Supporting Material

1. Derivation of Eqs. (14a) – (14k)

The methods of Laplace transform and finite-integral transform are applied to solve Eqs. (8) - (13). The former converts $\bar{h}(\bar{r},\bar{z},\bar{t})$ into $\hat{h}(\bar{r},\bar{z},p)$, $\partial\bar{h}/\partial\bar{t}$ in Eq. (8), (11a) into $p\hat{h}$, the integration in Eq. (11b) into $p\hat{h}/(p+a_2)$, and $\sin(\gamma\bar{t})$ in Eq. (10) into $\gamma/(p^2+\gamma^2)$ with the Laplace parameter *p*. The result of Eq. (8) in the Laplace domain can be written as

$$\frac{\partial^2\hat{h}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\hat{h}}{\partial\bar{r}}+\mu\frac{\partial^2\hat{h}}{\partial\bar{z}^2}=p\hat{h} \tag{A.1}$$

The transformed boundary conditions in *r* and *z* directions are expressed as

$$\frac{\partial\hat{h}}{\partial\bar{r}}=\begin{cases}\dfrac{\gamma}{p^2+\gamma^2} & \text{for} \quad \bar{z}_l\le\bar{z}\le\bar{z}_u \\ 0 & \text{outside screen interval}\end{cases} \quad \text{at} \quad \bar{r}=1 \tag{A.2}$$

$$\frac{\partial\hat{h}}{\partial\bar{z}}=-ap\hat{h} \quad \text{at} \quad \bar{z}=1 \ \text{ for IGD} \tag{A.3a}$$

$$\frac{\partial\hat{h}}{\partial\bar{z}}=-\frac{a_1p\hat{h}}{p+a_2} \quad \text{at} \quad \bar{z}=1 \ \text{ for DGD} \tag{A.3b}$$

$$\frac{\partial\hat{h}}{\partial\bar{z}}=0 \quad \text{at} \quad \bar{z}=0 \tag{A.4}$$

$$\lim_{\bar{r}\to\infty}\hat{h}(\bar{r},\bar{z},p)=0 \tag{A.5}$$

The finite-integral transform proposed by Latinopoulos (1985) is applied to $\hat{h}$ of Eqs. (A.1) - (A.5). It is defined as

$$\tilde{h}(\beta_n)=\Im\{\hat{h}(\bar{z})\}=\int_0^1\hat{h}(\bar{z})\ F\cos(\beta_n\bar{z})\,d\bar{z} \tag{A.6}$$

$$F=\sqrt{\frac{2(\beta_n^2+c^2)}{\beta_n^2+c^2+c}} \tag{A.7}$$

where β_n is the root of Eq. (15), and $c=ap$ for Eq. (A.3a) and $a_1p/(p+a_2)$ for Eq. (A.3b). On the basis of integration by parts, one can write

$$\Im\left\{\frac{\partial^2\hat{h}}{\partial\bar{z}^2}\right\}=\int_0^1\left(\frac{\partial^2\hat{h}}{\partial\bar{z}^2}\right)F\cos(\beta_n z)\,d\bar{z}=-\beta_n^2\tilde{h} \tag{A.8}$$

Note that Eq. (A.8) is applicable only for the no-flow condition specified at $\bar{z}=0$ (i.e., Eq. S.4) and third-type condition specified at $\bar{z}=1$ (i.e., Eq. A.3a or A.3b). The formula for the inverse finite-integral transform is defined as

$$\hat{h}(\bar{z}) = \Im^{-1}\{\tilde{h}(\beta_n)\} = \sum_{n=1}^{\infty} \tilde{h}(\beta_n) F \cos(\beta_n \bar{z}) \tag{A.9}$$

Using Eqs. (A.6) and (A.8) converts $\hat{h}(\bar{r},\bar{z},p)$ into $\tilde{h}(\bar{r},\beta_n,p)$ and $\partial^2\hat{h}/\partial\bar{z}^2$ in Eq. (A.1) into $-\beta_n^2\tilde{h}$ with $n \in (1,2,3,\ldots\infty)$. Eq. (A.1) then becomes an ordinary differential equation (ODE) denoted as

$$\frac{\partial^2\tilde{h}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\tilde{h}}{\partial\bar{r}} - \mu\beta_n^2\tilde{h} = p\tilde{h} \tag{A.10}$$

with the transformed Eqs. (A.5) and (A.2) written, respectively, as

$$\lim_{\bar{r}\to\infty} \tilde{h}(\bar{r},\beta_n,p) = 0 \tag{A.11}$$

$$\frac{\partial\tilde{h}}{\partial\bar{r}} = \frac{\gamma F}{\beta_n(p^2+\gamma^2)}(\sin(\bar{z}_u\beta_n) - \sin(\bar{z}_l\beta_n)) \quad \text{at} \quad \bar{r} = 1 \tag{A.12}$$

Solving Eq. (A.10) with (A.11) and (A.12) results in

$$\tilde{h}(\bar{r},\beta_n,p) = -\frac{\gamma F K_0(r\lambda)(\sin(\bar{z}_u\beta_n)-\sin(\bar{z}_l\beta_n))}{\beta_n\lambda K_1(\lambda)(p^2+\gamma^2)} \tag{A.13}$$

with

$$\lambda = \sqrt{p + \mu\ \beta_n^2} \tag{A.14}$$

Taking the inverse finite-integral transform to Eq. (A.13) using the formula in Eq. (A.9), the Laplace-domain solution is obtained as

$$\hat{h}(\bar{r},\bar{z},p) = 2\sum_{n=1}^{\infty} \tilde{h}(\bar{r},\beta_n,p)\cos(\beta_n\bar{z}) \tag{A.15}$$

with

$$\tilde{h}(\bar{r},\beta_n,p) = \tilde{h}_1(p)\cdot\tilde{h}_2(p) \tag{A.16}$$

$$\tilde{h}_1(p) = \frac{\gamma}{(p^2+\gamma^2)} \tag{A.17}$$

$$\tilde{h}_2(p) = -\varphi_1\varphi_2 \tag{A.18}$$

$$\varphi_1 = K_0(\bar{r}\lambda)(\sin(\bar{z}_u\beta_n) - \sin(\bar{z}_l\beta_n))/\big(\beta_n\lambda K_1(\lambda)\big) \tag{A.19}$$

$$\varphi_2 = (\beta_n^2 + c^2)/(\beta_n^2 + c^2 + c) \tag{A.20}$$

Using the Mathematica function InverseLaplaceTransform, the inverse Laplace transform for $\tilde{h}_1(p)$ in Eq. (A.17) is obtained as

$$\bar{h}_1(\bar{t}) = \sin(\gamma\bar{t}) \tag{A.21}$$

The inverse Laplace transform for $\tilde{h}_2(p)$ in Eq. (A.18) is written as

$$\tilde{h}_2(\bar{t}) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \tilde{h}_2(p)\ e^{p\bar{t}}dp \tag{A.22}$$

where ρ is a real number being large enough so that all singularities are on the left-hand side of the straight line from $(\rho, -i\infty)$ to $(\rho, i\infty)$ in the complex plane. The integrand $\tilde{h}_2(p)$ is a multiple-value function with a branch point at $p = -\mu\beta_n^2$ and a branch cut from the point along the negative real axis. In order to reduce $\tilde{h}_2(p)$ to a single-value function, we consider a modified Bromwich contour that contains a straight line $\overline{\mathrm{AB}}$, $\overline{\mathrm{CD}}$ right above the branch cut and $\overline{\mathrm{EF}}$ right below the branch cut, a semicircle with radius *R*, and a circle $\overset{\frown}{\mathrm{DE}}$ with radius r' in Fig. S1. According to the residual theory, Eq. (A.22) may be expressed as

$$\tilde{h}_2(\bar{t}) + \lim_{\substack{r'\to 0\\ R\to\infty}} \frac{1}{2\pi i}\Big[\int_B^C \tilde{h}_2(p)\ e^{p\bar{t}}dp + \int_C^D \tilde{h}_2(p)\ e^{p\bar{t}}dp + \int_D^E \tilde{h}_2(p)\ e^{p\bar{t}}dp +$$
$$\int_E^F \tilde{h}_2(p)\ e^{p\bar{t}}dp + \int_F^A \tilde{h}_2(p)\ e^{p\bar{t}}dp\Big] = 0 \tag{A.23}$$

where zero on the right-hand side (RHS) is due to no pole in the complex plane. The integrations for paths $\overset{\frown}{\mathrm{BA}}$ (i.e. $\int_B^C \tilde{h}_2(p)\ e^{p\bar{t}}dp + \int_F^A \tilde{h}_2(p)\ e^{p\bar{t}}dp$) with $R \to \infty$ and $\overset{\frown}{\mathrm{DE}}$ (i.e. $\int_D^E \tilde{h}_2(p)\ e^{p\bar{t}}dp$) with $r' \to 0$ equal zero. The path $\overline{\mathrm{CD}}$ starts from $p = -\infty$ to $p = -\mu\beta_n^2$ and $\overline{\mathrm{EF}}$ starts from $p = -\mu\beta_n^2$ to $p = -\infty$. Eq. (A.23) therefore reduces to

$$\tilde{h}_2(\bar{t}) = -\frac{1}{2\pi i}\left(\int_{-\infty}^{-\mu\beta_n^2} \tilde{h}_2(p^+)e^{p^+\bar{t}}dp + \int_{-\mu\beta_n^2}^{-\infty} \tilde{h}_2(p^-)e^{p^-\bar{t}}dp\right) \tag{A.24}$$

where p^+ and p^- are complex numbers right above and below the real axis, respectively. Consider $p^+ = \zeta e^{i\pi} - \mu\beta_n^2$ and $p^- = \zeta e^{-i\pi} - \mu\beta_n^2$ in the polar coordinate system with the origin at $(-\mu\beta_n^2, 0)$ in the complex plane. Eq. (A.24) then becomes

$$\tilde{h}_2(\bar{t}) = \frac{-1}{2\pi i}\int_0^\infty \tilde{h}_2(p^+)e^{p^+\bar{t}}dp - \tilde{h}_2(p^-)e^{p^-\bar{t}}d\zeta \tag{A.25}$$

where p^+ and p^- lead to the same result of $p_0 = -\zeta - \mu\beta_n^2$ for a given ζ; $\lambda = \sqrt{p + \mu\beta_n^2}$ equals $\lambda_0 = \sqrt{\zeta}i$ for $p = p^+$ and $-\lambda_0$ for $p = p^-$. Note that $\tilde{h}_2(p^+)\ e^{p^+\bar{t}}$ and

$\tilde{h}_2(p^-)\ e^{p^-\bar{t}}$ are in terms of complex numbers. The numerical result of the integrand in Eq. (A.25) must be a pure imaginary number that is exactly twice of the imaginary part of a complex number from $\tilde{h}_2(p^+)\ e^{p^+t}$ with $p^+ = p_0$ and $\lambda = \lambda_0$. The inverse Laplace transform for $\tilde{h}_2(p)$ can be written as

$$\tilde{h}_2(\bar{t}) = \frac{-1}{\pi}\int_0^\infty \mathrm{Im}\left(\varphi_1\varepsilon_2\ e^{p_0\bar{t}}\right)d\zeta \tag{A.26}$$

where $p = p_0$; $\lambda = \lambda_0$; φ_1 and ε_2 are respectively defined in Eqs. (A.19) and (14i). According to the convolution theory, the inverse Laplace transform for $\tilde{h}(\bar{r}, \beta_n, p)$ is

$$\tilde{h}(\bar{r}, \beta_n, \bar{t}) = \int_0^{\bar{t}} \tilde{h}_2(\bar{t}')\,\bar{h}_1(\bar{t} - \bar{t}')d\bar{t}' \tag{A.27}$$

where $\bar{h}_1(\bar{t} - \bar{t}') = \sin(\gamma(\bar{t} - \bar{t}'))$ based on Eq. (A.21); $\tilde{h}_2(\bar{t}')$ is defined in Eq. (A.26) with $\bar{t} = \bar{t}'$. Eq. (A.27) can reduce to

$$\tilde{h}(\bar{r}, \beta_n, \bar{t}) = \frac{-1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{\varphi_1\varepsilon_2\left(\gamma e^{p_0\bar{t}} - \gamma\cos(\gamma\bar{t}) - p_0\sin(\gamma\bar{t})\right)}{p_0^2 + \gamma^2}\right)d\zeta \tag{A.28}$$

Substituting $\tilde{h}(\bar{r}, \beta_n, p) = \tilde{h}(\bar{r}, \beta_n, \bar{t})$ and $\hat{h}(\bar{r}, \bar{z}, p) = \bar{h}(\bar{r}, \bar{z}, \bar{t})$ into Eq. (A.15) and rearranging the result leads to

$$\bar{h}(\bar{r}, \bar{z}, \bar{t}) = \frac{-2\gamma}{\pi}\sum_{n=1}^{\infty}\int_0^\infty \cos(\beta_n\bar{z})\exp(p_0\bar{t})\,\mathrm{Im}(\varepsilon_1\varepsilon_2)\,d\zeta + \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^\infty \cos(\beta_n\bar{z})\,\mathrm{Im}\left(\varepsilon_1\varepsilon_2(\gamma\cos(\gamma\bar{t}) + p_0\sin(\gamma\bar{t}))\right)d\zeta \tag{A.29}$$

where ε_1 and ε_2 are defined in Eqs. (14h) and (14i), respectively; the first RHS term equals $\bar{h}_{\exp}(\bar{r}, \bar{z}, \bar{t})$ defined in Eq. (14b); the second term is denoted as $\bar{h}_{\mathrm{SHM}}(\bar{r}, \bar{z}, \bar{t})$ defined in Eq. (14c). Finally, the complete solution is expressed as Eqs. (14a) – (14k).

2. Derivation of Eqs. (23a) – (23m)

Applying the Weber transform to $\bar{H}$ of Eqs. (18) – (22) yields

$$\tilde{H}(\xi) = \mathcal{W}\{\bar{H}\} = \int_1^\infty \bar{H}\ \bar{r}\ \Omega\ dr \tag{B.1}$$

where Ω is defined in Eq. (23g). With the integration by parts, the transform has the property that

$$\mathcal{W}\left\{\frac{\partial^2 \bar{H}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{H}}{\partial \bar{r}}\right\} = -\xi^2 \widetilde{H} - \frac{2}{\pi\xi}\frac{d\bar{H}}{d\bar{r}}\bigg|_{\bar{r}=1} \tag{B.2}$$

where ξ is the Weber parameter and the second RHS term represents the Neumann boundary condition, i.e., Eq. (19). The formula for the inversion can be written as

$$\bar{H} = \mathcal{W}^{-1}\{\widetilde{H}\} = \int_0^\infty \widetilde{H}\ \xi\ \Omega\ d\xi \tag{B.3}$$

Taking the transform to Eqs. (18) - (22) converts $\bar{H}$ into $\widetilde{H}$ and $\partial^2\bar{H}/\partial\bar{r}^2 + \bar{r}^{-1}\partial\bar{H}/\partial\bar{r}$ into $-\xi^2\widetilde{H} - 2/(\pi\xi)\partial\bar{H}/\partial\bar{r}|_{\bar{r}=1}$. The result is expressed as

$$\frac{\partial^2 \widetilde{H}}{\partial \bar{z}^2} - \lambda_w^2 \widetilde{H} = \begin{cases} 0 & \text{for} \quad \bar{z}_u < \bar{z} \le 1 \\ \frac{2}{\pi\mu\xi} & \text{for} \quad \bar{z}_l \le \bar{z} \le \bar{z}_u \\ 0 & \text{for} \quad 0 \le \bar{z} < \bar{z}_l \end{cases} \tag{B.4}$$

$$\frac{\partial \widetilde{H}}{\partial \bar{z}} = -i a \gamma \widetilde{H} \quad \text{at} \quad \bar{z} = 1 \tag{B.5}$$

$$\frac{\partial \widetilde{H}}{\partial \bar{z}} = 0 \quad \text{at} \quad \bar{z} = 0 \tag{B.6}$$

Eq. (B.4) can be separated as

$$\begin{cases} \partial^2\widetilde{H}_u/\partial\bar{z}^2 - \lambda_w^2\widetilde{H}_u = 0 \quad \text{for} \quad \bar{z}_u < \bar{z} \le 1 \\ \partial^2\widetilde{H}_m/\partial\bar{z}^2 - \lambda_w^2\widetilde{H}_m = 2/(\pi\mu\xi) \quad \text{for} \quad \bar{z}_l \le \bar{z} \le \bar{z}_u \\ \partial^2\widetilde{H}_l/\partial\bar{z}^2 - \lambda_w^2\widetilde{H}_l = 0 \quad \text{for} \quad 0 \le \bar{z} < \bar{z}_l \end{cases} \tag{B.7}$$

with the continuity requirements:

$$\begin{cases} \widetilde{H}_m = \widetilde{H}_u \\ \partial\widetilde{H}_m/\partial\bar{z} = \partial\widetilde{H}_u/\partial\bar{z} \end{cases} \quad \text{at} \quad \bar{z} = \bar{z}_u \tag{B.8}$$

$$\begin{cases} \widetilde{H}_l = \widetilde{H}_m \\ \partial\widetilde{H}_l/\partial\bar{z} = \partial\widetilde{H}_m/\partial\bar{z} \end{cases} \quad \text{at} \quad \bar{z} = \bar{z}_l \tag{B.9}$$

Solving Eq. (B.7) with (B.5), (B.6), (B.8), and (B.9) results in Eqs. (23h) – (23m). The solution of $\bar{H}$ defined in Eq. (23f) can be obtained by the formula Eq. (B.3) for the inverse Weber transform.

3. Derivation of Eq. (24)

Applying the finite Fourier cosine transform to the model, Eqs. (18) – (22) with S_y = 0 (i.e., $a = 0$) for the confined condition converts $\overline{H}$ into $\acute{H}$ and $\partial^2\overline{H}/\partial\bar{z}^2$ into $(m\pi)^2\acute{H}$ with m being an integer from 0, 1, 2, … ∞. The result is written as

$$\frac{\partial^2\acute{H}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\acute{H}}{\partial\bar{r}}-\lambda_m^2\acute{H}=0 \tag{C.1}$$

$$\frac{\partial\acute{H}}{\partial\bar{r}}=\begin{cases}\bar{z}_u-\bar{z}_l & \text{for} \quad m=0\\ \frac{1}{m\pi}(\sin(\bar{z}_u m\pi)-\sin(\bar{z}_l m\pi)) & \text{for} \quad m>0\end{cases} \quad \text{at} \quad \bar{r}=1 \tag{C.2}$$

$$\lim_{\bar{r}\to\infty}\acute{H}=0 \tag{C.3}$$

where $\lambda_m^2=\gamma i+\mu(m\pi)^2$, and $\bar{z}_u-\bar{z}_l$ results from $\lim_{m\to 0}(\sin(\bar{z}_u m\pi)-\sin(\bar{z}_l m\pi))/(m\pi)$ using L' Hospital's rule. Solving Eq. (C.1) with (C.2) and (C.3) results in

$$\acute{H}(\bar{r})=\frac{-K_0(\bar{r}\lambda_m)}{\lambda_m K_1(\lambda_m)}\times\begin{cases}\bar{z}_u-\bar{z}_l & \text{for} \quad m=0\\ \frac{1}{m\pi}(\sin(\bar{z}_u m\pi)-\sin(\bar{z}_l m\pi)) & \text{for} \quad m>0\end{cases} \tag{C.4}$$

After applying the inversion to Eq. (C.4) yields Eq. (24).

Figure

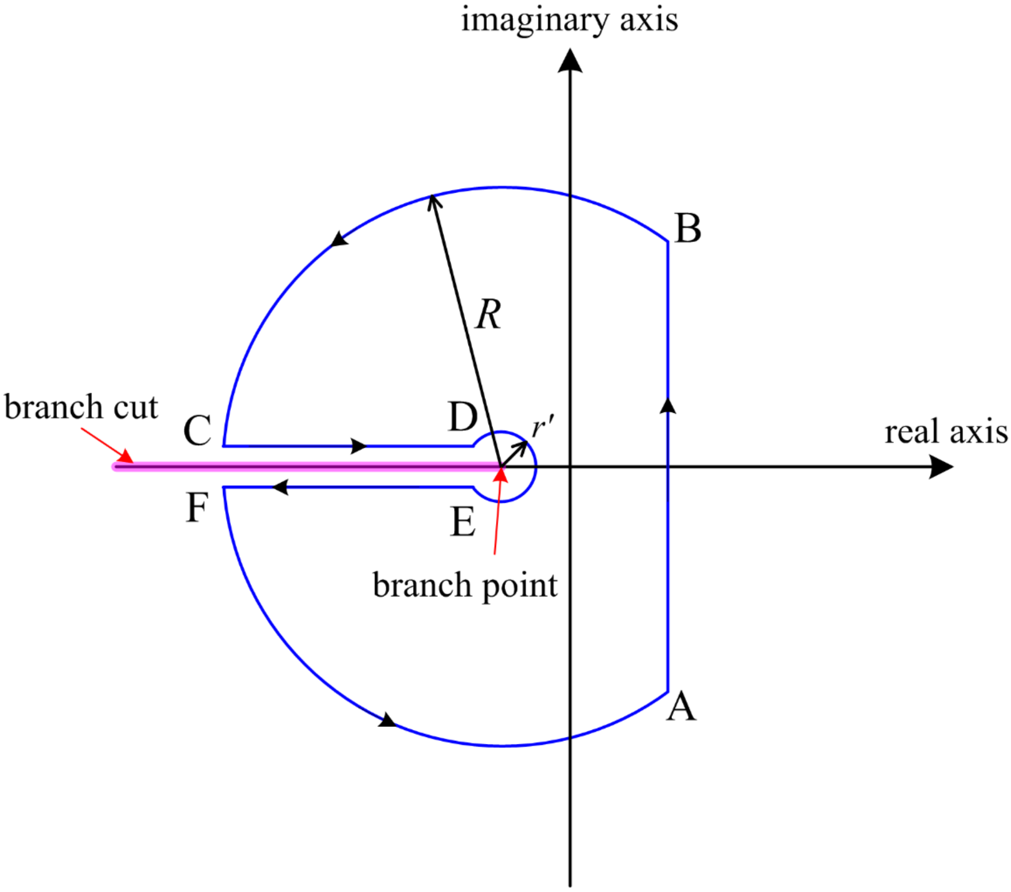

Figure S1. Modified Bromwich contour for the inverse Laplace transform to a multiple-value function with a branch point and a branch cut