A Numerical Model Based on Coupled One-Dimensional Richards and Boussinesq Equations

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An approximate numerical method to solve transient two-dimensional unsaturated-saturated subsurface flow problems is presented. Several one-dimensional vertical unsaturated column models are linked to a one-dimensional saturated flow model. The Dupuit-Forchheimer assumptions are used for the saturated flow, and the unsaturated flow is assumed to be strictly vertical. The systems are linked because solutions of the equation governing unsaturated flow determine recharge and storage information used to solve the equation governing saturated flow. In addition, the position of the water table locates the lower boundaries of the unsaturated models. The solution of a test problem using a linked model is compared with results from a rigorous two-dimensional model. The comparison is good at early times when recharge to the water table is small but poor at later times when recharge increases. Applications are then made to field-size problems. Results compare closely with an actual groundwater hydrograph. In addition, a linked model is used to show that in a specific humid climate watershed the type of vegetation does not significantly affect the groundwater regimen, whereas in a given arid climate watershed the type of vegetation would determine whether groundwater recharge occurred. For field-size problems where water table movement is relatively small, where the Dupuit-Forchheimer assumptions are valid, and where lateral unsaturated flow is not important, the linked model offers an efficient approximate way to solve unsaturated-saturated flow problems.

Several researchers, including *Rubin* [1968], *Taylor and Luthin* [1969], *Hornberger et al.* [1969], *Verma and Brutsaert* [1970], *Cooley* [1971], and *Freeze* [1969, 1971], have used a single equation to model unsaturated-saturated subsurface flow. This is the most general and rigorous approach. However, the difficulty of adapting the highly nonlinear governing equation to an efficient numerical technique seems to justify the use of a treatment that is less rigorous but computationally efficient and useful for investigating special hydrologic systems.

Hornberger et al. [1970] and *Zucker et al.* [1973] used an efficient numerical technique to solve the one-dimensional form of the Boussinesq equation and successfully applied their model to several saturated flow problems. *Lin* [1972] presented an efficient technique for solving the two-dimensional form of the Boussinesq equation. These studies suggest that a model based on a numerical solution of the Boussinesq equation accurately and efficiently simulates flow through many unconfined aquifers under a variety of boundary conditions.

In this study we modify the one-dimensional Boussinesq model by including the effects of flow phenomena in the overlying unsaturated zone. We accomplish this conceptually by routing moisture first through vertical unsaturated soil columns and then through the Boussinesq flow system. Flow through the unsaturated columns is simulated by using the one-dimensional (vertical) form of *Richards'* [1931] equation. The systems are linked, or coupled, because solutions of the Richards equation determine recharge and storage information used to solve the Boussinesq equation. If the vertical movement of the water table is relatively large, it is necessary to adjust continuously the position of the lower boundaries of the unsaturated soil columns. If the movement of the water table is relatively small, the lengths of the columns may be taken as being constant. In either case, two governing equations and two sets of auxiliary conditions must be satisfied simultaneously.

This study is aimed at designing a subsurface flow model that can be used routinely as a tool in predicting groundwater response to proposed changes in a watershed or its hydrologic inputs. It is hoped that operational efficiency will outweigh any loss of generality. The efficiency of the model recommends it as a component in a complete hydrologic response model that will include overland flow, unsaturated and saturated subsurface flow, and open channel flow. Ongoing research is directed toward completion of the watershed model.

This paper describes the components of the linked subsurface flow model and the method of linking. Unsaturated soil properties and assumptions inherent in the linking procedure are discussed. Applications are then made to unsaturated-saturated flow problems. The first example deals with a hypothetical two-dimensional aquifer for which subsurface flow and discharge to a draining stream are simulated by using a linked model. In the second example the same system is

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modeled by using a different soil. Water table positions are compared with those obtained by using a rigorous one-equation model developed by *Freeze* [1971]. In the third example a linked model is used to simulate subsurface flow in an idealized field situation. The results are compared with an actual groundwater hydrograph. The fourth example uses a linked model to study the effects of a vegetation change on groundwater storage in two different climates.

MATHEMATICAL MODEL

Differential equations. Figure 1 is an idealization of the unsaturated-saturated subsurface flow system modeled herein. It shows a section of an unconfined aquifer extending from a stream to a groundwater divide. The symmetry boundary at the groundwater divide is shown as an equivalent impermeable (no-flow) boundary.

Figure 1 shows also one of the several unsaturated soil columns that are linked to the unconfined aquifer. It extends from the water table ($z = 0$) to the land surface ($z = Z$). Because the origin of the z axis is always taken at the water table, the height of the soil column Z varies with the horizontal space coordinate x and in some cases with time t.

For a homogeneous medium and under the Dupuit assumptions the partial differential equation describing saturated unconfined flow [*Remson et al.*, 1971, p. 54] for the one-dimensional case is

$$K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = S(x, t)\frac{\partial h}{\partial t} - w(x, t) \qquad (1)$$

where

- h height of water table above the datum at the impermeable lower boundary;
- x horizontal space coordinate extending from 0 at the stream-aquifer boundary to L at the no-flow boundary at the divide;
- $S(x, t)$ a storage coefficient that is a function of x and t even in homogeneous media, as explained below;
- K saturated hydraulic conductivity;
- t time;
- $w(x, t)$ sink or source term.

The initial and boundary conditions for the saturated subsystem are

$$h(x, 0) = f(x) \qquad 0 \leq x \leq L \qquad (2a)$$

$$h(0, t) = h_0(t) \qquad t > 0 \qquad (2b)$$

$$\partial h/\partial x(L, t) = 0 \qquad t > 0 \qquad (2c)$$

where $h_0(t)$ is the height of the stream surface above datum and $f(x)$ is a prescribed initial function.

The partial differential equation describing vertical unsaturated flow through a soil column [*Remson et al.*, 1971, p. 35] is

$$\frac{\partial}{\partial z}\left[k(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right] = C(\psi)\frac{\partial \psi}{\partial t} \qquad (3)$$

where

- ψ unsaturated pressure head;
- z vertical space coordinate extending from 0 at the water table to Z at the top of the column (land surface);
- $k(\psi)$ unsaturated hydraulic conductivity;
- $C(\psi)$ specific moisture capacity, equal to $d\theta/d\psi$;
- θ soil moisture content;
- t time.

The initial and boundary conditions for the unsaturated subsystem are

$$\psi(z, 0) = f(z) \qquad 0 \leq z \leq Z \qquad (4a)$$

$$\psi(0, t) = 0 \qquad t > 0 \qquad (4b)$$

$$\partial \psi/\partial z\,(Z, t) = [R/k(\psi)] - 1 \qquad t > 0 \qquad (4c)$$

where R is the flux across the land surface and $f(z)$ is a prescribed initial function. If $R > 0$, the flux is inflow to the column (infiltration), and if $R < 0$, the flux is outflow (evaporation). Equation 4c is simply an expression of Darcy's law.

Equation 3 does not include a sink term to account for moisture depletion by roots distributed through the upper part of the soil profile. Where transpiration effects are significant, the system domain is bounded at the base of the 'belt of soil water' [*Meinzer*, 1923]. Drainage rates from the belt of soil water are computed by using the tables of *Thornthwaite and Mather* [1957] and are used as R in (4c).

Unsaturated soil properties. The values of $C(\psi)$ and $k(\psi)$ must be known to solve (3). For this study we use a soil model adapted from *Verma and Brutsaert* [1970]. The soil properties are represented by the following functions:

$$k = K\left[\frac{A}{A + (-\psi)^B}\right]^N \qquad (5a)$$

$$\theta = (\theta_0 - \theta_r)\left[\frac{A}{A + (-\psi)^B}\right] + \theta_r \qquad (5b)$$

$$C = \frac{d\theta}{d\psi} = \frac{(\theta_0 - \theta_r)AB(-\psi)^{B-1}}{[A + (-\psi)^B]^2} \qquad (5c)$$

where θ_0 is the soil moisture content at saturation, θ_r is the 'residual' soil moisture content when ψ has a very large negative value, and A, B, and N are the parameters that depend upon the soil type. Hysteresis effects are ignored.

NUMERICAL APPROXIMATIONS

Boussinesq equation. Equation 1 may be rewritten as

$$\frac{\partial^2 h}{\partial x^2} = \frac{S(x, t)}{Kh}\frac{\partial h}{\partial t} - \frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{w(x, t)}{Kh} \qquad (6)$$

This equation is approximated and solved by using the predictor-corrector technique of *Douglas and Jones* [1963]. The predictor (written for the time step from n to $n + ½$) is

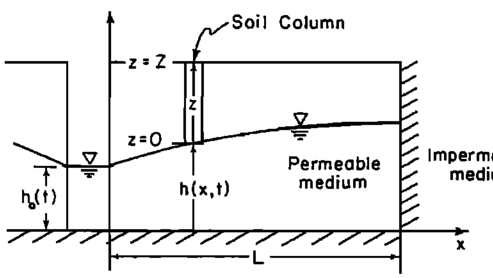

Fig. 1. Idealization of the unsaturated-saturated subsurface flow problem.

$$\frac{h_{i-1}^{n+1/2} - 2h_i^{n+1/2} + h_{i+1}^{n+1/2}}{(\Delta x)^2} = \frac{S_i^n}{Kh_i^n}\left(\frac{h_i^{n+1/2} - h_i^n}{\Delta t/2}\right) - \frac{1}{h_i^n}\left(\frac{h_{i+1}^n - h_{i-1}^n}{2\,\Delta x}\right)^2 - \frac{w_i^n}{Kh_i^n} \quad (7)$$

where i is the horizontal space index, n is the time index, Δx is the increment in space, and Δt is the increment in time. The corrector (written for the time step from n to $n + 1$) is

$$\frac{1}{2}\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{(\Delta x)^2} + \frac{h_{i-1}^n - 2h_i^n + h_{i+1}^n}{(\Delta x)^2}\right) = \frac{S_i^n}{Kh_i^{n+1/2}}\left(\frac{h_i^{n+1} - h_i^n}{\Delta t}\right) - \frac{1}{h_i^{n+1/2}}\left(\frac{h_{i+1}^{n+1/2} - h_{i-1}^{n+1/2}}{2\,\Delta x}\right)^2 - \frac{w_i^n}{Kh_i^{n+1/2}} \quad (8)$$

Because the variation in S is small, it is acceptable and more convenient to use S_i^n instead of $S_i^{n+1/2}$ in (8). Equation 6 is solved through successive applications of (7) and (8) as the solution is marched through time.

Richards equation. The partial differential equation describing vertical unsaturated flow (3) also is solved by using the suggestions of *Douglas and Jones* [1963]. The predictor (written for the time step from n to n + ½) is

$$\frac{1}{\Delta z}\left[k_{j+1/2}^n\left(\frac{\psi_{j+1}^{n+1/2} - \psi_j^{n+1/2}}{\Delta z} + 1\right) - k_{j-1/2}^n\left(\frac{\psi_j^{n+1/2} - \psi_{j-1}^{n+1/2}}{\Delta z} + 1\right)\right] = C_j^n\left(\frac{\psi_j^{n+1/2} - \psi_j^n}{\Delta t/2}\right) \quad (9)$$

where j is the vertical space index, Δz is the increment in space, $k_{j+1/2} = (k_j + k_{j+1})/2$, and $k_{j-1/2} = (k_j + k_{j-1})/2$. The corrector (written for the time step from n to $n + 1$) is

$$\frac{1}{2\,\Delta z}\left[k_{j+1/2}^{n+1/2}\left(\frac{\psi_{j+1}^{n+1} - \psi_j^{n+1}}{\Delta z} + 1\right) - k_{j-1/2}^{n+1/2}\left(\frac{\psi_j^{n+1} - \psi_{j-1}^{n+1}}{\Delta z} + 1\right) + k_{j+1/2}^n\left(\frac{\psi_{j+1}^n - \psi_j^n}{\Delta z} + 1\right) - k_{j-1/2}^n\left(\frac{\psi_j^n - \psi_{j-1}^n}{\Delta z} + 1\right)\right] = C_j^{n+1/2}\left(\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t}\right) \quad (10)$$

Equations 9 and 10 are well suited for use with the particular boundary conditions in this problem.

LINKING

In the usual applications of (1) it is assumed that the uptake of water by or release of water from the unsaturated zone in response to a moving water table can be modeled by taking $S(x, t)$ as being equal to a constant, defined as

$$\bar{S} = \theta_0 - \theta_f \quad (11)$$

where θ_f is the moisture content at the 'field capacity' for the soil.

Childs [1960] pointed out that maintaining S constant accurately represents the effects of unsaturated flow only in special circumstances. For example, if (11) is used to define S, the moisture content at the soil surface always equals θ_f. This equality may not hold if infiltration or evaporation occurs at the surface. Furthermore, if S is constant, it follows that the soil moisture profile can be assumed to maintain a constant shape, merely being displaced vertically as it adjusts to the changing position of the water table.

dos Santos and Youngs [1969] found that for their sand tank experiments a good approximation to S could be obtained from

$$\bar{S} = \theta_0 - \theta_s \quad (12)$$

where θ_s is the moisture content at the soil surface when steady state conditions are reached. However, for small values of t the true S is either higher or lower than $\bar{S}$, depending on initial conditions.

In the present study we define S as

$$S(x, t) = \theta_0 - \theta_m(x, t) \quad (13)$$

where θ_m is the minimum soil moisture content below the depth from which moisture may be removed directly by evapotranspiration. For large values of t, θ_m approaches the θ_s of *dos Santos and Youngs* [1969]. We assume that S is constant within a single time step. However, because θ_m is different in each unsaturated column and because θ_m changes with time, S does vary with x and t. For example, consider the following cases. If the soil is relatively dry initially, θ_m is small, and consequently S is large as computed by (13). In response to a rising water table the soil becomes wet, and θ_m increases while S decreases and at steady state becomes constant. On the other hand, if the soil is initially relatively wet, θ_m is large, and S is small. In response to a falling water table the soil drains, and θ_m decreases while S increases and at steady state becomes constant.

The second term on the right-hand side of (1) is $w(x, t)$, i.e., the rate of drainage out of each unsaturated column. This is found from the materials balance at the end of each time step.

The linking procedure is described below. If the movement of the water table is relatively large, it is desirable to satisfy internal boundary conditions between the two systems by adjusting the lower boundaries of the unsaturated columns after every time step, as described in step 2 below. If the movement of the water table is relatively small, this step can be omitted.

1. The saturated model is solved for the water table profile at t^n by using S^{n-1} and w^{n-1}.
2. If adjustment is necessary, the lower boundaries of the unsaturated columns are adjusted to correspond with the new water table position determined in step 1.
3. The unsaturated column models are solved for pressure potential heads at t^n. Values of θ^n are determined from (5*b*).
4. A materials balance is performed to determine w^n for each unsaturated column; $\theta_m{}^n$ is located in each column, and S^n is determined from (13).
5. Time is advanced by increasing the time index. Steps 1–5 are repeated until the simulation is completed.

To satisfy internal boundary conditions between coupled systems, it is usually desirable to iterate several times before increasing the time index [*Bredehoeft and Pinder*, 1970; *Pinder and Sauer*, 1971; *Freeze*, 1972]. In the model presented here the internal boundary is the water table. Iteration proved unnecessary if the time step did not exceed a certain limit unique to the problem being solved. If the time step exceeded this limit, several iteration cycles were necessary within each time step. It proved more economical to use smaller time steps without iteration.

ASSUMPTIONS AND LIMITATIONS

The theory behind the linked model is based on three assumptions:

1. The Dupuit-Forchheimer assumptions for saturated flow are valid.
2. Unsaturated flow is strictly vertical.
3. The values $w(x, t)$ and $S(x, t)$ are constant throughout a single time step Δt, and good approximations to S can be obtained through the use of (13).

A fourth assumption is necessary if the lower boundaries of the unsaturated columns are adjusted after every time step:

4. The continuous adjustment of the unsaturated columns to changes in the position of the water table can be discretized by assuming an instantaneous change in the position of the water table at the beginning of the time step. (Consequently, the unsaturated systems will adjust to the rise or fall of the water table throughout the time step.)

Because of assumption 4 and the implication that it is desirable to avoid large changes in the water table profile within a time step, a model whose unsaturated columns have moving lower boundaries may require the use of very small time steps. When the movement of the water table is relatively small, it is an acceptable approximation to use unsaturated columns with stationary lower boundaries. In this way much larger time steps can be used.

In this study we used hypothetical soil curves that do not display hysteresis. However, hysteretic functions can be incorporated into the model.

APPLICATIONS

Example 1. To test the linked model, the following problem was formulated. Infiltration is introduced at a constant rate of 0.01 cm/min into a two-dimensional box containing a mass of soil 150 cm high and 300 cm long. No flow is permitted through the base or the right-hand side of the box. A constant head of 50.0 cm is maintained at the left-hand side. Initial conditions are those of equilibrium for zero infiltration. The water table is allowed to rise until it approaches the analytic steady state position. The infiltration source is then removed, and the water table declines to its initial position.

We used a hypothetical soil with the following values of the parameters in (5): $K = 0.096$ cm/min, $\theta_0 = 0.342$, $\theta_r = 0.089$, $A = 20.0$ cm, $B = 0.50$, and $N = 10.0$. For this case, where the movement of the water table is relatively large, it was necessary to adjust the lengths of the unsaturated columns through time.

The saturated system approaches the analytic solution of the steady state form of (1) at $t = 3380$ min, when the unsaturated columns are near steady state. Figure 2*a* shows the rising water table profiles and the analytic steady state solution. Figure 2*b* shows the water table profile as it declines from the analytic steady state position and approaches static equilibrium after approximately 2320 min.

The above results were obtained by using a horizontal node spacing of 50 cm and a vertical node spacing of 4 cm. A variable time step ranging from 0.1 min to 4.0 min was used. The total computer time for both runs was 16 min on an IBM 360/67. This is only slightly less than the time required to solve the same problem by using a rigorous single equation model (R. A. Freeze, personal communication, 1973).

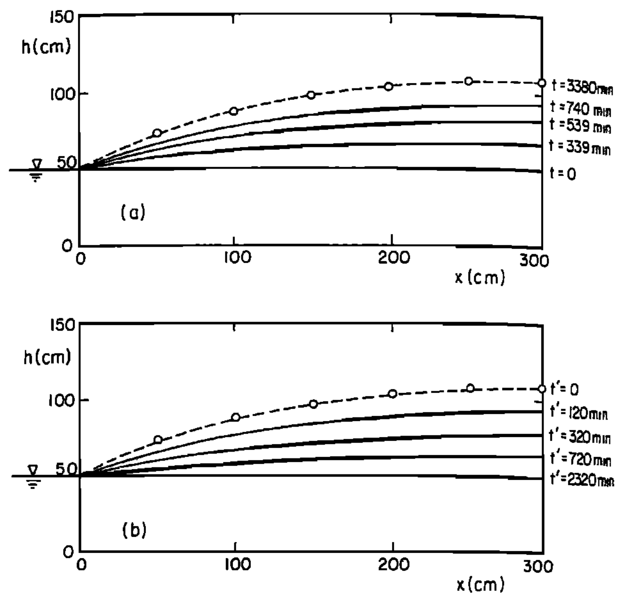

Fig. 2. Results for example 1. Analytic steady state solution of (1), where $w = 0.01$ cm/min (dashed lines); linked model steady state solution (circles); and linked model transient solutions (solid lines). (*a*) Rising water table profiles; (*b*) falling water table profiles.

Example 2. Example 1 demonstrates that the linked model is internally consistent, because the transient model reaches the analytic steady state solution. To compare the linked model with a rigorous two-dimensional model, the problem described in example 1 was solved again for a different soil type and compared with a solution computed by R. A. Freeze (personal communication, 1973), using a rigorous single-equation model [*Freeze*, 1971].

The soil curves for this simulation (Figure 3) were obtained from R. A. Freeze (personal communication, 1973) rather than from (5*a*) and (5*b*). The values for K and θ_0 in example 2 are the same as those for K and θ_0 for the soil used in example 1, but the shapes of the curves are different. Figure 4 shows the results of the rising water table simulation for the linked model in comparison with those obtained from Freeze's model.

Freeze found that for this problem, unsaturated flow has a significant lateral component, as shown by Figure 5. It follows that the linked model assumption of strictly vertical flow could be expected to introduce errors. However, when $t \leq 611$ min, the comparison between the two models is good for the following reasons. At early times the unsaturated columns of the linked model are relatively dry, unsaturated conductivities are low, and consequently, recharge to the water table from each column averages less than 60% of R. Therefore differences in model behavior because of lateral flow do not significantly affect the slowly rising water table. At later times, as the soil continues to become wet, conductivities in the unsaturated columns of the linked model increase, recharge to the water table increases, and the water table begins to rise rapidly. After 611 min the soil columns of the linked model have become wet enough to transmit most of the infiltrated moisture directly to the water table, a steady, rapid rise resulting. In the rigorous two-dimensional model, considerable moisture flows laterally through the unsaturated zone, and after delay it reaches the water table closer to the stream. At $t = 1611$ min, Freeze's solution shows that $k > 0.5K$ in about 98% of the unsaturated zone. Thus at $t = 1611$ min, when the recharge to the water table in the linked model

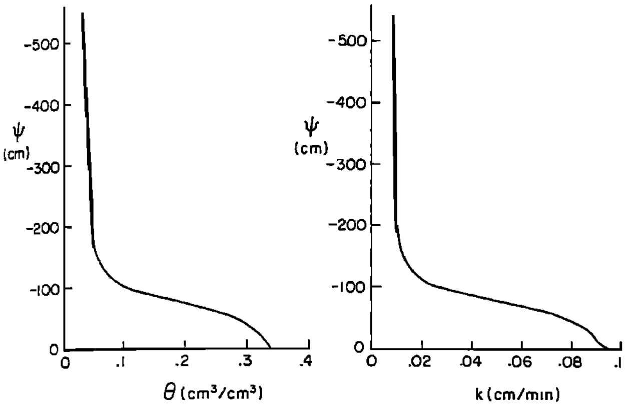

Fig. 3. Soil curves for example 2 (R. A. Freeze, personal communication, 1973); $K = 0.096$ cm/min, and $\theta_0 = 0.342$.

averages about 90% of R, the linked model solution deviates significantly from Freeze's solution, as is shown in Figure 4.

Figure 5 shows the lines of total potential and the water table positions as computed by R. A. Freeze (personal communication, 1973). Figure 5*a* shows the configuration at t = 611 min, and Figure 5*b* shows the configuration at t = 1611 min. In both cases the Dupuit assumptions are fair approximations. However, in both cases, unsaturated flow has a significant lateral component. Ongoing experiments have demonstrated that when lateral unsaturated flow is unimportant, the validity of the linked model depends on how well the Dupuit assumptions are fulfilled. Because one of the assumptions requires that a nearly horizontal water table exist, it follows that the linked model operates with least error for very low water table gradients.

In this example the infiltration rate is 10% of K, and the length of the system is only twice its height, the result being the development of a high, lateral unsaturated potential gradient to transmit the infiltrated moisture. In field situations the ratios of infiltration rate to K are generally lower, and the length to height ratio is higher, so that lateral unsaturated potential gradients are probably low. Hence it is likely that in field situations, lateral unsaturated flow is relatively unimportant. Ongoing research is aimed at establishing criteria to predict when lateral unsaturated flow becomes important.

Example 3. Zucker et al. [1973] reported the 16-year average hydrograph for the Hulsart well near Old Bridge, New Jersey. Using their data, we simulated the Hulsart watershed by using a linked model. To simulate long periods of time (yearly cycles), the lengths of the unsaturated columns were taken as being constant because in this case the total annual water table fluctuation in the Hulsart watershed is only about 60 cm.

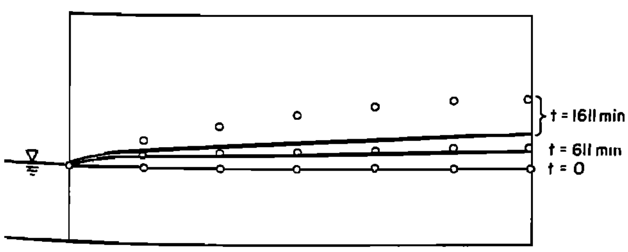

Fig. 4. Linked model results (circles) compared with results from R. A. Freeze's (personal communication, 1973) two-dimensional model (solid lines) for example 2.

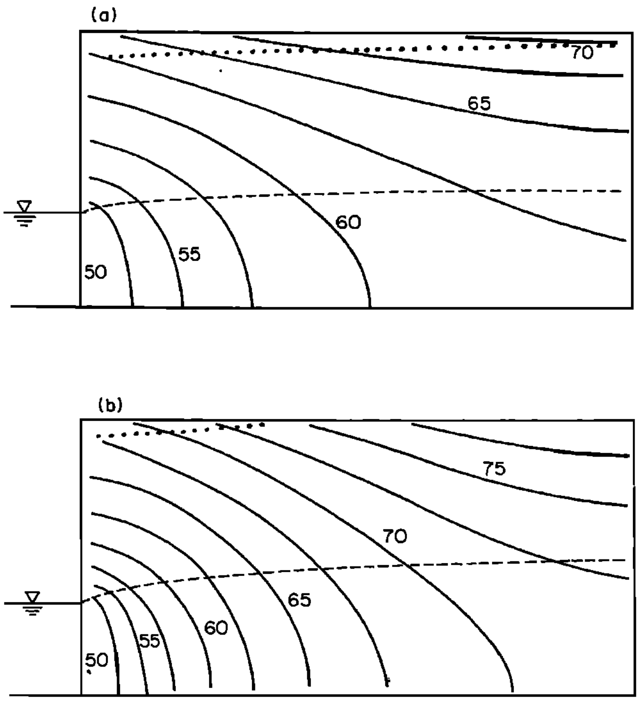

Fig. 5. Contours of the total potential ($\psi + z$) and the position of the water table (dashed lines) as computed by R. A. Freeze (personal communication, 1973) for example 2. Above the dotted line, $k < 0.5K$, and below the dotted line, $k > 0.5K$. (*a*) Time t = 611 min; (*b*) t = 1611 min.

Although the actual aquifer thickness is unavailable, *Zucker et al.* [1973] found that a value of 6.09 m for the vertical distance from the stream surface to the base of the aquifer worked well for their model. Accepting this figure and obtaining relative topography from a topographic map, we fixed the lengths of the unsaturated columns, as is shown in Table 1. The Hulsart well is 152.40 m from the stream, and the divide is 274.32 m from the stream.

The recharge data given by *Zucker et al.* [1973] were derived by using climatic data from Runyon, New Jersey, and assuming a mature forest vegetation cover in fine sandy soil. Soil moisture surpluses were computed by using the tables of *Thornthwaite and Mather* [1957]. In our model we assume that the vegetation cover is rooted in fine sandy soil and overlies the unsaturated columns. Drainage rates from this uppermost zone, *Meinzer*'s [1923] belt of soil water, are given below in centimeters per month.

November	5.77
December	9.30
January	9.88
February	7.62
March	9.09
April	5.11
May	0.86
June	0.00
July	0.00
August	0.00
September	0.00
October	0.00

We used the soil model described by (5) with K = 0.349 cm/min and an average S = 0.34, values used by *Zucker et al.* [1973]. The soil parameters are: K = 0.349 cm/min, θ_0 = 0.451, θ_r = 0.001, A = 40.0 cm, B = 1.00, and N = 5.10.

TABLE 1. System Dimensions for Example 3

Distance From Stream, m	Distance From Datum to Ground Surface, m	Length of Unsaturated Column, m
45.72	7.84	1.75
91.44	8.34	2.25
137.16	9.54	3.45
182.88	10.34	4.25
228.60	10.84	4.75
274.32	11.34	5.25

Figure 6 shows the results of the simulation using the above data. The simulated hydrograph follows the actual hydrograph very closely. Figure 6 also shows the hydrograph obtained by using a model similar to that of *Zucker et al.* [1973]. Because their model is a saturated model only and does not have input from unsaturated model components, it is able to reproduce the shape of the hydrograph but not the lag effect caused by the time necessary for flow through the unsaturated zone.

We do not believe that keeping the lengths of the unsaturated columns constant significantly influenced the results, because the water table movement in the Hulsart watershed is relatively small. Experiments with small-scale models like the one described in example 1 indicate that when relative water table movement is small, adjusting the lower boundaries of the unsaturated columns does not significantly affect the water table profiles generated. Moreover, because the ratio of the infiltration rate to K is always low (less than 0.07%) and the length to height ratio is high (about 30 to 1), lateral unsaturated flow is probably insignificant.

In this model, $\Delta x = 45.72$ m, $\Delta z = 0.10$ m, and $\Delta t = 0.05$ month. One year of simulated time requires approximately 4 min of computer time. This is significantly less than the time that would be required for a rigorous single-equation model or a linked model whose unsaturated columns have time-dependent lower boundaries.

Example 4. The effects of vegetation modifications were studied for two climatic regimens. It was assumed that the forest vegetation cover of example 3, having a 25.40-cm maximum soil moisture storage capacity, was changed to shallow-rooted crops having 5.08-cm maximum soil moisture storage capacity for sandy soil. The soil moisture surpluses generated by the climatic balance differ from the previous case only in the month of November and then by only 2.41 cm. The annual groundwater recharge is 47.63 cm from the forest soil and 50.05 cm from the crop-covered soil. In this region of relatively abundant precipitation (117 cm/yr), the summer soil moisture deficits in the forest soil and the shallow-cropped soil are restored to maximum soil moisture storage capacity in the fall by amounts of infiltrated precipitation that differ by only 2.41 cm. Then they produce similar amounts of soil moisture surplus available for groundwater recharge during the remaining wetter and cooler months.

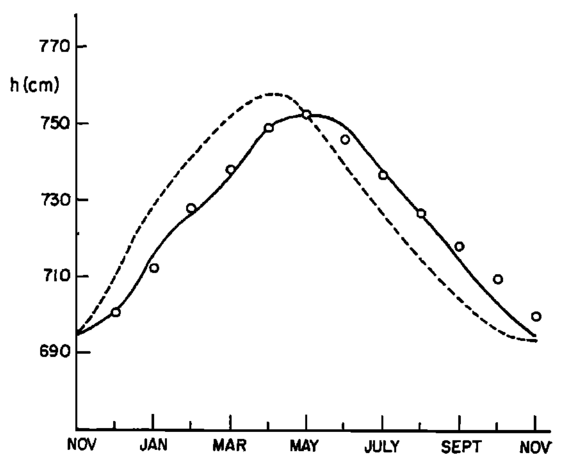

Fig. 6. Hulsart hydrograph: the actual hydrograph (solid line) was simulated by using a linked model (circles). The hydrograph obtained by using a saturated model without input from unsaturated models (dashed line) is also shown.

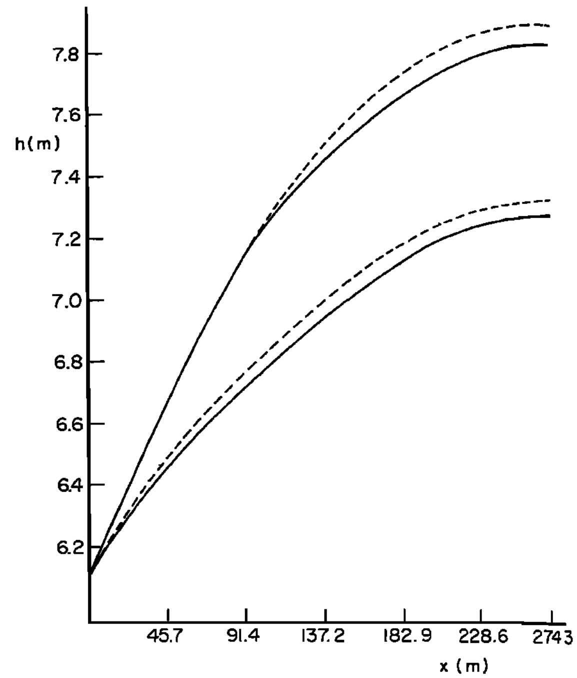

Fig. 7. Highest (mid-May) and lowest (mid-November) monthly water table profiles in the Hulsart watershed under a mature forest cover (solid lines) and under shallow-rooted crops (dashed lines).

The solid lines in Figure 7 show the highest (May) and lowest (November) monthly water table profiles under a mature forest cover. The dashed lines show the highest and lowest water table profiles under the shallow-cropped soil. The maximum difference in water table height resulting from differences in vegetative cover is at the divide and amounts to only 7.1 cm.

A hypothetical watershed having the Hulsart geometry and aquifer properties but located at Palo Alto, California, was considered next. The vegetative cover was to be rooted in silt loam soil. If the vegetation consisted of deep-rooted shrubs or pasture with a 25.4-cm maximum soil moisture storage capacity, the scanty rainfall (40.6 cm/yr) would never restore soil moisture storage after severe summer depletion. and no groundwater recharge would occur. On the other hand, if the watershed were devoted to shallow-rooted crops with a maximum soil moisture storage capacity of 12.7 cm, the rainfall would be sufficient to restore depleted soil moisture storage in January and produce 5.20 cm of surplus for groundwater recharge in February and March.

Figure 8 shows the highest (June–July) and lowest (January–February) water table profiles under shallow-rooted crops at Palo Alto, the February and March moisture surpluses from the climatic materials balance being used. The

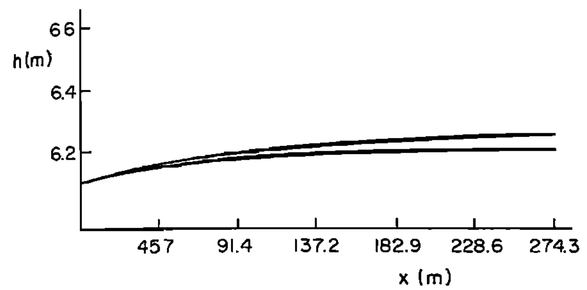

Fig. 8. Highest (June-July) and lowest (January-February) monthly water table profiles under shallow-rooted crops in a hypothetical watershed near Palo Alto, California.

water table profiles are much flatter than those obtained for the New Jersey example. Figure 9 shows the annual variation in drainage w from unsaturated columns centered at $x = 91.4$, 182.9, and 274.3 m. It shows that near the stream, where the unsaturated zone is relatively thin, the February–March recharge is routed quickly through the unsaturated zone and reaches the water table throughout February, March, April, and part of May. Because the unsaturated zone thickens as one moves toward the divide, the peak recharge is delayed until April at $x = 182.9$ m and until June at the divide ($x = 274.3$ m). The total annual recharge (the area under the curve) is the same from each of the three columns.

Of course, no groundwater recharge occurs under the deep-rooted shrubs or pasture in the hypothetical watershed, as has been verified by field measurements [*Dickason*, 1970]. In such arid and semiarid situations, significant groundwater recharge occurs mainly where surface waters have been concentrated in channels, and this situation has not been included in the above model.

Conclusion

Our experiments have shown that the linked model is internally consistent. Comparison of linked model results with results from a rigorous one-equation model showed that when lateral unsaturated flow is significant, the linked model is not valid, and a rigorous model must be used. In laboratory models, lateral unsaturated flow can be important. However, in field situations where there are low water table gradients, lateral unsaturated flow is probably negligible.

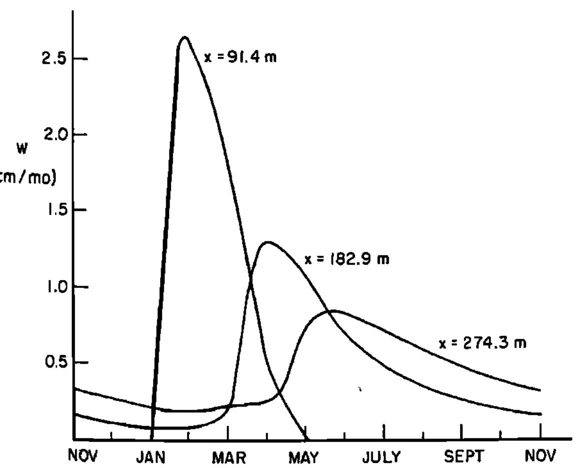

Fig. 9. Annual variation in drainage w from unsaturated columns centered at $x = 91.4$, 182.9, and 274.3 m for a hypothetical watershed near Palo Alto, California.

For problems in which the water table movement is relatively large, it is necessary to vary the lengths of the soil columns with time and to use small time steps, which limit the use of a linked model. On the other hand, if the water table movement is relatively small, it is unnecessary to adjust continuously the lower boundaries of the unsaturated columns. Then it is possible to use large time steps and to simulate long-period movements in field-size models. This model offers an efficient, approximate way to solve unsaturated-saturated subsurface flow problems and has great potential as a tool for predicting the hydrologic effects of urbanization changes. The study of the effects of vegetation changes in the Hulsart watershed demonstrates this capacity.

Notation

- A, B soil parameters used in the expressions for $C(\psi)$, $\theta(\psi)$, and $k(\psi)$.
- C specific moisture capacity, equal to $d\theta/d\psi$, L^{-1}.
- h height of the water table above datum, L.
- h_0 height of the stream surface above datum, L.
- i, j indexes in the x and z dimensions.
- k unsaturated hydraulic conductivity, LT^{-1}.
- K saturated hydraulic conductivity, LT^{-1}.
- L length of the one-dimensional aquifer, L.
- n time index.
- N soil parameter used in the expression for $k(\psi)$.
- R infiltration rate at the land surface, LT^{-1}.
- S storage coefficient, dimensionless.
- $\bar{S}$ *Child's* [1960] storage coefficient, dimensionless.
- $\bar{\bar{S}}$ *dos Santos and Youngs'* [1969] storage coefficient, dimensionless.
- t time, T.
- w sink or source term, LT^{-1}.
- x horizontal space coordinate, L.
- z vertical space coordinate, L.
- Z height of an unsaturated column, L.
- Δt increment in time, T.
- $\Delta x, \Delta z$ increments in space, L.
- θ soil moisture content, dimensionless.
- θ_f soil moisture content at the field capacity for the soil, dimensionless.
- θ_m minimum soil moisture content below the depth affected by evapotranspiration, dimensionless.
- θ_0 soil moisture content at saturation, dimensionless.
- θ_r residual soil moisture content when ψ has a very large negative value, dimensionless.
- θ_s soil moisture content at the soil surface when steady state conditions are reached, dimensionless.
- ψ unsaturated pressure head, L.

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