were compared with the published paper of Troch et al. (2004), and the numerical solutions were compared with Troch et al. (2003). The comparison results are in a good agreement, and thus which validates both the solutions. In addition, we also made a comparison between the numerical model and the analytical solutions to check the correctness of the analytical solutions, and conversely, the analytical solutions can also benefit the verification of the numerical model. To develop a more popular numerical model associated with the map of GIS is our goal for future research.

Furthermore, this present analytical solution is obtained for a linearized equation, but the present numerical solution is for a nonlinear equation which was described on Line 177 (original version of MS) “a numerical model was developed to solve the original nonlinear equation, Eq. (4)”. Both solutions are to different governing equations, so there are discrepancies in between. For the numerical solution by a finite difference method (F.D.M.) to the same LINEARIZED equation, the results are given below:





It shows that the numerical solutions are equal to the analytical solutions based on the same linearized governing equations and same scenarios, thus justifying that the present analytical solutions are correct.

We have tried our best to do the research, and prepared the present manuscript for a long journey. Thank you very much for your precious comments.

Specific comments:

L41, “by means of isotope study”: Please delete these words.

Response: Thank you for your comment. These words have been deleted. Please see Line 41.

L77, “The ground surface is vegetation free, …”: Please discuss the potential effects of vegetation.

Response: Thank you for your comment. The vegetation effect on the hill-storage is a good topic. The potential effects of vegetation might be beneficial to the water storage. Please allow us to do this in future research. The present study only discusses vegetation free surface.

L98, Equation (6): The n here should not be mixed with the n for drainable porosity.

Response: Thank you for your comment. We will change the upper limit of summation n to M. Please see Line 101.

L102, Equation (7): s/w=nh=bnD, because b<1, so h<D? But D is the average depth, how can h be less than its average everywhere?

Response: Thank you for your comments. In a real world, the groundwater h might be greater than the average depth D, but in this study h<D is the limitation. In other words, only pore water storage was considered.

L103, "where b is a fitting parameter …": Please show more detail for the method used in tuning b.

Response: Thank you for your comment. We will add the following explanation" which is determined by trial and error to better fit the results of the numerical model." Please see Lines 106-107.

L194, Equation (37): Please show more detail how to use Taylor series expansion to transform the Eq (13) to the Eq(37).

Response: Thank you for your comment. We have added more detail about using Taylor series expansion above Eq. (37). Please see Lines 198-201.

From linear extrapolation, we have

$$\begin{cases} {s_\alpha}^j(1) = {s_\alpha}^j(0) - \frac{\Delta x}{2}{s_\alpha}^{j\prime}(0) + \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2 {s_\alpha}^{j\prime\prime}(0) + O(\Delta x)^3 \\ {s_\alpha}^j(2) = {s_\alpha}^j(0) + \frac{\Delta x}{2}{s_\alpha}^{j\prime}(0) + \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2 {s_\alpha}^{j\prime\prime}(0) + O(\Delta x)^3 \\ {s_\alpha}^j(3) = {s_\alpha}^j(0) + \frac{3\Delta x}{2}{s_\alpha}^{j\prime}(0) + \frac{1}{2!}\left(\frac{3\Delta x}{2}\right)^2 {s_\alpha}^{j\prime\prime}(0) + O(\Delta x)^3 \end{cases}$$

Eliminating ${s_\alpha}^{j\prime}(0)$ and ${s_\alpha}^{j\prime\prime}(0)$ with boundary condition ${s_\alpha}^j(0) = 0$, we can obtain

$${s_\alpha}^j(1) = -2{s_\alpha}^j(2) + \frac{1}{3}{s_\alpha}^j(3)$$

L203-232: What is the major difference between this work and Torch et al. (2003, 2004)? The authors should particularly stress it in the manuscript because the similarity is too high in my view based on the current description.

Response: Thank you for your comment. As mentioned in the introduction section, Troch et al. (2003) solved the linearized and simplified Boussinesq equation using the finite difference method to discretize the space and the multistep solver to deal with time. Troch et al. (2004) analytically solved the linearized Boussinesq equation with uniform rainfall recharge by using the Laplace transform. However, in this study, we evaluate the hillslope storage with variable width under temporally varied rainfall recharge, not uniform recharge. This is more realistic in real situation. Thus, the source term R=R(t) (see Eq. (6)) in the governing equation makes it difficult to be solved by an analytical approach. We used the generalized integral transform instead of Laplace transform to avoid the difficulty which inverse Laplace transform might encounter. This is a new contribution. We will stress it in the conclusion section. Please see Lines 311-315.

L263: “Theta = 5%”: Is the theta angle of slope? How to understand the symbol of percentage?

Response: Thank you for your comment. Yes, theta is the angle of slope. Theta = 5% means Theta = 0.05 in radian. Please see Line 269.