Let's assume a DA-hydrodynamic modeling framework with l parameters ($p = 1,2, \ldots, l$), m states ($s = 1,2, \ldots, m$) and n observations ($j = 1,2, \ldots, n$). The following EnKF equations are described in accordance with the flowchart shown in Figure 2. In the EnKF, parameter samples can be generated by adding the noise of τ_t^i ~~η_t~~ with covariance ~~Σ_t^θ~~ σ_t^θ to the prescribed parameters.

$$\theta_{t+1}^{i-} = \theta_t^{i+} + \tau_t^i \qquad \tau_t^i \sim N\left(0, \sigma_t^\theta \cancel{, \eta_{t+1}}\right) \quad \cancel{\forall \quad \eta_{t+1} = \Sigma_{t+1}^{\theta}.} \tag{3}$$

Using θ_{t+1}^{i-} and forcing data, a model state ensemble and predictions are generated, respectively.

$$x_{t+1}^{i-} = f\left(x_t^{i+}, u_t^i, \theta_{t+1}^{i-}\right) + \omega_t^i \qquad \omega_t^i \sim N(0, \sigma_t^x \cancel{Q_t}) \quad \cancel{\forall \quad Q_t = \Sigma_t^x.} \tag{4}$$

$$\hat{y}_{t+1}^i = h\left(x_{t+1}^{i-}, \theta_{t+1}^{i-}\right) + \nu_{t+1}^i \qquad \nu_{t+1}^i \sim N\left(0, \sigma_{t+1}^{\hat{y}} \cancel{R_{t+1}}\right) \quad \cancel{\forall \quad R_{t+1} = \Sigma_{t+1}^{\hat{y}}.} \tag{5}$$

where x_t, u_t, θ_t and $\hat{y}_t$ ~~y_t~~ are the vector of the uncertain state variables, forcing data, model parameters and ~~observation data~~model prediction at time step t, respectively. ω_t and ν_t represent~~s~~ the model state and prediction errors due to the imperfect modeling~~, and ν_t is the measurement error~~. Most often, ω_t and ν_t are assumed to be white noises with mean zero and covariance σ_t^x ~~Q_t~~ and $\sigma_{t+1}^{\hat{y}}$ ~~R_t~~, respectively. In addition, the two noises ω_t and ν_t are assumed to be independent.

Then we update the parameter ensemble members using the standard Kalman filter equation:

$$\theta_{t+1}^{i+} = \theta_{t+1}^{i-} + K_{t+1}^\theta\left(y_{t+1}^i - \hat{y}_{t+1}^i\right) \tag{6}$$

where $\hat{y}_{t+1}^i$ and ~~$\hat{y}_{t+1}^i$~~ y_{t+1}^i are the model ~~simulation~~ prediction and observations, respectively, and $K_{t+1}^\theta \in \mathbb{R}^{l \times n}$ is the Kalman gain matrix for correcting the parameter trajectories obtained by:

$$\cancel{K_{t+1}^\theta = \Sigma_{t+1}^{\theta y}\left[\Sigma_{t+1}^{yy} + R_{t+1}'\right]^{-1} \quad (7)}$$

$$K_{t+1}^\theta = \sigma_{t+1}^{\theta y}\left[\sigma_{t+1}^{yy} + R_{t+1}'\right]^{-1} \tag{7}$$

where $\sigma^{\theta y}_{t+1}$ $\cancel{\Sigma^{\theta y}_{t+1}}$ $\in \mathbb{R}^{l\times n}$ is the cross-covariance matrix of parameter ensemble and prediction ensemble (Eq. 6). Unlike other studies, and for more realistic characterization of observation and model errors here the correlation between the errors associated with n observation data are accounted for during the assimilation process. Therefore, the covariance matrix $R'_t \in \mathbb{R}^{n\times n}$ is a nonzero matrix, such that the values in the diagonal represent the error variance associated with each observation data (R_{t+1}) and all elements lower/upper the main diagonal denote the cross covariance between different observations (Eq. ~~7~~9). $\sigma^{yy}_{t+1} \in \mathbb{R}^{n\times n}$ $\cancel{\Sigma^{yy}_{t} \in \mathbb{R}^{n\times n}}$ is also a similar covariance matrix with the inclusion of error correlation between the model simulations (Eq. 8).

$$\sigma^{\theta y}_{t+1}(p,j)\ \cancel{\Sigma^{\theta y}_{t+1}(p,j).} = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\theta^{i-}_{t+1}(p) - E[\theta^{-}_{t+1}(p)]\right)\left(\hat{y}^{i}_{t+1}(j) - E[\hat{y}_{t+1}(j)]\right)\right] \tag{8}$$

$$R'_{t+1}(j,j') = \begin{cases} R_{t+1} & j = j' \\ \frac{1}{N}\sum_{i=1}^{N}\left[\left(y^{i}_{t+1}(j) - E[y_{t+1}(j)]\right)\left(y^{i}_{t+1}(j') - E[y_{t+1}(j')]\right)\right] & j \neq j' \end{cases} \tag{9}$$

$$\sigma^{yy}_{t+1}(j,j')\ \cancel{\Sigma^{yy}_{t+1}(j,j').} = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\hat{y}^{i}_{t+1}(j) - E[\hat{y}_{t+1}(j)]\right)\left(\hat{y}^{i}_{t+1}(j') - E[\hat{y}_{t+1}(j')]\right)\right] \tag{10}$$

$$E[\theta^{-}_{t+1}] = \frac{1}{N}\sum_{i=1}^{N}\theta^{i-}_{t+1} \tag{11}$$

$$E[\hat{y}_{t+1}] = \frac{1}{N}\sum_{i=1}^{N}\hat{y}^{i}_{t+1} \tag{12}$$

Now using the updated parameter, the new model state trajectories (state forecasts) and prediction trajectories are generated:

$$x^{i-}_{t+1} = f\left(x^{i+}_{t}, u^{i}_{t}, \theta^{i+}_{t+1}\right) + \omega^{i}_{t} \qquad \omega^{i}_{t} \sim N(0, \sigma^{x}_{t}\ \cancel{\Sigma^{x}_{t}}) \quad \cancel{\forall \quad Q_t = \Sigma^{x}_{t+1}} \tag{13}$$

$$\hat{y}^i_{t+1} = h\left(x^{i-}_{t+1}, \theta^{i+}_{t+1}\right) + v^i_{t+1} \qquad v^i_{t+1} \sim N\left(0, \sigma^{\hat{y}}_{t+1} \cancel{\Sigma^{y}_{t+1}}\right) \qquad \cancel{\forall \quad R_{t+1} = \Sigma^{y}_{t+1}} \tag{14}$$

Model states ensemble is similarly updated as follows:

$$x^{i+}_{t+1} = x^{i-}_{t+1} + K^x_{t+1}\left(y^i_{t+1} - \hat{y}^i_{t+1}\right) \tag{15}$$

$$y^i_{t+1} = y^i_{t+1} + \eta^i_{t+1}\cancel{v^i_{t+1}} \qquad \eta^i_{t+1}\cancel{v^i_{t+1}} \sim N(0, R_{t+1}) \qquad \cancel{\forall \quad R_{t+1} = \Sigma^{y}_{t+1}} \tag{16}$$

where $K^x_{t+1} \in \mathbb{R}^{m \times n}$ is the Kalman gain for correcting the state trajectories and is obtained by:

$$\cancel{K^x_{t+1} = \Sigma^{xy}_{t+1}\left[\Sigma^{yy}_{t+1} + R'_{t+1}\right]^{-1} \qquad (17)}$$

$$K^x_{t+1} = \sigma^{xy}_{t+1}\left[\sigma^{yy}_{t+1} + R'_{t+1}\right]^{-1}$$

$$\sigma^{xy}_{t+1}(s,j)\,\cancel{\Sigma^{xy}_{t+1}(s,j)} = \frac{1}{N}\sum_{i=1}^{N}\left[\left(x^{i-}_{t+1}(s) - E\left[x^{-}_{t+1}(s)\right]\right)\left(\hat{y}^i_{t+1}(j) - E\left[\hat{y}_{t+1}(j)\right]\right)\right] \tag{18}$$