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Short communication

An easy and efficient combination of the Mixed Finite Element Method and the Method of Lines for the resolution of Richards' Equation

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ABSTRACT

In this work, the Mixed Hybrid Finite Element (MHFE) method is combined with the Method Of Lines (MOL) for an accurate resolution of the Richard's Equation (RE). The combination of these methods is often complicated since hybridization requires a discrete approximation of the time derivative whereas with the MOL, it should remain continuous. In this paper, we use the new mass lumping technique developed in Younes et al. [Younes, A., Ackerer, P., Lehmann, F., 2006. A new mass lumping scheme for the mixed hybrid finite element method. International Journal for Numerical Methods in Engineering 67, pp. 89–107.] for the MHFE method. With this formulation, the MOL is easily implemented and sophisticated time integration packages can be used without significant amount of work.
Numerical simulations are performed on both homogeneous and heterogeneous porous media to show the efficiency and robustness of the developed scheme.

1. Introduction

Accurate numerical simulation of infiltration in the vadose zone remains a challenge, especially in the presence of sharp wetting fronts. This problem is often modeled using Richard's Equation (RE) and closed by nonlinear constitutive relations between pressure head, hydraulic conductivity and water content (De Marsily, 1986; Hillel, 1980). In this work, we solve the RE using efficient advanced approximations for both spatial and temporal discretizations in order to reduce the excessive computational requirement while maintaining the accuracy.

For the spatial discretization, we use the Mixed Hybrid Finite Element (MHFE) method since it is locally conservative and produces an accurate and consistent velocity field even for highly heterogeneous domains (Brezzi and Fortin, 1991). This method has been successfully employed during the last few years for solving the RE (Bause and Knabner, 2004; Farthing et al., 2003; Bergamaschi and Putti, 1999).

For the temporal discretization, we use higher order methods via the Method Of Lines (MOL). With the MOL, we discretize first spatial derivatives and then integrate in time the semi-discrete problem as a system of Ordinary Differential Equations (ODEs) or Differential Algebraic Equations (DAEs). The advantage of the MOL is that the temporal accuracy can be specified by the user and therefore the error checking, robustness, order selection and time step adaptativity features available in sophisticated ODE/DAE codes can be applied to the time integration of the Partial Differential Equations (PDEs) (see Tocci et al., 1997). In the context of porous media, the MOL was shown to be very effective for the resolution of the nonlinear RE (Li et al., 2007; Miller et al., 2006; Lee et al., 2004; Matthews et al., 2004; Farthing et al., 2003; Kees and Miller, 2002; Tocci et al., 1998, 1997).

The combination of the MOL with the MHFE method was shown to be more complicated than with the standard methods (Farthing et al., 2003, 2002). Indeed, the hybridization technique requires a discrete approximation of the time derivative whereas the MOL requires a continuous temporal derivative.

We show in this work that these difficulties can be circumvented by using the Lumped MHFE method (LMHFE). This method, initially developed by Younes et al. (2006) to reduce unphysical oscillations for saturated flow simulations, is extended here for unsaturated flow simulations.

2. The mathematical model

Water flow in unsaturated porous media can be described by Richards' Equation which combines the mass conservation equation and the Darcy–Buckingham's law. The mass conservation equation can be formulated in several ways (Celia et al., 1990; De Marsily, 1986; Milly, 1985; Huyakorn and Pinder, 1983). The common mixed form is:

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$$\frac{\partial\theta}{\partial t}+S_sS_w(\theta)\frac{\partial H}{\partial t}+\nabla\cdot\vec{q}=f \tag{1}$$

and the pressure-head based form is:

$$(c(h)+S_sS_w(\theta))\frac{\partial H}{\partial t}+\nabla\cdot\vec{q}=f \tag{2}$$

where H and h are respectively the hydraulic and pressure head such as $H=h+z$, z is the depth taken positive upward, S_s is the specific storage, S_w is the relative saturation of the aqueous phase ($S_w=\theta/\theta_s$), θ and θ_s are respectively the volumetric and saturated water content, c is the specific moisture capacity ($c=d\theta/dh$), f is the source/sink term and $\vec{q}$ is the water velocity given from the Darcy's law:

$$\vec{q}=-K(h)\nabla H \tag{3}$$

where K is the hydraulic conductivity.

The interdependencies of pressure head, hydraulic conductivity and water content are characterized using constitutive relations. In this work, we use the standard model of Van Genuchten (1980):

$$S_e(h)=\frac{\theta-\theta_r}{\theta_s-\theta_r}=\begin{cases}\frac{1}{(1+|\alpha h|^n)^m} & h<0\\ 1 & h\geq 0\end{cases} \tag{4}$$

where θ_r is the residual volumetric water content, α is a parameter related to the mean pore size, n is a parameter for the uniformity of the pore size distribution and $m=1-1/n$. For the conductivity–saturation relationship we use the Mualem's model (1976) given by:

$$K(S_e)=K_sS_e^{1/2}\left\{1-\left(1-S_e^{1/m}\right)^m\right\}^2 \tag{5}$$

where S_e is given by equation (4) and K_s is the saturated conductivity.

Often the pressure-head form (2) is avoided since the mixed form (1) presents better conservation properties. However, it was demonstrated in (Miller et al., 1998; Tocci et al., 1997) that higher order numerical integration is effective for solving the pressure-head form of RE. This form was used with the MOL for solving the RE in (Miller et al., 1998, 2006; Tocci et al., 1998, 1997; Li et al., 2007) and will be adopted in this work.

The systems (2) and (3) are solved with the corresponding initial and boundary conditions which can be of Dirichlet or Neumann type. The domain is discretized with unstructured triangular meshes. These meshes are suitable for practical problems with complex geometry and local mesh refinement.

3. Numerical discretization

3.1. The spatial discretization

The spatial discretization is based upon the LMHFE method. In the following, we recall the main stages in order to use this formulation within a MOL context.

The velocity inside the element E is approximated using the lowest-order Raviart-Thomas space (Brezzi and Fortin, 1991):

$$\vec{q_E}=\sum_{i=1}^{3}Q_{E,i}\vec{\omega_{E,i}} \tag{6}$$

where $Q_{E,i}$ denotes the flux leaving E through the ith edge, taken positive outward.

The vectorial basis function $\vec{\omega_{E,i}}$ verifies,

$$\int_{E_j}\vec{\omega_{E,i}}\vec{\eta_{E_j}}=\delta_{ij} \tag{7}$$

$\vec{\eta_{E_j}}$ being the unit outward vector normal to the edge E_j of the element E.

The variational formulation of (3) using (7) leads to:

$$\begin{aligned}\int_E K_E^{-1}\vec{q_E}\vec{\omega_{E,i}}&=\sum_{j=1}^{3}Q_{E,j}\int_E\vec{\omega_{E,i}}K_E^{-1}\vec{\omega_{E,j}}=-\int_E\nabla H\vec{\omega_{E,i}}\\&=H_E-TH_{E,i}\end{aligned} \tag{8}$$

where H_E and $TH_{E,i}$ are respectively, the mean hydraulic head on the element E and on the edge E_i. K_E the value of the parameter K in the element E.

Using the local matrix $M_{E,ij}=\int_E\vec{\omega_{E,i}}K_E^{-1}\vec{\omega_{E,j}}$, we obtain

$$Q_{E,i}=\sum_{j=1}^{3}M_{E,ij}^{-1}(H_E-TH_{E,j}) \tag{9}$$

In the numerical codes, the hydraulic conductivity K_E of the element E is evaluated using the arithmetic mean of conductivities at element edges (Belfort and Lehmann, 2005).

With the LMHFE method, we approximate the flux by (see Younes et al. (2006) for details):

$$Q_{E,i}=\overline{Q}_{E,i}+\frac{Q_{E,s}}{3}-\frac{|E|}{3}(c_E+S_{s,E}S_{w,E})\frac{dTH_{E,i}}{dt} \tag{10}$$

where $Q_{E,s}$ is the sink/source term over the element E (of area $|E|$), $S_{s,E}$, $S_{w,E}$ and c_E are respectively the specific storage, the relative saturation of the aqueous phase and the specific moisture capacity of E.

In (10), $\overline{Q}_{E,i}$ is the flux corresponding to the stationary problem without sink/source terms,

$$\overline{Q}_{E,i}=\sum_{j=1}^{3}N_{E,ij}TH_{E,j} \tag{11}$$

Finally, the general expression of the flux (10) writes,

$$Q_{E,i}=\sum_{j=1}^{3}N_{E,ij}TH_{E,j}+\frac{Q_{E,s}}{3}-\frac{|E|}{3}(c_E+S_{s,E}S_{w,E})\frac{dTH_{E,i}}{dt} \tag{12}$$

The final system with the LMHFE method is obtained when using continuities of flux and hydraulic head between adjacent elements: each line i of the global matrix is formed by $Q_{E,i}+Q_{E',j}=0$ and $TH_{E,i}=TH_{E',i}$ where E and E' are the two elements sharing the edge i which leads to,

$$\begin{aligned}&\frac{|E|}{3}(c_E+S_{s,E}S_{w,E})\frac{dTH_{E,i}}{dt}+\frac{|E'|}{3}\left(c_{E'}+S_{s,E'}S_{w,E'}\right)\frac{dTH_{E',i}}{dt}\\&\quad=\sum_{j=1}^{3}N_{E,ij}TH_{E,j}+\sum_{j=1}^{3}N_{E',ij}TH_{E',j}+\frac{Q_{E,s}}{3}+\frac{Q_{E',s}}{3}\end{aligned} \tag{13}$$

Contrarily to the standard MHFE method, the hybridization procedure with the LMHFE method is performed without discretizing the temporal derivative which allows to obtain the system (13) of ODEs. Note also that the lumped formulation avoids unphysical oscillations for triangular meshes without additional numerical errors (see Younes et al., 2006).

3.2. Time integration

The time integration of the previous system of ODEs is performed with the DLSODIS solver (Double precision Livermore Solver for Ordinary Differential Equations – Implicit form and Sparse matrix) (Seager and Balsdon, 1982; Hindmarsh, 1980).

Table 1
Materials Properties.

Parameters	Homogeneous	Heterogeneous	
		Upper Layer	Lower Layer
θ_r	0.01	0.0001	0.045
θ_s	0.3	0.399	0.43
$\alpha(\mathrm{cm}^{-1})$	0.033	0.0174	0.145
n	4.1	1.3757	2.68
$K(\times 10^{-4}\ \mathrm{cm\ s}^{-1})$	97.22	3.45	82.5
$S_s(\times 10^{-10}\ \mathrm{cm}^{-1})$	1	1	1

DLSODIS is a variable time step size, variable order integrator of ODE/DAE system given in the implicit form ($A(t,y)dy/dt = G(t,y)$). The solver is used with the fixed coefficient implementation of the Backward Difference Formula (BDF).

The algebraic nonlinear system is solved with the modified Newton iteration (Radhakrishnan and Hindmarsh, 1993) where the Jacobian matrix is calculated numerically using finite difference approximation and the column grouping technique of Curtis et al. (1974) (see Hindmarsh (1982)).

DLSODIS uses the direct solver CDRV (Eisenstat et al., 1982, 1977) to solve linear systems that arise. All the matrices involved (mass and Jacobian matrices) are assumed to be sparse.

Note that periodically, the DSLODIS solver attempts to change the step size and/or the method order to minimize computational work while maintaining prescribed accuracy. The local accuracy is specified to the code by both a relative ε_r and absolute ε_a local error tolerances. All tolerances are fixed to 10^{-6} in this work. The implementation of DLSODIS requires the following subroutines:

- Subroutine for the calculation of the mass matrix $A(t,y)$, which is simple in our case since A is a diagonal matrix given by:

$$\boldsymbol{A}_{i,i} = \frac{|E|}{3}\left(c_E + S_{s,E}S_{w,E}\right) + \frac{|E'|}{3}\left(c_{E'} + S_{s,E'}S_{w,E'}\right) \tag{14}$$

- Subroutine for the calculation of the second member $G(t,y)$, which is given by:

$$G_i = \sum_{j=1}^{3} N_{E,ij}TH_{E,ij} + \sum_{j=1}^{3} N_{E',ij}TH_{E',j} + \frac{Q_{E,s}}{3} + \frac{Q_{E',s}}{3} \tag{15}$$

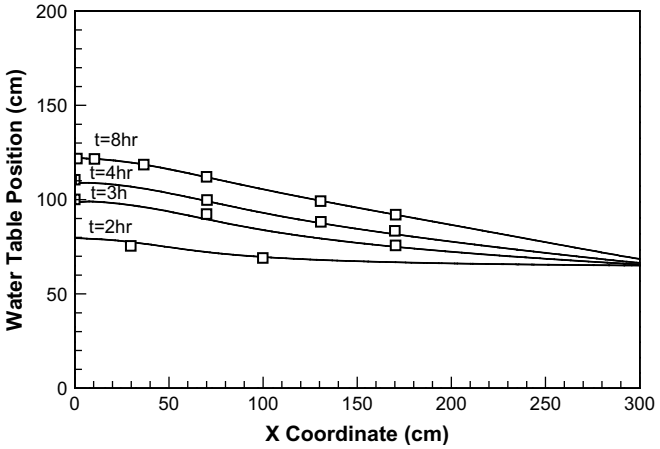

Fig. 1. Comparison between experiment (□) and numerical (—) results for the variably water-table problem.

Table 2
CPU time, total number of time steps (*ndt*) and number of linear system resolutions (*nl*) required by MF_PI, MOL_1 and MOL_V for the variably water-table problem.

	CPU (s)	*ndt*	*nl*
MF_PI	187.65	1925	10,777
MOL_1	90.68	6126	308
MOL_V	25.28	507	148

- Optionally, a subroutine to provide the solver with the sparsity structure of the Jacobian.

4. Numerical experiments

The performance of the LMHFE/MOL scheme is studied using the transient variably water-table recharge problem of Vauclin et al. (1979) and the heterogeneous bidimensional infiltration problem of Huang et al. (1996).

The test cases are simulated using three numerical codes:

- MF_PI (Mixed Form with Picard Iteration): In this code the mixed form of RE given by equations (1) and (3) is solved using the LMHFE method with the fixed point (Picard) linearization. The water content is expanded using a first-order Taylor series with respect to the hydraulic head (Celia et al., 1990). The time step management during the simulation is of heuristic type and the tolerance on the pressure head is fixed to 10^{-6}.
- MOL_1 (First order with MOL): The order of the integration method with the DLSODIS solver is fixed at one. The code in this configuration only adapts the time step size.
- MOL_V (variable order with MOL): The system of ODEs is solved with DLSODIS using variable time step size and variable order.

4.1. The transient variably water-table recharge problem

This test case is based on the laboratory experiments performed by Vauclin et al. (1979) to evaluate the transient position of the water table in a laboratory scale soil box. The domain consisted of a rectangular soil slab of 600 cm × 200 cm with an initial horizontal water table located at a height of 65 cm from the base. A recharge of 355 cm/day was applied over a width of 100 cm in the centre of the soil surface. Because of the symmetry, only the right hand side of the domain needs to be modelled with a no flow boundary prescribed along the axis of symmetry. No flow boundaries were also imposed along the lower and upper boundaries with the exception of the recharge zone. For the right face, a fixed head of 65 cm is imposed below the initial water table, and no flow boundary above the initial water table. The soil system is assumed to be initially at hydrostatic equilibrium with respect to the water table throughout the flow domain. The material properties are obtained from Clement et al. (1994) and given in Table 1.

The simulation of this test case is performed using a triangular mesh of 1990 elements. Fig. 1 shows the experimental and the simulated water table at different times. Results with the three codes are in very good agreement with the experimental results.

Table 2 gives the total CPU time, the number of time steps (*ndt*) and the number of linear solutions performed (*nl*) with the three codes. These results show that MOL_V is more efficient than MOL_1 and MF_PI for the following reasons: (i) MOL_V allows large time steps because the time integration can use several previous solutions and (ii) the number of linear systems solved by MOL_V is reduced since the iteration matrix can be approximated from the previous iterate and/or time level whenever possible (see Radhakrishnan and Hindmarsh (1993) for details).

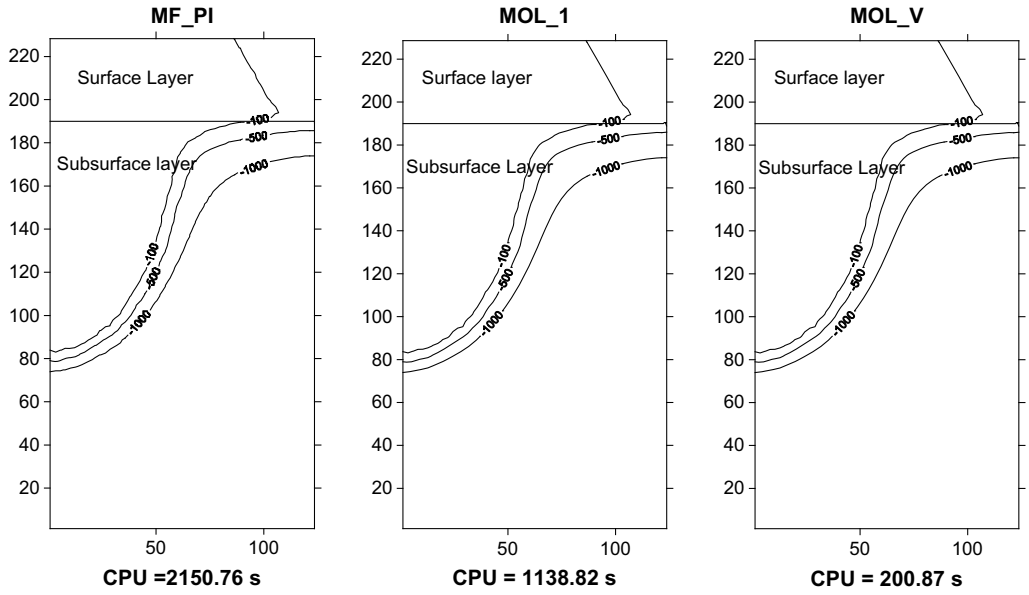

Fig. 2. Results for the bidimensional heterogeneous infiltration problem.

4.2. The bidimensional heterogeneous infiltration problem

This test case was previously investigated by Huang et al. (1996). The problem involves infiltration under a constant head boundary condition into an initially dry heterogeneous porous media. The spatial domain is 125 cm width and 230 cm depth.

The initial problem of Huang et al. (1996) contains two horizontal layers: a surface clayey soil layer (40 cm) and a subsurface loamy soil layer (190 cm). In the following and in order to increase the soil contrast in the domain, the loamy subsurface layer is replaced by a sandy layer. The properties of theses layers are given in Table 1. Boundary conditions are a constant pressure head (-10 cm) in a strip of 20 cm at the inflow boundary ($0 \leq x \leq 20$ cm) and a constant pressure head (-10^4 cm) at the bottom of the domain. The other sides are impervious. The final simulation time is 22 days.

This problem is simulated using a triangular mesh of 1700 elements. The results of the three codes are in very good agreement (Fig. 2). However, large differences can be observed between the three CPU times. As previously, MOL_V runs much faster than both MOL_1 and MF_PI. The gain with MOL_V is more pronounced for this heterogeneous problem. Indeed, in this case the MF_PI code requires very small time steps to reach the convergence. This highlights the efficiency and robustness of the LMHFE/MOL scheme to handle infiltration problems with highly heterogeneous porous media.

The effect of the number of unknowns on the total CPU time is also studied with the three codes. To this aim, 4 runs are performed with each code by increasing the number of elements (1700, 2400, 4000, 16,150 respectively). Fig. 3 highlights the efficiency of MOL_V especially for fine meshes.

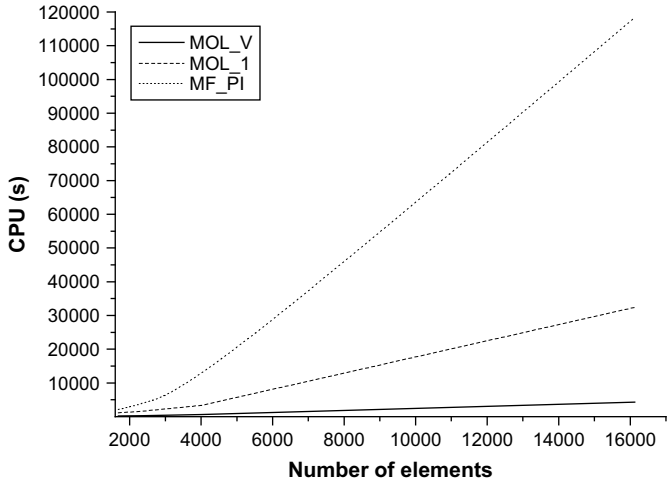

Fig. 3. The variation of CPU time as function of number of elements.

5. Conclusion

In this work, the nonlinear RE is solved using a combination of LMHFE method and the MOL. Contrarily to the standard formulation of MHFEs, the lumped formulation can be easily implemented with existing sophisticated ODE codes.

The LMHFE/MOL scheme was used to simulate homogeneous and heterogeneous infiltration problems. The numerical results highlight the efficiency of the developed scheme especially for heterogeneous domains and fine meshes.

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