



The Heterogeneous Discrete Generalized Nash Model for Flood Routing

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Abstract. The topographic heterogeneity of the rivers has great effects on the river flood routing. The discrete generalized Nash model (DGNM), developed on the basis of the Nash's instantaneous unit hydrograph (IUH), is a lumped model that can't reflect the spatial heterogeneity of the river topography. The heterogeneous DGNM (HDGNM) with a consideration of such heterogeneity has been developed by the conceptual interpretation of the DGNM. Two compositions of the downstream outflow generated by the recession of the old water stored in the river channel and the discharge of the new water from upstream inflow were deduced respectively with the help of the heterogeneous IUH and the corresponding heterogeneous S curve. The HDGNM is finally expressed as a linear combination of the inflows and outflows, whose weight coefficients are calculated by the heterogeneous S curve. The HDGNM expands the application scope, and becomes more applicable, especially in river reaches where the river slopes and cross-sections change greatly. The middle Hanjiang River was selected as a case study to test the model performance. It is suggested that the HDGNM performs better than the DGNM, with higher model efficiency and smaller relative error in the simulated flood hydrographs.

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1 Introduction

In rainfall-runoff modelling, the instantaneous unit hydrograph (IUH) proposed by Nash (Nash, 1957) is one of the most widely used methods for overland flow routing. Under the assumption that 25 watershed was represented as a cascade of equal linear reservoirs, IUH was obtained in a form of gamma distribution with two parameters – n , the number of linear reservoirs, and K , reservoir storage coefficient. The linear cascade concept has greatly promoted the development of the flow routing theory. Many linear cascade – based models have been developed since then. However, as a lumped model, IUH cannot reflect the spatial heterogeneity of rainfall and landforms. Great efforts have been 30 made to make IUH be semi-distributed or distributed mainly by two approaches. The first approach has been mostly performed by replacing the equal reservoirs in the IUH with unequal ones. Dooge (1959) conceptualized the watershed as a combination of unequal linear reservoirs and linear channels, and developed a general theory for unit hydrograph. Singh (1964) derived the IUH using a nonlinear model considering the overland and channel flow components separately, in which two unequal linear 35 reservoirs with different storage coefficients were used. To solve the flow routing in urban areas, Diskin et al. (1978) proposed an urban parallel cascade IUH model by representing the basin system as the combination of two parallel branches having a series of equal linear reservoirs. Bhunya et al. (2005) developed a hybrid model by splitting the single linear reservoir into two serially connected reservoirs of unequal storage coefficients (one hybrid unit), and obtained the analytical expression of 40 the model for two hybrid units in series. Later, to consider the translation time, Singh et al. (2007) extended this hybrid model by inserting a linear channel between each hybrid unit. Bhunya et al. (2008) formulated a rainfall-runoff model incorporating a variable storage coefficient instead in the two-



reservoir Nash cascade model. Li et al. (2008) derived the IUH with different K values for each
reservoir using the Laplace transform and developed a general rule for the equation of the IUH of any
45 order. The other approach is to divide the watershed into a number of subwatersheds to consider the
nonuniformly distributed rainfall. For example, based on the structure of a stream network, Wang and
Chen (1996) divided the watershed into a number of subwatersheds and obtained the outflow
hydrograph of each subwatershed based on the concept of linear cascade reservoirs. This linear,
spatially distributed model can be capable of predicting runoff from non-uniformly distributed rainfall
50 and geographical conditions over an entire watershed. Similarly, Wan et al. (2016) divided the
watershed into subareas by isochrones, and established an independent linear reservoir-channel
cascade model in each subarea. Finally, the generalized concentration curve that can be applied to large
heterogeneous watersheds was derived. All of these modifications of IUH have made a certain
improvement in the rainfall-runoff modeling.

55 As a general method of flow routing, Nash's IUH is also applicable to river flow routing, which
has been done independently by Kalinin and Milyukov (1958), also known as Characteristic Reach
method. Nash's IUH can also be obtained by solving the n th order differential equation of a linear
system with the zero initial conditions (Chow, 1988). Zero initial conditions represent that the linear
reservoirs in the Nash cascade model are empty at initial time, or equivalently the initial river storages
60 are empty when IUH is applied in river flow routing, which does not match the fact. To consider the
influence of the initial state, Szollosi-Nagy (1982) formulated a state-space description of the Nash
cascade model in a matrix form, whereby the initial storage of the river system should be estimated
separately via observability analysis (Szollosi-Nagy, 1987). Szilagyi (2003) then extended this model



to a sample-data system framework and made some modifications to make it more applicable (Szilagyi,
65 2006; Szilagyi and Laurinyecz, 2014). Recently, Yan et al. (2015) exactly solved the n th order
differential equation of the Nash cascade model with the same non-zero initial condition, and obtained
the generalized Nash model (GNM) with a simpler expression, in which the initial state was directly
included and should not be estimated separately anymore. To make the GNM be applied easily to the
sample-data system, Yan et al. (2019) further discretized its analytical expression by introducing a
70 variable S_n -curve, and obtained the discrete generalized Nash model (DGNM). The DGNM expresses
the outflow as a linear combination of the old water stored in the river reach and new water from the
upstream inflow. The DGNM is based on the lumped IUH that it cannot reflect the spatial
heterogeneity of the river topography too. So the DGNM is less applicable for those the topography
changes large along the river. In this case, the unequal reservoir concept should be used instead.
75 However, under the non-zero initial conditions, the solving of high order differential equation of the
Nash cascade model with unequal storage parameters will become very difficult, which makes the
generalization of the Nash model impossible by directly solving the differential equation. A new way
is proposed in this paper to obtain the heterogeneous DGNM (HDGNM) through the conceptual
interpretation and mathematical derivation of the DGNM.

80 2 Conceptual interpretation of the DGNM

According to Yan et al. (2019), the calculation formula of the DGNM is

$$O_{t+1} = \sum_{j=0}^{n-1} \sum_{i=0}^j \frac{(-1)^i}{j!} C_j^i (1 - S_{n-j}) O_{t-i} + S_n I_t + \left(1 - \frac{K}{\Delta t} \sum_{i=1}^n S_i \right) \Delta I_{t+1} \quad (1)$$



Where C_j^i is the combination formula; $O_{t \pm i}$ represents the downstream outflow at time $t \pm i\Delta t$; Δt is the time interval; I_t represents the upstream inflow at time t ; ΔI_{t+1} represents the inflow increment
85 during the time interval $[t, t + \Delta t]$; n and K are model parameters; S_i can be computed by

$$S_i = 1 - e^{-\frac{\Delta t}{K}} \sum_{j=0}^{i-1} \frac{1}{j!} \left(\frac{\Delta t}{K} \right)^j \quad (2)$$

According to the definition of the S curve, S_i represents the outflow generated by the unit continuous inflow after the routing of a series of i reservoirs at the end of the time. The calculation formula of the DGNM shows that the downstream outflow is composed of three items. The first item
90 is the recession flow of the current water storage capacity in the channel, which is the superposition of the flow generated by the current storage of each reservoir routed by the subsequent reservoirs. The second term is the recession flow generated by the current inflow I_t routed by river channel, or equivalently by a series of n cascade linear reservoirs. According to the definition of S_i as well as the storage-discharge relation of the linear reservoir, KS_i represents the water stored in each reservoir for
95 a unit continuous inflow, and $\sum_{i=1}^n K S_i / \Delta t$ represents the ratio of water stored in the channel during the period Δt . Then, $1 - \sum_{i=1}^n K S_i / \Delta t$ represents the ratio of water discharges from the channel. So the third term is the outflow generated by the inflow increment during the time interval $[t, t + \Delta t]$ after the channel routing. In summary, the downstream outflow is generated by the old water stored in the river channel and the new water from upstream inflow. Part of the new water flows out of the
100 downstream section and becomes one part of the outflow, the other part remains in the river channel to supplement the old water. The old water recedes and becomes the other component of the outflow. In such circulation, the outflow process of the downstream section is formed. Through the conceptual interpretation of the DGNM, the downstream outflow is physically generated by the old water stored



in the river reach and new water from the upstream inflow, and formally expressed as a linear
 105 combination of the inflows and outflows, whose weight coefficients are calculated by the S curve that
 gives another way to deduce the HDGNM by introducing the heterogeneous S curve.

3 Heterogeneous S curve

The routing storage capacity of the basin is affected by geographical features and has spatial
 heterogeneity. The storage routing effect of the basin is equated to a cascade reservoir routing in Nash's
 110 IUH, and each reservoir has the same storage coefficient. This generalization has certain rationality
 for basins with homogeneous topography variation. But for the basins with large topographic changes,
 the spatial difference in storage routing effect will be more significantly affected by topography and
 geomorphology. To consider such spatial heterogeneity in storage routing effect, Dooge (1959)
 conceptualized the watershed as a combination of unequal linear reservoirs and linear channels, and
 115 developed a general theory for unit hydrograph. Li et al. (2008) further deduced the IUH with different
 storage parameters, here we call it heterogeneous IUH (HIUH) to distinguish with Nash' IUH:

$$u_n(t) = \sum_{j=1}^n \frac{K_j^{n-2}}{\prod_{i=1, i \neq j}^n (K_j - K_i)} e^{-\frac{t}{K_j}} \quad (3)$$

where $K_i (i = 1, \dots, n)$ is the storage parameter of the i -th reservoir (the numbers here are sorted in
 the forward direction, that is, the most upstream reservoir is the number 1, and the most downstream
 120 reservoir is the number n). Correspondingly, the heterogeneous S curve formed by HIUH is (Li et al.,
 2008)

$$S_n(t) = \int_0^t u_n(t) dt = \sum_{j=1}^n \frac{K_j^{n-1}}{\prod_{i=1, i \neq j}^n (K_j - K_i)} \left(1 - e^{-\frac{t}{K_j}} \right) \quad (4)$$



where $S_n(t)$ represents the outflow of the n th reservoir yielded by a continuous unit upstream inflow.

If further define the storage curve, we obtain

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$$R_n(t) = \int_t^{+\infty} u_n(t) dt = \sum_{j=1}^n \frac{K_j^{n-1} e^{-\frac{t}{K_j}}}{\prod_{i=1, i \neq j}^n (K_j - K_i)} \quad (5)$$

where $R_n = 1 - S_n(t)$, represents the detention storage of the n th reservoir yielded by a continuous unit upstream inflow.

The HIUH is a more accurate generalization of the watershed storage routing, and is a theoretical expansion of the Nash's IUH. With consideration of the spatial heterogeneity in the storage routing, HIUH is especially applicable to the basin with large topographic changes. The DGNM is developed on the basis of the Nash's IUH, which leads to its theoretical limitations when applied to the river reach with large changes of cross-sections and slopes. The introduction of HIUH can reflect the difference of flood routing in each sub-river, thus can improve the flood simulation precision in the whole river reach theoretically.

135 **4 Derivation of the heterogeneous DGNM**

The conceptual interpretation of the DGNM shows that the downstream outflow is generated by the old water stored in the channel and the new water from upstream inflow, denoted by O^{old} and O^{new} respectively, we have

$$O_{t+1}^{old} = \sum_{j=0}^{n-1} \sum_{i=0}^j \frac{(-1)^i}{j!} C_j^i (1 - S_{n-j}) O_{t-i} \quad (6)$$

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$$O_{t+1}^{new} = S_n I_t + \left(1 - \frac{K}{\Delta t} \sum_{i=1}^n S_i \right) \Delta I_{t+1} \quad (7)$$



For the linear reservoir system with unequal storage parameters, S_i represents the outflow of the i th reservoir yielded by a continuous unit upstream inflow, then $K_i S_i$ represents the water stored in each reservoir for a unit continuous inflow, and $\sum_{i=1}^n K_i S_i / \Delta t$ represents the ratio of water stored in the channel during the period Δt , based on the former conceptual interpretation of the O^{new} . Hence, 145 for the linear reservoir system with unequal storage parameters, the outflow generated by "new water" can be obtained by replacing the storage parameter K and S curve in Eq. (7) with variable K_i and heterogeneous S curve, respectively, i. e.

$$O_{t+1}^{\text{new}} = S_n I_t + \left(1 - \frac{1}{\Delta t} \sum_{i=1}^n K_i S_i \right) \Delta I_{t+1} \quad (8)$$

Therefore, on the basis of the conceptual interpretation of the DGNM, the outflow O^{new} formed 150 by the new water can be deduced directly. But it seems impossible to obtain the outflow O^{old} by directly using the heterogeneous S curve instead of in Eq. (7) due to the coefficient of S_{n-j} is also varying with it. For the sake of simplicity, we assume that the most downstream reservoir is numbered 1, and the most upstream reservoir is numbered n , that is to say, the n reservoirs are reversely numbered. Then the storage routing equation of the j -th reservoir can be obtained from the water balance equation:

$$155 \quad K_j \frac{dO_j(t)}{dt} = O_{j+1}(t) - O_j(t) \quad (9)$$

It can be known from Eq.(9) that the outflow of each reservoir at the current time is as follows:

$$O_1(t) = O(t)$$

$$O_2(t) = O(t) + K_1 O'(t)$$

$$O_3(t) = O(t) + (K_1 + K_2) O'(t) + K_1 K_2 O''(t)$$

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$$O_j(t) = O(t) + \sum_{p=1}^{j-1} \prod_{r_p > \dots > r_1=1}^{j-1} (K_{r_1} \dots K_{r_p}) O^{(p)}(t) \quad (10)$$

Based on the physical interpretation of the GNM (Yan et al., 2015), the recession flow of the current water storage in river channel is the superposition of the recession flow generated by the current water storage in each reservoir. According to the conception of linear reservoir, the current water storage of the j -th reservoir is $K_j O_j(t)$, which can be treated as an instantaneous inflow into each reservoir, then the outflow at the end of the period generated by that is $K_j O_j(t) u_j(\Delta t)$. Based on the principle of superposition, the outflow at the end of the period formed by the current water storage of all reservoirs is

$$\begin{aligned} O_{t+1}^{old} &= \sum_{j=1}^n K_j O_j(t) u_j(\Delta t) \\ &= \sum_{j=1}^n K_j u_j(\Delta t) \left[O(t) + \sum_{p=1}^{j-1} \prod_{r_p > \dots > r_1=1}^{j-1} (K_{r_1} \dots K_{r_p}) O^{(p)}(t) \right] \end{aligned} \quad (11)$$

The formula shows that the recession process can finally be expressed as a linear combination of 0~(n-1) derivatives of the current time $O(t)$, which is

$$O_{t+1}^{old} = \sum_{p=0}^{n-1} A_p O_t^{(p)} \quad (12)$$

where $A_p (p = 0, \dots, n-1)$ is the coefficient of p -th order derivative of $O(t)$, then we have (detailed derivation is provided in Appendix)

$$A_p = \begin{cases} 1 - S_n, & p = 0 \\ \sum_{r_p > \dots > r_1=1}^{n-1} (K_{r_1} \dots K_{r_p}) (S_{r_p} - S_n), & p > 0 \end{cases} \quad (13)$$

A_p is derived in the case of reverse numbering, and its calculation formula shows that K_i has symmetry. Therefore, in the case of forward numbering, or equivalently, let K_1, \dots, K_n replace



K_n, \dots, K_1 respectively, the formula for calculating A_p is still the same as Eq.(15), thus ensuring that
 180 O^{old} and O^{new} are calculated under the same numbering system. To further discretize the term O^{old}
 in Eq. (12), the derivatives of $O(t)$ are approximated by the following backward finite difference
 method.

$$O_t^{(p)} = \frac{1}{\Delta t^p} \sum_{i=0}^p (-1)^i C_p^i O_{t-i} \quad (14)$$

Substituting equation (13) and (14) into equation (12), we obtain

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$$O_{t+1}^{\text{old}} = \sum_{p=0}^{n-1} \sum_{i=0}^p \frac{(-1)^i}{\Delta t^p} C_p^i A_p O_{t-i} \quad (15)$$

According to the conceptual interpretation of the DGNM, the downstream outflow is jointly
 produced by the old water stored in the river channel and the new water from upstream inflow, then
 we have

$$O_{t+1} = \sum_{j=0}^{n-1} \sum_{i=0}^j \frac{(-1)^i}{\Delta t^j} C_j^i A_j O_{t-i} + S_n I_t + \left(1 - \frac{1}{\Delta t} \sum_{i=1}^n K_i S_i \right) \Delta I_{t+1} \quad (16)$$

190 Eq. (16) is the calculation formula of HDGNM, and is also the discrete solution of linear cascade
 model with unequal reservoirs under non-zero initial conditions. Like the HIUH, with a consideration
 of the spatial heterogeneity in the storage routing, HDGNM should be more applicable to the river
 reach with large changes of cross-sections and slopes.

5 Case study

195 To test the applicability of the HDGNM, the river reach between gauging stations Huangjiagang
 and Xiangyang in the Hanjiang River of China is selected as a case study. Hanjiang River is one of the
 most important tributaries of the Yangtze River of China. The Danjiangkou reservoir, located on the



upper Hanjiang River, is the water source of the middle route of China's south-to-north water transfer project. Since the opening of the first phase of the middle route on Dec. 12, 2014, the Danjiangkou reservoir has supplied a total of 25.5 billion cubic meters of water to drier north areas in China, thereby benefiting 58 million people in Beijing, Tianjin, Hebei province, and Henan province. The studied river reach with a length of 105 kms is located in the middle Hanjiang River, where the Huangjiagang hydrological station is located at 6 kms downstream of the Danjiangkou dam site and serves as the outflow control station of the Danjiangkou reservoir. The interval basin area between Huangjiagang and Xiangyang is 8044 km². The sketch map of the middle Hanjiang River and the studied river reach are shown in Fig. 1. The studied river reach is located in the hilly and plain areas, where hills, terraces, artificial narrows and wide valleys distribute alternatively, and showing obvious lotus root node shape on the plane. The main channels in wide sections have large swings and many beaches, but become single in narrow sections, which makes a large change of the shape in the sections along the river reach, as shown in Fig. 1. The mean slopes of sub-reaches Huangjiagang - Guanghua, Guanghua - Taipingdian, Taipingdian - Niushou and Niushou - Xiangyang are 0.000176, 0.000276, 0.000221 and 0.000214 (Gong, 1982), respectively. It is indicated that the channel slope of the studied river reach changes largely, especially from sub-reach 1 to sub-reach 2. In short, the topography of the studied river reach varies greatly. For such significant spatial heterogeneity, the proposed HDGNM should be more applicable as interpreted above.

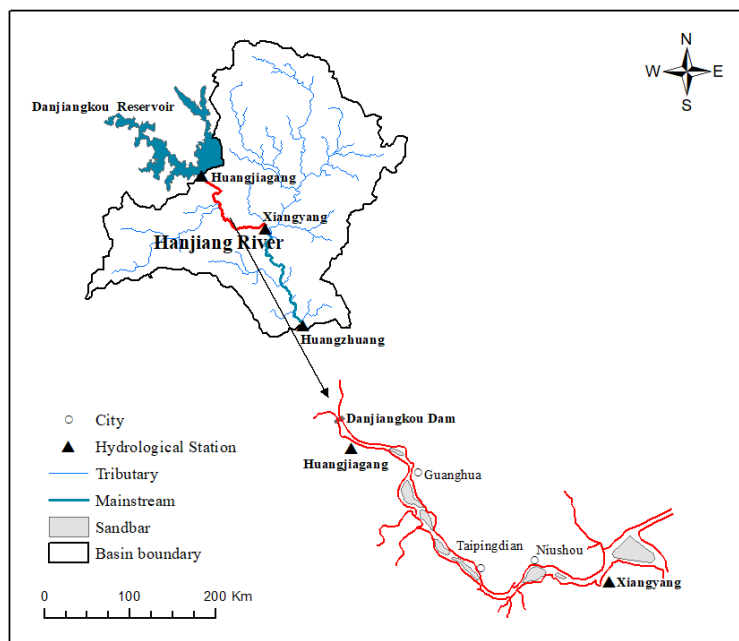


Fig.1 The sketch map of the middle Hanjiang River and the studied river reach

According to the flood data of the Huangjiagang and Xiangyang hydrological stations from the year 1974 to 2011, 10 floods with a low proportion of the lateral inflows (time interval $\Delta t = 3h$) were selected, of which 8 floods were used for model calibration, and the other 2 floods were used for validation. In order to demonstrate the simulation effect and test the forecast capability of the HDGNM, the DGNM was selected for comparison. For application, the model parameters in these two models should be calibrated by using the observed flood data. The SCE-UA algorithm, with the advantages of robust and reliable performance, global search capability, has become a commonly used optimization method for hydrological model parameters (Duan et al., 1994), and hence was used to obtain the optimized parameters of these two models. Take the root mean square error of the observed 8 floods from the year 1974 to 2005 as the objective function, and run the SCE-UA algorithm to minimize this objection function, we obtain the optimized parameters of $n = 3$, $K = 3.51h$ for the DGNM and



$n = 3$, $K_1 = 1.58\text{h}$, $K_2 = 8.80\text{h}$, $K_3 = 1.59\text{h}$ for the HDGNM. The parameter K reflects the
 230 difference in the storage capacity of linear reservoirs. The three linear reservoirs of the DGNM are
 equivalent, in essence, it is a homogenization of the topographical differences of the sub-reaches.
 While two of the three linear reservoirs of the HDGNM are approximately the same, and the other is
 quite different from the two. Therefore, it can more objectively reflect the influence of topographical
 differences on the storage routing of the river channel. Theoretically, it can improve the accuracy of
 235 the flood routing in river channel. The performances of these two models were assessed by the
 following two commonly used statistics (Wu et al. 2012):

(1) Relative error (RE) of peak discharge

$$RE = \frac{|O_{p,est} - O_{p,obs}|}{O_{p,obs}} \times 100\% \quad (17)$$

in which $O_{p,est}$ and $O_{p,obs}$ are the estimated and observed peak discharge, respectively.

240 (2) Nash–Sutcliffe efficiency coefficient (E_{NS}).

$$E_{NS} = 1 - \frac{\sum_{t=1}^T (O_{t,est} - O_{t,obs})^2}{\sum_{t=1}^T (O_{t,obs} - \bar{O}_t)^2} \quad (18)$$

in which $O_{t,est}$ and $O_{t,obs}$ are estimated and observed discharge at time t , respectively. \bar{O}_t represents
 the mean of observed discharge.

Table 1 The accuracy evaluation results of DGNM and HDGNM

Period	Data	DGNM		HDGNM	
		$RE(\%)$	E_{NS}	$RE(\%)$	E_{NS}
Calibration	October 1974	0.91	0.9930	4.27	0.9861
	September 1975	3.69	0.9918	2.87	0.9950
	July 1980	1.62	0.9043	1.08	0.9284
	August 1981	5.26	0.9699	3.47	0.9747
	September 1981	7.82	0.9655	4.45	0.9806



	September 1984	0.99	0.9774	1.78	0.9860
	September 2003	2.54	0.9655	0.77	0.9715
	October 2005	8.72	0.9845	0.63	0.9963
	Average	3.94	0.9690	2.42	0.9773
	July 2007	0.62	0.9707	0.66	0.9775
Validation	September 2011	2.48	0.9889	0.32	0.9928
	Average	1.55	0.9798	0.49	0.9852

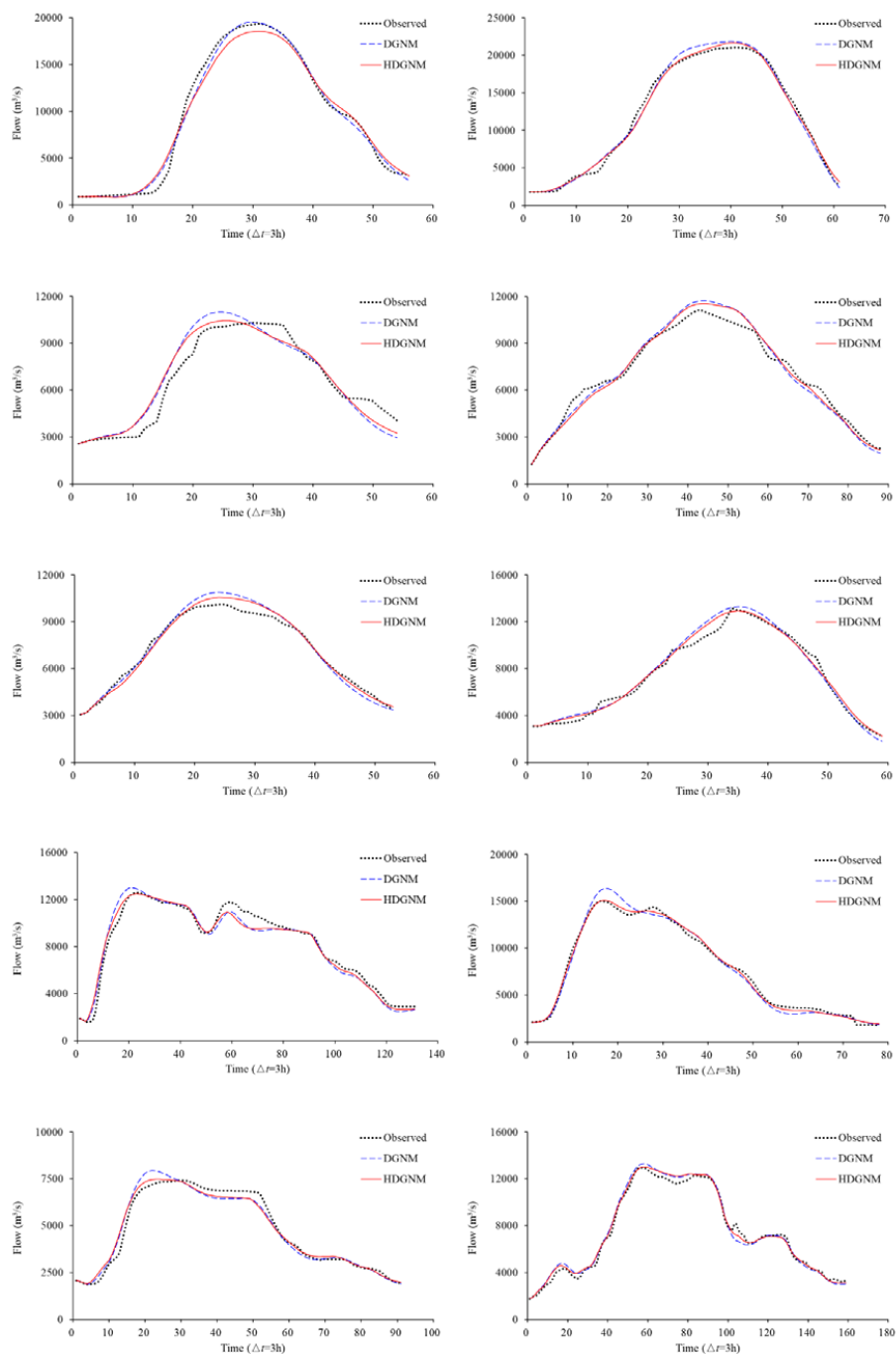
245 To further test the validity of the parameter estimations from calibration, the verification experiment was also conducted. The other 2 observed floods in the year 2007 and 2011 were adopted to verify the calibration results. The accuracy evaluation results of these two models in calibration and validation periods were both shown in Table 1. In calibration period, the average values of RE and E_{NS} were 3.94% and 0.9690 for DGNM, respectively. Compared with the DGNM, the HDGNM has made

250 some improvements in the simulation. The average value of RE has reduced to 2.42% and that of E_{NS} has increased to 0.9773. The similar improvements can be found in the validation period with values from 1.55% to 0.49% for RE , and from 0.9798 to 0.9852 for E_{NS} , respectively. Comparison of the observed and simulated hydrographs for the selected 10 floods was shown in Fig. 2. The HDGNM with a consideration of the topographical heterogeneity of the river reach, makes the simulated

255 hydrographs much closer to the measured flood hydrographs, especially near the flood peaks. Except for the October 1974 flood event, the simulations of other floods have different improvements. The simulation results show that, compared with the DGNM, the spatial difference of the storage parameter is considered, thus the HDGNM is more applicable, especially in the river reaches where the river slopes and cross - sections change greatly.



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Fig. 2 Comparison of the observed and simulated hydrographs for the selected 10 floods



6 Conclusions

The heterogeneous DGNM for flood routing was deduced indirectly by conceptual interpretation of the DGNM. It is suggested that the downstream outflow is generated by the recession of the old water stored in river channel and the discharge of the new water from upstream inflow, and can formally expressed as a linear combination of the inflows and outflows, whose weight coefficients are calculated by the *S* curve. When such old water and new water are routed by a series of unequal reservoirs, the DGNM becomes to HDGNM. Hence, these two compositions of the outflow were deduced respectively. The discharging part produced by the new water can be easily deduced with the help of the heterogeneous *S* curve. The recession part produced by the old water is obtained by the superposition of the recession process for each linear reservoir, which can be calculated by the impulse response of the current stored water with the help of HIUH. At last, the HDGNM is expressed as a linear combination of the inflows and outflows, whose weight coefficients are calculated by the heterogeneous *S* curve.

The proposed HDGNM was applied to a reach of the middle Hanjiang River with large changes of river slopes and cross-sections. A different linear reservoir with a much larger storage coefficient was detected from the conceptualized reservoirs in the studied river reach. Considered of the topographical heterogeneity of the river reach, the HDGNM performs better than the DGNM, with higher model efficiency and smaller relative error in the simulated flood hydrographs. The HDGNM enriches the existing generalized Nash flow routing theory and becomes more applicable, especially in river reaches where the river slopes and cross-sections change greatly. The river storage is conceptually as a series of unequal linear reservoirs, thus the HDGNM may have the potential for



semi-distributed modelling, e.g. river flow routing with multiple tributaries inflows, which will be further studied in the future.

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Data Availability

The data that support the findings of this study are available from the corresponding author upon
295 reasonable request.

Author contributions

Baowei Yan contributed to the development of the model and revised the manuscript. Huining Jiang wrote the first draft of the manuscript. Zhengkun Li collected and preprocessed the data. Jun Zhang and Wenfa Yang provided recommendations for the data analysis, participated in discussions
300 about the results.

Competing interests

The authors declare that they have no competing interests.

Appendix: Derivation of the Coefficient A_p

In the derivation of the coefficient A_p , the following identity holds for any integer $m \in (1, n]$ and
305 any time t

$$K_m u_m(t) = R_m(t) - R_{m-1}(t) \quad (\text{A.1})$$

The proof is as follows



$$\begin{aligned}
 R_m(t) - R_{m-1}(t) &= \sum_{j=1}^m \frac{K_j^{m-1} e^{-\frac{t}{K_j}}}{\prod_{i=1, i \neq j}^m (K_j - K_i)} - \sum_{j=1}^{m-1} \frac{K_j^{m-2} e^{-\frac{t}{K_j}}}{\prod_{i=1, i \neq j}^{m-1} (K_j - K_i)} \\
 &= \frac{K_1^{m-1} e^{-\frac{t}{K_1}}}{(K_1 - K_2) \cdots (K_1 - K_m)} + \cdots + \frac{K_{m-1}^{m-1} e^{-\frac{t}{K_{m-1}}}}{(K_{m-1} - K_1) \cdots (K_{m-1} - K_m)} + \frac{K_m^{m-1} e^{-\frac{t}{K_m}}}{(K_m - K_1) \cdots (K_m - K_{m-1})} \\
 &\quad - \frac{K_1^{m-2} e^{-\frac{t}{K_1}}}{(K_1 - K_2) \cdots (K_1 - K_{m-1})} - \cdots - \frac{K_{m-1}^{m-2} e^{-\frac{t}{K_{m-1}}}}{(K_{m-1} - K_1) \cdots (K_{m-1} - K_{m-2})} \\
 &= \frac{K_m K_1^{m-2} e^{-\frac{t}{K_1}}}{(K_1 - K_2) \cdots (K_1 - K_m)} + \cdots + \frac{K_m K_{m-1}^{m-2} e^{-\frac{t}{K_{m-1}}}}{(K_{m-1} - K_1) \cdots (K_{m-1} - K_m)} + \frac{K_m K_m^{m-2} e^{-\frac{t}{K_m}}}{(K_m - K_2) \cdots (K_m - K_{m-1})} \\
 &= K_m \sum_{j=1}^m \frac{K_j^{m-2} e^{-\frac{t}{K_j}}}{\prod_{i=1, i \neq j}^m (K_j - K_i)} \\
 &= K_m u_m(t) \tag{A.2}
 \end{aligned}$$

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According to Eq. (11) and Eq. (A1), the coefficient of $O(t)$ can be calculated as follows

$$A_0 = \sum_{j=1}^n K_j u_j(\Delta t) = K_1 u_1 + R_2 - R_1 + \cdots + R_n - R_{n-1} = R_n \tag{A.3}$$

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where u_j and R_j denote $u_j(\Delta t)$ and $R_j(\Delta t)$, respectively. Then the coefficient of first order derivative of $O(t)$ can be derived as

$$\begin{aligned}
 A_1 &= \sum_{j=2}^n K_j u_j(\Delta t) \sum_{r=1}^{j-1} K_r \\
 &= K_1 (R_2 - R_1) + (K_1 + K_2)(R_3 - R_2) + \cdots + \sum_{r=1}^{n-1} K_r (R_n - R_{n-1}) \\
 &= \sum_{r=1}^{n-1} K_r (R_n - R_r), \tag{A.4}
 \end{aligned}$$

320

and the coefficient of second order derivative of $O(t)$ can be derived as



$$\begin{aligned}
 A_2 &= \sum_{j=3}^n K_j u_j (\Delta t) \sum_{r_2 > r_1 = 1}^{j-1} K_{r_1} K_{r_2} \\
 &= K_1 K_2 (R_3 - R_2) + (K_1 K_2 + K_1 K_3 + K_2 K_3) (R_4 - R_3) + \dots + \sum_{r_2 > r_1 = 1}^{n-1} K_{r_1} K_{r_2} (R_n - R_{r_1}) \\
 &= \sum_{r_2 > r_1 = 1}^{n-1} K_{r_1} K_{r_2} (R_n - R_{r_2}) \tag{A.5}
 \end{aligned}$$

325 Similarly, we can obtain:

$$A_p = \sum_{r_p > \dots > r_1 = 1}^{n-1} (K_{r_1} \dots K_{r_p}) (R_n - R_{r_p}) \tag{A.6}$$

Further, if the relation between S_m and R_m , i.e., $S_m + R_m = 1$ is used, the coefficient A_p can be calculated by Eq. (13).

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