

## S1 Introduction

The following items support the main text of the manuscript:

- 1) Figure S1 provides the locations of the UK and US reference catchments used in the empirical test of the catchment trajectory conjecture.
- 5 2) Table S1 lists studies that have developed relationships for the catchment-specific parameters,  $n$  or  $w$ , in terms of biophysical features, as well as their explicit equations.
- 3) Section S2 provides supplemental information for empirically testing for Budyko curve trajectories.
- 4) Section S3 provides supplemental information for the non-uniqueness of the parametric Budyko equations.
- 5) Section S4 provides supplemental information for interpreting  $n$  and  $w$  as proxy variables for the evaporative index.
- 10 6) Figure S2 illustrates  $n$  and  $w$  as proxy variables for the evaporative index.

## S2 Supplemental information for empirically testing for Budyko curve trajectories

Here we give more specific details of the methods used in the empirical test of the catchment trajectory conjecture, which consisted of following steps:

- 1) Computing the time series of  $E_0$  for the US reference catchments
- 15 2) Calculating the expected Budyko curve catchment trajectories
- 3) Calculating the multiple realizations of actual catchment trajectories
- 4) Statistically comparing actual and expected catchment trajectories

### S2.1 Computing the time series of $E_0$ for the US reference catchments

The 660 US references catchments used in the empirical test of the catchment trajectory conjecture were from the  
20 CAMELS dataset (Addor et al., 2017; Newman et al., 2015). Daily time series of  $T_{max}$  and  $T_{min}$  were available for each watershed, which allowed for the calculation of  $E_0$  daily time series using the Hargreaves potential evaporation equation (Hargreaves and Allen, 2003; Lu et al., 2005);

$$E_0 = \begin{cases} [0.023]R_a \left[ \frac{T_{max}+T_{min}}{2} + 17.8 \right] \sqrt{T_{max} + T_{min}}, & \frac{T_{max}+T_{min}}{2} > -17.8 \\ 0, & \frac{T_{max}+T_{min}}{2} \leq -17.8 \end{cases}, \quad (S1)$$

where  $E_0$  is the potential evapotranspiration in mm/day,  $T_{max}$  is the maximum daily temperature in degrees Celsius,  $T_{min}$  is  
25 the minimum daily temperature in degrees Celsius, and  $R_a$  is the extraterrestrial radiation in MJ/m<sup>2</sup>/day. Daily  $R_a$  estimates were calculated using the catchments' latitude and the equations for "extraterrestrial radiation for daily periods" (Allen et al. (1998).

## S2.2 Calculating the expected Budyko curve catchment trajectories

The expected Budyko curve trajectories were calculated by fitting Eq. (5) of the main text to the long-term values of  $\frac{\bar{E}_0}{\bar{P}}$  and  $\frac{\bar{E}}{\bar{P}}$  for each catchment. Long-term values of  $\frac{\bar{E}_0}{\bar{P}}$  and  $\frac{\bar{E}}{\bar{P}}$  were calculated from  $\bar{E}_0$ ,  $\bar{P}$ , and  $\bar{E}$ , which were estimated using the entire period of record of daily  $Q$ ,  $P$ , and  $E_0$  values. Specifically,

$$5 \quad \bar{P} = \frac{1}{N} \sum_{i=1}^N P_i, \quad (S2)$$

$$\bar{E}_0 = \frac{1}{N} \sum_{i=1}^N E_{0i}, \quad (S3)$$

and

$$\bar{E} = \bar{P} - \bar{Q} = \bar{P} - \frac{1}{N} \sum_{i=1}^N Q_i, \quad (S4)$$

where  $P_i$ ,  $E_{0i}$ , and  $Q_i$  are the daily values of  $P$ ,  $E_0$ , and  $Q$ , respectively, and  $N$  is the total number of days in the catchment's record. The expected Budyko curve trajectories are defined by Eq. (5) with a fitted value of the catchment-specific parameter  $n$  obtained by numerically solving

$$\left[ \frac{\frac{\bar{E}_0}{\bar{P}}}{\left[1 + \left(\frac{\bar{E}_0}{\bar{P}}\right)^n\right]^{\frac{1}{n}}} - \frac{\bar{E}}{\bar{P}} \right]^2 = 0. \quad (S5)$$

The fitted  $n$  values were used to generate the expected location of the catchments in Budyko space for any value of  $\frac{\bar{E}_0}{\bar{P}}$ .

## S2.3 Calculating the multiple realizations of actual catchment trajectories

15 The actual trajectories that the reference catchments took through Budyko space were determined by calculating the temporally sequential values of  $\frac{\bar{E}_0}{\bar{P}}$  and  $\frac{\bar{E}}{\bar{P}}$ , using averaging windows varying from 1 year to the maximum number of whole years in the catchment's record (ranging between 1 to 11 years and 1 to 56 years). For example, if a reference catchment had a period of record of length 25.5 years, it would have 25 realizations of the actual trajectory it took through Budyko space, one for each averaging window between 1 and 25 years. This resulted in a sum total of 2904 realizations from the 81 reference  
20 catchments.

The actual Budyko space trajectories, consisting of temporally sequential values of  $\frac{\bar{E}_0}{\bar{P}}$  and  $\frac{\bar{E}}{\bar{P}}$ , were calculated from temporally sequential values  $\bar{E}_0$ ,  $\bar{P}$ , and  $\bar{E}$ , which were estimated using the records of daily  $Q$ ,  $P$ , and  $E_0$  values. Specifically,

$$\mathcal{X}_M = \left\{ \left( \frac{\frac{1}{M} \sum_{i=1}^M E_{0i}}{\frac{1}{M} \sum_{i=1}^M P_i}, \frac{\frac{1}{M} \sum_{i=1}^M P_i - \frac{1}{M} \sum_{i=1}^M Q_i}{\frac{1}{M} \sum_{i=1}^M P_i} \right), \left( \frac{\frac{1}{M} \sum_{i=2}^{M+1} E_{0i}}{\frac{1}{M} \sum_{i=2}^{M+1} P_i}, \frac{\frac{1}{M} \sum_{i=2}^{M+1} P_i - \frac{1}{M} \sum_{i=2}^{M+1} Q_i}{\frac{1}{M} \sum_{i=2}^{M+1} P_i} \right), \dots, \left( \frac{\frac{1}{M} \sum_{i=N-M+1}^N E_{0i}}{\frac{1}{M} \sum_{i=N-M+1}^N P_i}, \frac{\frac{1}{M} \sum_{i=N-M+1}^N P_i - \frac{1}{M} \sum_{i=N-M+1}^N Q_i}{\frac{1}{M} \sum_{i=N-M+1}^N P_i} \right) \right\}, \quad (S6)$$

where  $\mathcal{K}_M$  is actual Budyko space trajectory realization for averaging window size  $M$ , comprised of the set,  $\{ \}$ , of temporally sequential values of  $\frac{\bar{E}_0}{\bar{P}}$  and  $\frac{\bar{E}}{\bar{P}}$ , ( , ).  $P_i, E_{0i}, Q_i$ , and  $N$  are the same as defined in Sect. S2.2.

#### S2.4 Statistically comparing actual and expected catchment trajectories

The catchment trajectory conjecture was tested for each realization of the reference catchments' actual trajectories (i.e.,  $\mathcal{K}_M$ ) by statistically comparing them to the corresponding expected Budyko curve trajectories using the non-parametric sign test (Holander and Wolfe, 1973). The sign test is sensitive to the presence of consistent differences between paired observations and requires few assumptions, which makes it widely applicable. Specifically, for the test between the expected and actual Budyko space catchment trajectories, the assumptions are: 1)  $Z_k = Y_k - X_k$  and  $Z_k = \theta + \epsilon_k$ ; where  $Y_k = \frac{\frac{1}{M} \sum_{i=k}^{M+k-1} P_i - Q_i}{\frac{1}{M} \sum_{i=k}^{M+k-1} P_i}$  and  $X_k = \frac{\frac{\frac{1}{M} \sum_{i=k}^{M+k-1} E_{0i}}{\frac{1}{M} \sum_{i=k}^{M+k-1} P_i}}{1 + \left( \frac{\frac{1}{M} \sum_{i=k}^{M+k-1} E_{0i}}{\frac{1}{M} \sum_{i=k}^{M+k-1} P_i} \right)^{n \frac{1}{n}}}$  are the paired actual and expected evaporative index observations, respectively;

$P_i, E_{0i}, Q_i, N$ , and  $M$  are the same as defined in Sect. S2.2 and Sect. S2.3;  $n$  is the fitted value of the catchment-specific parameter obtained from Eq. (S5) in Sect. S2.2;  $k = 1, 2, 3, \dots, N - M + 1$ ;  $\theta$  is the size of the unknown consistent differences between actual and expected trajectories; and  $\epsilon_k$  is the unobserved random effects associated with each pair of actual and expected evaporative index observations. 2)  $\{\epsilon_k\}$  are mutually independent. 3) Each  $\epsilon_k$  comes from a continuous population with a median zero, though not necessarily the same distribution.

The null hypothesis for the sign test is,

$$H_0: \theta = 0, \quad (S7)$$

which is that there is no consistent difference between the actual and expected Budyko space trajectories. To test  $H_0$ , we first compute the statistic,

$$B = \sum_{k=1}^{N-M+1} \chi_k, \quad (S8)$$

where  $\chi_k = \begin{cases} 1, & Z_k > 0 \\ 0, & Z_k < 0 \end{cases}$ . Under the null hypothesis,  $B$  is distributed binomially, specifically,  $B \sim \frac{(N-M+1)!}{B!(N-M+1-B)!} \left[ \frac{1}{2} \right]^{N-M+1}$ , which has the cumulative distribution function,

$$p = I_{\frac{1}{2}}(N - M + 1 - B, 1 + B), \quad (S9)$$

where  $I_x(, )$  is the regularized incomplete beta function. We reject the null hypothesis at the 0.95 confidence level, in favor of the alternative hypothesis,  $H_1: \theta \neq 0$  (i.e., there is a consistent difference between the actual and expected Budyko space trajectories), if,  $I_{\frac{1}{2}}(N - M + 1 - B, 1 + B) < 0.025$  or  $I_{\frac{1}{2}}(N - M + 1 - B, 1 + B) > 0.975$ .

### S3 Supplemental information for the non-uniqueness of the parametric Budyko equations

#### S3.1 Generating parametric Budyko equations

Here we provide a method to generate single parameter Budyko equations and use it to produce Eq. (14) and (15) in the main text. It is important to note that the equations generated through this method are arbitrary and under-determined in the same way as Eq. (5) and (6), therefore, we do not advocate their use as a predictive tool within catchment hydrology. Their purpose is only to illustrate the non-uniqueness of single parameter Budyko equations (such as Eq. (5) and (6)).

To be a valid parametric Budyko equation analogous to Eq. (5) and (6) in the main text, an equation should: 1) approach 0 and 1 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively; 2) have its first derivative approach 1 and 0 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively; 3) asymptotically approach the energy and water limits as the catchment-specific parameter approaches infinity; and 4) asymptotically approach zero as the catchment-specific parameter approaches its lower bound; Any other single-parameter equation in addition to Eq. (5) and (6) that has these properties could be an equally valid parametric “Budyko equation”.

To generate valid parametric Budyko equations, first, we assume the aridity index,  $\phi$ , is the first moment of an underlying distribution which describes the frequency of observing a given instantaneous aridity,  $\phi$ , for a catchment (i.e.,  $\phi = \bar{\phi}$ ). This instantaneous aridity variable is always positive or zero (i.e.,  $\phi \geq 0$ ). From  $\phi$  and conservation of energy and mass we can define an instantaneous evaporative variable as,

$$\varepsilon = \begin{cases} \phi, & \phi < 1 \\ 1, & \phi \geq 1 \end{cases}. \quad (\text{S10})$$

The evaporative index is the first moment of the instantaneous evaporative distribution (i.e.,  $\frac{\bar{\varepsilon}}{\bar{p}} = \bar{\varepsilon}$ ), which can be written as,

$$\frac{\bar{\varepsilon}}{\bar{p}} = \bar{\varepsilon} = \rho_{\phi \geq 1} \bar{\Phi}_{\phi \geq 1} + \rho_{\phi < 1} \bar{\Phi}_{\phi < 1}, \quad (\text{S11})$$

where  $\rho_{\phi \geq 1}$  is the truncated zeroth moment of the instantaneous aridity distribution for  $\phi \geq 1$ ,  $\bar{\Phi}_{\phi \geq 1}$  is the truncated first moment of the instantaneous aridity distribution for  $\phi \geq 1$ ,  $\rho_{\phi < 1}$  is the truncated zeroth moment of the instantaneous aridity distribution for  $\phi < 1$ , and  $\bar{\Phi}_{\phi < 1}$  is the truncated first moment of the instantaneous aridity distribution for  $\phi < 1$ . Since,  $\bar{\Phi}_{\phi \geq 1} = 1$  and  $\rho_{\phi \geq 1} = 1 - \rho_{\phi < 1}$ , Eq. (S11) can be further simplified to,

$$\frac{\bar{\varepsilon}}{\bar{p}} = \bar{\varepsilon} = 1 - \rho_{\phi < 1} + \rho_{\phi < 1} \bar{\Phi}_{\phi < 1}. \quad (\text{S12})$$

Equation (S12) can be used to generate distinct valid parametric Budyko equations by specifying a particular non-negative two-parameter distribution for  $\phi$  and analytically solving  $\rho_{\phi < 1}$  and  $\bar{\Phi}_{\phi < 1}$  in terms of  $\phi$  and a second, appropriately chosen catchment-specific parameter,  $q$ .

### S3.2 Choice of $\phi$ distribution and $q$ for Eq. (14)

To produce Eq. (14) in the main text, we use Eq. (S12) with the choice of a two-parameter distribution for  $\phi$ , the gamma distribution,

$$\phi \sim \frac{\beta^\alpha}{\Gamma(\alpha)} \phi^{\alpha-1} e^{-\beta\phi}, \quad (\text{S13})$$

5 where the shape parameter,  $\alpha > 0$ , the rate parameter,  $\beta > 0$ , and  $\Gamma(\cdot)$  is the gamma function. The parameters  $\rho_{\phi < 1}$  and  $\bar{\Phi}_{\phi < 1}$  of Eq. (S12) are defined as follows for the gamma distribution,

$$\rho_{\phi < 1} = \frac{\gamma(\alpha, \beta)}{\Gamma(\alpha)}, \quad (\text{S14})$$

and

$$\bar{\Phi}_{\phi < 1} = \frac{\Gamma(\alpha)}{\gamma(\alpha, \beta)} \frac{\alpha \gamma(\alpha+1, \beta)}{\beta \Gamma(\alpha+1)}, \quad (\text{S15})$$

10 where  $\gamma(\cdot, \cdot)$  is the lower incomplete gamma function.

Next, we choose an appropriate attribute of the distribution of  $\phi$  as the catchment-specific parameter,  $q$ , specifically,

$$q = \alpha. \quad (\text{S16})$$

The mean of the gamma distribution can be expressed in terms of  $\alpha$  and  $\beta$ , as  $\phi = \frac{\alpha}{\beta}$ . Solving these equations for  $\alpha$  and  $\beta$  gives the gamma distribution parameters in terms of  $\phi$  and  $q$ ,

$$15 \quad \alpha = q, \quad (\text{S17})$$

and

$$\beta = \frac{\alpha}{\phi} = \frac{q}{\phi}, \quad (\text{S18})$$

Substituting Eq. (S14), (S15), (S17), and (S18) into Eq. (S12) and letting  $q = q_n$  yields,

$$\frac{\bar{E}}{\bar{P}} = 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_n+1, \frac{q_n}{\phi})}{\Gamma(q_n+1)} \phi \right], \quad (\text{S19})$$

20 which is Eq. (14) of the main text.

Now, we test the Eq. (14) to determine if it has the required properties to be a valid parametric Budyko equation analogous to Eq. (5) and (6). First, Eq. (14) should approach 0 and 1 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively. We can test these conditions by taking the appropriate limits of Eq. (S19),

$$\lim_{\phi \rightarrow 0} \left[ 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right] \right] = 1 - 1 + (1)(0) = 0, \quad (\text{S20})$$

and

$$\lim_{\phi \rightarrow \infty} \left[ 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right] \right] = \lim_{\phi \rightarrow \infty} \left[ 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{q_n \left[ \frac{q_n}{\phi} \right]^{q_n} e^{-\frac{q_n}{\phi}}}{\Gamma(q_{n+1})} \right] \right] = 1 - 0 + 0 = 1, \quad (\text{S21})$$

which are the required values. Second, the first derivative of Eq. (14) should approach 1 and 0 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively. We can test these conditions by taking the appropriate limits of the derivative of Eq. (S19),

$$\frac{d\bar{E}}{d\phi} = \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})}, \quad (\text{S22})$$

which gives,

$$\lim_{\phi \rightarrow 0} \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \right] = 1, \quad (\text{S23})$$

and

$$\lim_{\phi \rightarrow \infty} \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \right] = 0, \quad (\text{S24})$$

which are the required values. Third, Eq. (14) should asymptotically approach the energy and water limits as the catchment-specific parameter approaches infinity. We test this condition by taking the limit of Eq. (S19) as  $q_n \rightarrow \infty$ , which gives,

$$\lim_{q_n \rightarrow \infty} \begin{cases} 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right], & \phi < 1 \\ 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right], & \phi \geq 1 \end{cases} = \begin{cases} 1 - 1 + (1)(\phi), & \phi < 1 \\ 1 - 0 + (0)(\phi), & \phi \geq 1 \end{cases} = \begin{cases} \phi, & \phi < 1 \\ 1, & \phi \geq 1 \end{cases}, \quad (\text{S25})$$

the water and energy limits. Finally, Eq. (14) should asymptotically approach zero as the catchment-specific parameter approaches its lower bound. By taking the limit of Eq. (S19) as  $q_n \rightarrow 0$ , we can test to see that it asymptotically approaches zero,

$$\lim_{q_n \rightarrow 0} \begin{cases} 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right], & \phi < 1 \\ 1 - \left[ \frac{\gamma(q_n, \frac{q_n}{\phi})}{\Gamma(q_n)} \right] + \left[ \frac{\gamma(q_{n+1}, \frac{q_n}{\phi})}{\Gamma(q_{n+1})} \phi \right], & \phi \geq 1 \end{cases} = \begin{cases} 1 - 1 + (0)(\phi), & \phi < 1 \\ 1 - 1 + (0)(\phi), & \phi \geq 1 \end{cases} = \begin{cases} 0, & \phi < 1 \\ 0, & \phi \geq 1 \end{cases}. \quad (\text{S26})$$

Thus, Eq. (14) satisfies all requirements for a valid parametric Budyko equation.

### S3.3 Choice of $\phi$ distribution and $q$ for Eq. (15)

To produce Eq. (15) in the main text, we use Eq. (S12) with the choice of a two-parameter distribution for  $\phi$ , the Nakagami distribution,

$$5 \quad \phi \sim \frac{2m^m}{\Gamma(m)\Omega^m} \phi^{2m-1} e^{-\frac{m}{\Omega}\phi^2}, \quad (\text{S27})$$

where the shape parameter,  $m > 0$ , and the spread parameter,  $\Omega > 0$ . The parameters  $\rho_{\phi < 1}$  and  $\bar{\Phi}_{\phi < 1}$  of Eq. (S12) are defined as follows for the Nakagami distribution,

$$\rho_{\phi < 1} = \frac{\gamma(m, \frac{m}{\Omega})}{\Gamma(m)}, \quad (\text{S28})$$

and

$$10 \quad \bar{\Phi}_{\phi < 1} = \frac{\Gamma(m)}{\gamma(m, \frac{m}{\Omega})} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \sqrt{\frac{\Omega}{m}} \frac{\gamma(m + \frac{1}{2}, \frac{m}{\Omega})}{\Gamma(m + \frac{1}{2})}. \quad (\text{S29})$$

Next, we choose an appropriate attribute of the distribution of  $\phi$  as the catchment-specific parameter,  $q$ , specifically,

$$q = m + 1. \quad (\text{S30})$$

The mean of Nakagami distribution can be expressed in terms of  $m$  and  $\Omega$  as,

$$\phi = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \sqrt{\frac{\Omega}{m}}. \quad (\text{S31})$$

15 Solving these equations for  $m$  and  $\Omega$  gives the Nakagami distribution parameters in terms of  $\phi$  and  $q$ ,

$$m = q - 1, \quad (\text{S32})$$

and

$$\Omega = \frac{\phi^2 [q-1] \Gamma(q-1)}{\Gamma(q - \frac{1}{2})}, \quad (\text{S33})$$

Substituting Eq. (S28), (S29), (S32), and (S33) into Eq. (S12) and letting  $q = q_w$  yields,

$$20 \quad \frac{\bar{E}}{\bar{P}} = 1 - \left[ \frac{\gamma\left(q_w - 1, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma\left(q_w - \frac{1}{2}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - \frac{1}{2})} \phi \right], \quad (\text{S34})$$

which is Eq. (15) of the main text.

Now, we test the Eq. (15) to determine if it has the required properties to be a valid parametric Budyko equation analogous to Eq. (5) and (6). First, Eq. (15) should approach 0 and 1 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively. We can test these conditions by taking the appropriate limits of Eq. (S34),

$$\lim_{\phi \rightarrow 0} \left[ 1 - \left[ \frac{\gamma \left( q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma \left( q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi \right] = 1 - 1 + (1)(0) = 0, \quad (\text{S35})$$

5 and

$$\lim_{\phi \rightarrow \infty} \left[ 1 - \left[ \frac{\gamma \left( q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma \left( q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi \right] = 1 - 0 + \lim_{\phi \rightarrow \infty} \left[ \frac{2 \left[ \frac{\Gamma(q_w - \frac{1}{2})}{\Gamma(q_w - 1)} \right]^{[q_w - \frac{1}{2}]} e^{-\frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}}}{[\phi]^{[2q_w - 1]} \Gamma(q_w - \frac{1}{2})} \right] = 1 + 0 = 1, \quad (\text{S36})$$

which are the required values. Second, the first derivative of Eq. (15) should approach 1 and 0 in the humid (i.e.  $\phi \rightarrow 0$ ) and arid (i.e.  $\phi \rightarrow \infty$ ) limits, respectively. We can test these conditions by taking the appropriate limits of the derivative of Eq. (S34),

$$10 \quad \frac{d\bar{E}}{d\phi} = \frac{\gamma \left( q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - \frac{1}{2})}, \quad (\text{S37})$$

which gives,

$$\lim_{\phi \rightarrow 0} \left[ \frac{\gamma \left( q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - \frac{1}{2})} \right] = 1, \quad (\text{S38})$$

and

$$\lim_{\phi \rightarrow \infty} \left[ \frac{\gamma \left( q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)} \right)}{\Gamma(q_w - \frac{1}{2})} \right] = 0, \quad (\text{S39})$$

15 which are the required values. Third, Eq. (15) should asymptotically approach the energy and water limits as the catchment-specific parameter approaches infinity. We test this condition by taking the limit of Eq. (S34) as  $q_w \rightarrow \infty$ , which gives,



$$\lim_{q_w \rightarrow \infty} \begin{cases} 1 - \left[ \frac{\gamma\left(q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma\left(q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi, & \phi < 1 \\ 1 - \left[ \frac{\gamma\left(q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma\left(q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi, & \phi \geq 1 \end{cases} = \begin{cases} 1 - 1 + (1)(\phi), & \phi < 1 \\ 1 - 0 + (0)(\phi), & \phi \geq 1 \end{cases} = \begin{cases} \phi, & \phi < 1 \\ 1, & \phi \geq 1 \end{cases}, \quad (\text{S40})$$

the water and energy limits. Finally, Eq. (15) should asymptotically approach zero as the catchment-specific parameter approaches its lower bound. By taking the limit of Eq. (S34) as  $q_w \rightarrow 1$ , we can test to see that it asymptotically approaches zero,

$$5 \lim_{q_w \rightarrow 1} \begin{cases} 1 - \left[ \frac{\gamma\left(q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma\left(q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi, & \phi < 1 \\ 1 - \left[ \frac{\gamma\left(q_w^{-1}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - 1)} \right] + \left[ \frac{\gamma\left(q_w^{-\frac{1}{2}}, \frac{\Gamma(q_w - \frac{1}{2})}{\phi^2 \Gamma(q_w - 1)}\right)}{\Gamma(q_w - \frac{1}{2})} \right] \phi, & \phi \geq 1 \end{cases} = \begin{cases} 1 - 1 + (0)(\phi), & \phi < 1 \\ 1 - 1 + (0)(\phi), & \phi \geq 1 \end{cases} = \begin{cases} 0, & \phi < 1 \\ 0, & \phi \geq 1 \end{cases}. \quad (\text{S41})$$

Thus, Eq. (15) satisfies all requirements for a valid parametric Budyko equation.

## S4 Supplemental information for interpreting $n$ and $w$ as proxy variables for the evaporative index

### S4.1 Level curve expressions of the dependence between $n$ , $w$ , and the evaporative index

10 Here we formally express the proxy nature of  $n$  or  $w$  and  $\frac{\bar{E}}{\bar{P}}$  in terms of level curves. Level curves can be expressed in the form,

$$L_c(f) = \{(x_1, x_2) | f(x_1, x_2) = c\}, \quad (\text{S42})$$

which describes the set,  $\{ \}$ , of values of the real variables  $x_1$  and  $x_2$  given a function,  $f$ , that is equal to a constant,  $c$ . Applying this general form to the parametric Budyko equations for a particular  $\phi$  yields,

$$L_\phi(f_3) = \left\{ \left( n, \frac{\bar{E}}{\bar{P}} \right) \mid f_3 \left( n, \frac{\bar{E}}{\bar{P}} \right) = \phi \right\}, \quad (\text{S43})$$

15 and

$$L_\phi(f_4) = \left\{ \left( w, \frac{\bar{E}}{\bar{P}} \right) \mid f_4 \left( w, \frac{\bar{E}}{\bar{P}} \right) = \phi \right\}, \quad (\text{S44})$$

where

$$f_3 \left( n, \frac{\bar{E}}{\bar{P}} \right) = \left[ \frac{\bar{E}}{\bar{P}} \right] [1 + (\phi)^n]^{\frac{1}{n}}, \quad (\text{S45})$$

and

$$f_4\left(w, \frac{\bar{E}}{\bar{P}}\right) = \frac{\bar{E}}{\bar{P}} - 1 + (1 + (\phi)^w)^{\frac{1}{w}}. \quad (\text{S46})$$

Eq. (S43) and (S44) directly define the dependent relationship between  $n$  and  $w$  and  $\frac{\bar{E}}{\bar{P}}$ , respectively. These level curves are plotted in Fig. S2.

5

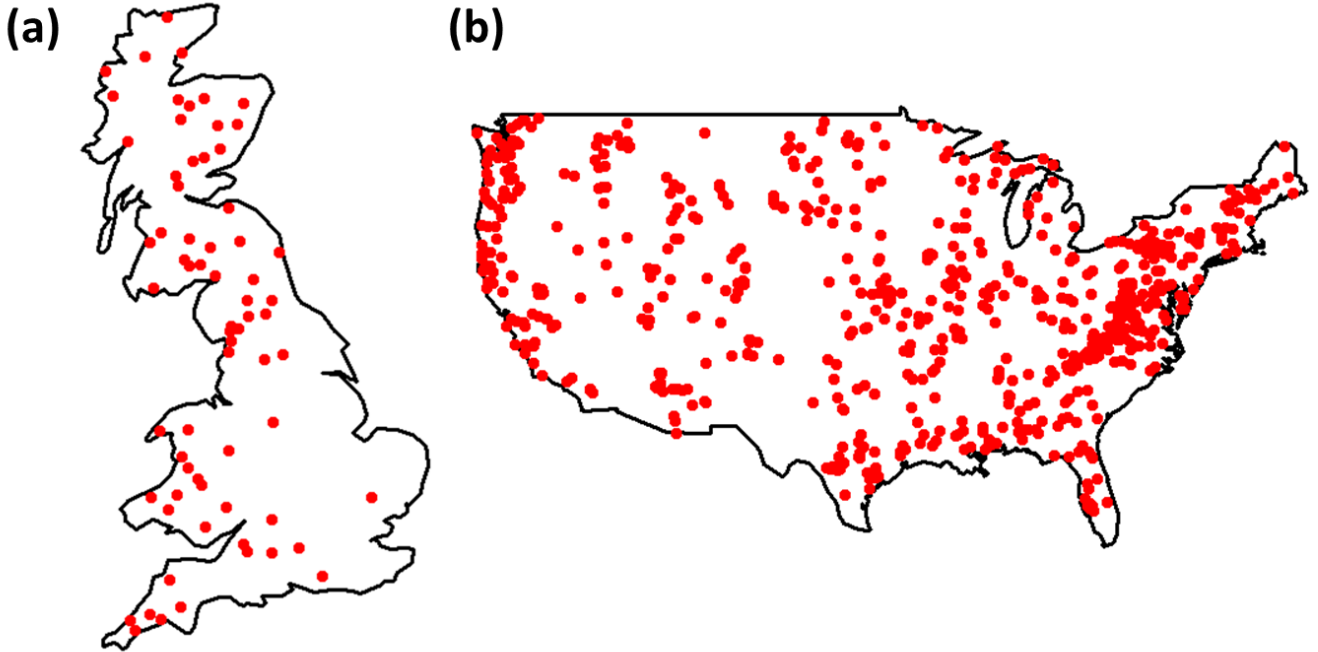
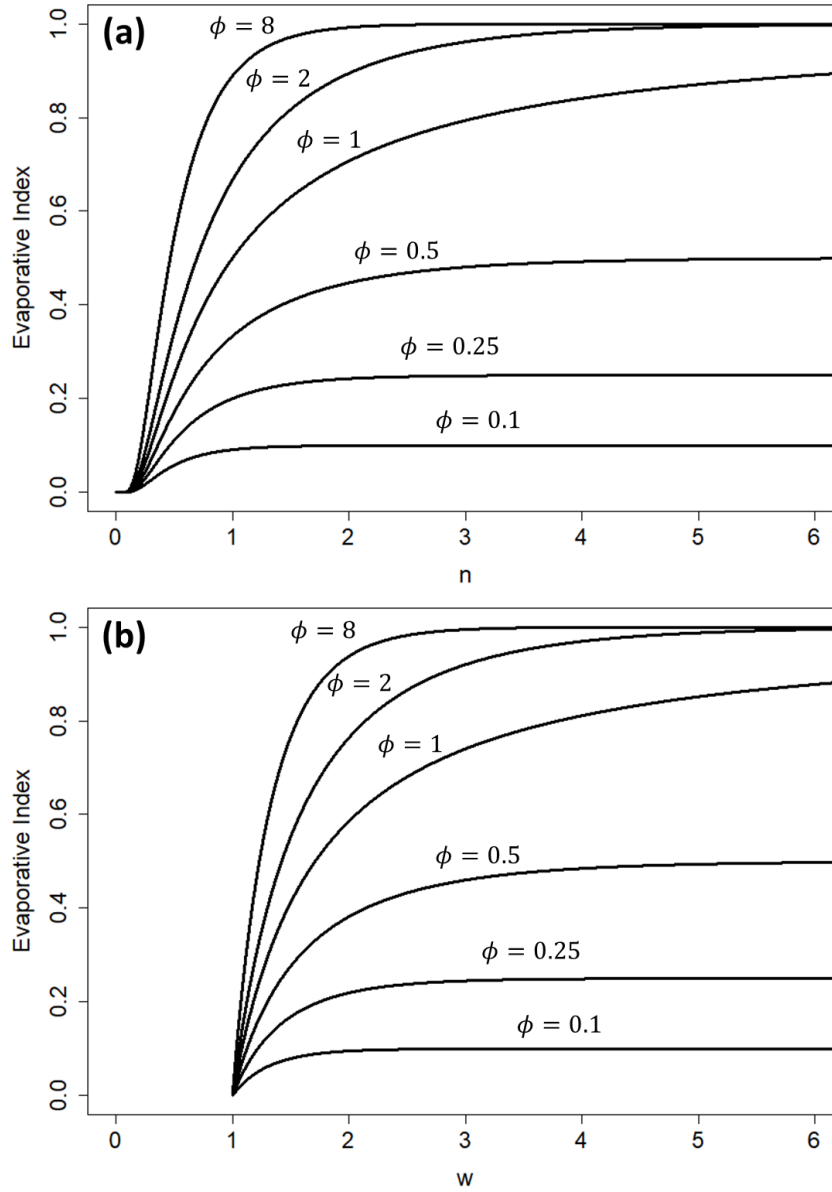


Figure S1: Reference catchments (red dots) used in the empirical test of the catchment trajectory conjecture. The reference catchments are known to have stable unchanging land use over the length of their  $Q$ ,  $P$ , and  $E_0$  records. (a) Great Britain with the locations of 68 UK reference catchments from the CAMELS-GB and UKBN2 datasets. (b) The US with the locations of the 660 reference catchments from the CAMELS dataset.

10



**Figure S2: Illustration of the catchment-specific parameters as proxy variables for the evaporative index. For a given aridity index ( $\phi$ ) the evaporative index completely determines the catchment-specific parameter ( $w$  or  $n$ ) through a one-to-one relationship. (a) Level curves described by Eq. (S43). (b) Level curves described by Eq. (S44).**

**Table S1: Existing relationships for the catchment-specific parameter in terms of biophysical features. Variables in each of the expressions are defined within the table. Complicated statistical expressions not amenable to tabulation are listed as a general function,  $F(\ )$ , of the statistically significant variables, with details available in the cited work.**

Functional Forms	Parameters	Methodology	Reference
$w = 1 + 8.852 \left(\frac{K_s}{\bar{I}_r}\right)^{-0.368} \left(\frac{S_{max}}{\bar{E}_0}\right)^{0.436} e^{-4.464 \tan \beta}$ <p style="text-align: center;">and</p> $w = 2.947 - 0.55 \left(\frac{K_s}{\bar{I}_r}\right) + 0.882 \left(\frac{S_{max}}{\bar{E}_0}\right) - 2.096 \tan \beta$	$K_s$ , saturated hydraulic conductivity $\bar{I}_r$ , mean precipitation intensity in 24 hours $S_{max}$ , plant extractable water capacity $\bar{E}_0$ , mean annual potential evapotranspiration $\tan \beta$ , average slope	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by stepwise regression analysis	(Yang et al., 2007)
$n = 5.755 \left(\frac{K_s}{\bar{I}_r}\right)^{-0.368} M^{0.292} e^{-5.428 \tan \beta}$ <p style="text-align: center;">and</p> $n = 2.721 \left(\frac{K_s}{\bar{I}_r}\right)^{-0.393} M^{-0.301} e^{4.351 \tan \beta}$	$K_s$ , saturated hydraulic conductivity $\bar{I}_r$ , mean precipitation intensity in 24 hours $M$ , vegetation coverage $\tan \beta$ , average slope	Optimization of $n$ values to measured catchment $\bar{E}$ , followed by stepwise regression analysis	(Yang et al., 2009)
$n = 0.21 \left(\frac{\kappa Z_r}{\alpha}\right) + 0.60$	$\kappa$ , fractional plant-available water holding capacity $Z_r$ , effective rooting depth $\alpha$ , mean depth per storm event	Optimization of $n$ values to a soil moisture model at single aridity index value followed by linear regression	(Donohue et al., 2012)
$w = 2.36(M) + 1.16$	$M$ , vegetation coverage	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by linear regression	(Li et al., 2013)

**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$w = F(PFC, PFE, Rr, ASD, CV_p)$	<i>PFC</i> , percentage of forest coverage	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by multivariate adaptive regression splines (MARS)	(Shao et al., 2012)
	<i>PfE</i> , number of months that peak precipitation follows peak potential evapotranspiration		
	<i>Rr</i> , relief ratio		
	<i>ASD</i> , average storm depth		
	<i>CV<sub>p</sub></i> , coefficient of variation of precipitation		
$w = (-0.09322)lat + (0.13085)CTI$ $+ (1.31697)NDVI$ $+ (0.00003)A$ $+ (-0.00018)elev$ $+ 5.05722$ and $w = (-0.01042)lat + (0.146186)CTI$ $+ (2.81063)NDVI$ $+ 0.69387$ and $w = (-0.03288)lat + (-0.09311)slp$ $+ (1.12312)NDVI$ $+ (-0.00205)long$ $+ (-0.00026)elev$ $+ 3.50412$	<i>lat</i> , absolute latitude of basin center	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by stepwise multiple linear regression (MLR)	(Xu et al., 2013)
	<i>long</i> , basin center longitude		
	<i>slp</i> , slope gradient		
	<i>CTI</i> , compound topographic index		
	<i>NDVI</i> , normalized difference vegetation index		
	<i>A</i> , drainage area		
	<i>elev</i> , elevation		

**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$w = F(lat, slp, NDVI, long, elev)$	<i>lat</i> , absolute latitude of basin center	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by stepwise multiple linear regression (MLR) to select input variables to train a neural network	(Xu et al., 2013)
	<i>long</i> , basin center longitude		
	<i>slp</i> , slope gradient		
	<i>NDVI</i> , normalized difference vegetation index		
$n = 0.7078 \left( \frac{\kappa Z_r}{\alpha} \right)^{0.5945}$	<i>elev</i> , elevation	Optimization of <i>n</i> values to a soil moisture model followed by curve fitting	(Cong et al., 2015)
	$\kappa$ , fractional plant-available water holding capacity		
	$Z_r$ , effective rooting depth		
$n = a \ln \left[ \left( \frac{\kappa Z_r}{\alpha} \right)^2 \right] + b \ln \left[ \frac{\kappa Z_r}{\alpha} \right] + c$	$\alpha$ , mean depth per storm event	Optimization of <i>n</i> values to a soil moisture model followed by curve fitting	(Yang et al., 2016; Zhang et al., 2018)
	where,		
$\begin{cases} a = -0.003 \left( \log_{10} \frac{\bar{E}_0}{\bar{P}} \right)^2 + 0.247 \log_{10} \frac{\bar{E}_0}{\bar{P}} + 0.19 \\ b = -0.022 \frac{\bar{E}_0}{\bar{P}} + 0.403 \\ c = -0.071 \left( \log_{10} \frac{\bar{E}_0}{\bar{P}} \right)^2 - 0.332 \log_{10} \frac{\bar{E}_0}{\bar{P}} + 0.761 \end{cases}$	$\kappa$ , fractional plant-available water holding capacity	Optimization of <i>n</i> values to a soil moisture model followed by curve fitting	(Yang et al., 2016; Zhang et al., 2018)
	$Z_r$ , effective rooting depth		
	$\alpha$ , mean rainfall depth		
$w = 1 + (3.683)M^{0.798}e^{-(0.246)S}$	<i>M</i> , vegetation coverage	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by least squares regression.	(Ning et al., 2017)
	<i>S</i> , climate seasonality index		

**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$w = F\left(\frac{AWC}{P}, rCMS, slope, SF\right)$	<i>AWC</i> , available soil water holding capacity	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by stepwise forward regression to build a generalized additive model	(Abatzoglou and Ficklin, 2017)
	<i>rCMS</i> , relative cumulative moisture surplus		
	<i>slope</i> , mean watershed slope		
$n = (2.100)CI + (-0.157)SIM + (-0.284)PDSI + (-1.203)SI + (-45.4)Rr + 1.299$	<i>SF</i> , fraction of annual precipitation falling as snow	Optimization of <i>n</i> values to measured catchment $\bar{E}$ , followed by multiple stepwise regression (MSR)	(Xing et al., 2018)
	<i>CI</i> , precipitation concentration index		
	<i>SIM</i> , Milly's index of seasonality		
	<i>PDSI</i> , Palmer drought severity index		
	<i>SI</i> , seasonality index of precipitation		
$n = F(M, ASD, Cv, SI, PDSI, IA, CA)$	<i>Rr</i> , relief ratio	Optimization of <i>n</i> values to measured catchment $\bar{E}$ , followed by multivariate adaptive regression splines (MARS)	(Xing et al., 2018)
	<i>M</i> , vegetation coverage		
	<i>ASD</i> , average storm depth		
	<i>Cv</i> , coefficient of variation of precipitation		
	<i>SI</i> , seasonality index of precipitation		
	<i>PDSI</i> , Palmer drought severity index		
	<i>IA</i> , effective irrigated area ratio		
<i>CA</i> , cultivated land area ratio			

**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$\omega = 1.8077 + (0.1856)CTI - (1.4551)NDVI + (0.4489)RFL + (0.5542)SI$	<i>CTI</i> , compound topographic index	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by multiple linear regression	(Bai et al., 2019)
	<i>NDVI</i> , normalized difference vegetation index		
	<i>RFL</i> , the ratio of farmland area to total basin area		
$w = F(CTI, NDVI, RFL, SI)$	<i>SI</i> , seasonality index of precipitation	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by calibration of an artificial neural network	(Bai et al., 2019)
	<i>CTI</i> , compound topographic index		
	<i>NDVI</i> , normalized difference vegetation index		
	<i>RFL</i> , the ratio of farmland area to total basin area		
$w = 1 + (3.67)M^{0.27}e^{-(0.21)SAI}$	<i>SI</i> , seasonality index of precipitation	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by a stepwise regression method.	(Ning et al., 2019)
	<i>M</i> , vegetation coverage		
$w = 1 + (25.03)M^{0.43}f_s^{0.71}$	<i>SAI</i> , climate seasonality and asynchrony index	Optimization of <i>w</i> values to measured catchment $\bar{E}$ , followed by a stepwise regression method and parameter reduction due to collinearity.	(Ning et al., 2019)
	<i>M</i> , vegetation coverage		
	<i>f<sub>s</sub></i> , fraction of precipitation falling as snow		



**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$w \text{ or } n = F(X_1(t), X_2(t), X_3(t))$	$X_1(t)$ , time varying per capita gross regional product $X_2(t)$ , time varying mean annual temperature $X_3(t)$ , time varying mean annual precipitation	Optimization of $w/n$ values to measured catchment $\bar{E}$ , followed by a hierarchical Bayesian calibration of constant and time varying regression models for the catchment-specific parameter	(Zhang et al., 2019)
$w = -(2.23) \ln[S_t] - (4.34) \ln[NDVI_t]$ $- (9.77) \ln[T_{max,t}]$ $+ (0.82) \ln[P_t] - 27.55$	$S_t$ , 11-year average of annual sunshine duration $NDVI_t$ , 11-year average of annual normalized difference vegetation index $T_{max,t}$ , 11-year average of annual maximum temperature $P_t$ , 11-year average of annual precipitation	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by a stepwise regression method.	(Li et al., 2020b)
$w = -2.328 - (8.220)URBN + (7.429) \ln[IA]$ $+ (1.211)e^{CA} + (5.324)e^T$ $- (2.069)e^{PET}$	$URBN$ constructed land area rate $IA$ , irrigated area rate $CA$ , cultivated area rate $T$ , temperature $PET$ , potential evapotranspiration	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by a stepwise regression method. This catchment-specific parameter for the “Budyko-like” equation of (Zhang et al., 2001)	(Li et al., 2020a)
$\Delta w = (0.63)\Delta F - 0.03$	$\Delta F$ , change in forest cover	Optimization of $\Delta w$ values to measured catchment $\bar{\Delta E}$ , followed by linear regression.	(Ning et al., 2020a)

**Table S1: Continued.**

Functional Forms	Parameters	Methodology	Reference
$w = 1 + (4.49)M^{0.73}e^{-(0.33)SAI}$	<p><math>M</math>, vegetation coverage</p> <p><math>SAI</math>, climate seasonality and asynchrony index</p>	Modification of an expression from (Ning et al., 2017).	(Ning et al., 2020b)
	<p><math>\frac{E}{P}</math>, evaporative index</p> <p><math>\frac{E_p}{P}</math>, aridity index</p> <p><math>SI</math>, seasonality index of precipitation</p> <p><math>SM</math>, Soil moisture of the basin</p> <p><math>BF</math>, baseflow of the basin</p> <p><math>NDVI</math>, normalized difference vegetation index</p>		
<p><math>w =</math></p> <p><math>F\left(\frac{E}{P}, \frac{E_p}{P}, SI, BF, SM, NDVI, AO, PDO, Ni\tilde{no} 3.4, Sunspots, EIA\right)</math></p>	<p><math>AO</math>, Arctic Oscillation</p> <p><math>PDO</math>, Pacific Decadal Oscillation</p> <p><math>Ni\tilde{no} 3.4</math>, The area averaged SST from 5°S to 5°N and 170°–120°W calculated from the HadISST1 data</p> <p><math>Sunspots</math>, Solar activity</p> <p><math>EIA</math>, Effective irrigated area in the study area</p>	Optimization of $w$ values to measured catchment $\bar{E}$ , followed by a correlation analysis and partial correlation analysis.	(Zhao et al., 2020)