Associate Editor

Reviewers provide positive comments on the paper. Authors are required to address these comments with a point to point reply. Particularly, authors are encouraged to further elaborate the need and new contribution of the study, to make a concise revision among sections (checking the rightness of equations, reducing a number of equations), and to include additional in depth analysis on results.

Thank you for handling the review process of this manuscript. We have revised the manuscript based on the reviews and prepared the responses to comments.

The number of equations has been reduced by moving Section 2.1 (including equations 7, 8, 9) to Appendix A, moving equation 21 to Appendix B, deleting equations 11, 25, 29, and 31. All the equations have been double-checked.

For elaborating the need and new contribution of this study, the introduction section has been revised. The following texts have been added to clearly state the scientific gap:

(Lines 57-66) "Even though the SCS-CN method has been interpreted as a saturation excess runoff model in the literature, there is a knowledge gap for the direct linkage between the SCS-CN method and the Xinanjiang/VIC type of model based on a probability distribution function for the spatial variability of soil water storage capacity. If the SCS-CN method is a saturation excess runoff model, is there a distribution function for soil water storage capacity which leads to the SCS-CN method? If yes, what is the probability density function (PDF)? This is an unsolved research question. The objective of this paper is to fill this knowledge gap, i.e., discovering the distribution function for soil water storage capacity which leads to the SCS-CN method. This is a procedure of inverse modeling, i.e., identifying the distribution function of saturation excess runoff model for a known functional form of runoff generation."

The following sentences have been added in the introduction section:

(Lines 67-78) "Meanwhile, the identification of the new distribution function is intrigued by the linkage between the SCS-CN method and Budyko equation [*Budyko*, 1974]. …. The representation of runoff generation in the Budyko-type of framework facilitates the identification of the new distribution function for soil water storage capacity leading to the SCS-CN method."

Anonymous Referee #3

This is an interesting work to unify the SCS-CN and VIC type model. I have to acknowledge that I didn't check each equation in the manuscript, and assume that they are all correctly presented.

Thank you for your constructive comments. All the equations in the manuscript have been double checked and they are correct.

I have some concerns the author could consider to address:

1. The motivation of the work is not clear to me. In my opinion, a scientific gap should be filled by a new published work. The author presented a lot of work on equation derivation, but the scientific gap in the current community filled by this work is not clear to me. The author may consider to rephrase the introduction section and make his motivation more visible.

   Thank you for your suggestion. The introduction section has been revised. The following texts have been added to clearly state the scientific gap:

   (Lines 57-66) "Even though the SCS-CN method has been interpreted as a saturation excess runoff model in the literature, there is a knowledge gap for the direct linkage between the SCS-CN method and the Xinanjiang/VIC type of model based on a probability distribution function for the spatial variability of soil water storage capacity. If the SCS-CN method is a saturation excess runoff model, is there a distribution function for soil water storage capacity which leads to the SCS-CN method? If yes, what is the probability density function (PDF)? This is an unsolved research question. The objective of this paper is to fill this knowledge gap, i.e., discovering the distribution function for soil water storage capacity which leads to the SCS-CN method. This is a procedure of inverse modeling, i.e., identifying the distribution function of saturation excess runoff model for a known functional form of runoff generation."

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2. Line 65: "The objective of this paper is to unify..''. Why should we unify SCS-CN and VIC type model? What are the individual drawbacks of SCS-CN and VIC type model for the application of hydrological modeling? Line 53:"Bartleet et al [2016b] unified …". What is the new contribution of this paper to Bartleet et al [2016b]?

   This sentence has been revised as (Lines 79-80): "The identified new distribution function for soil water storage capacity will unify the SCS-CN method and VIC type of model." The objective of the paper has been revised as (Lines 63-64): "The objective of this paper is to fill this knowledge gap, i.e., discovering the distribution function for soil water storage capacity which leads to the SCS-CN method."

The comparisons (or drawbacks) of SCS-CN and VIC type of distribution are summarized in Section 4.

The sentence has been revised as (Lines 53-56): "*Bartlett et al.* [2016b] proposed an event-based probabilistic storage framework for unifying TOPMODEL, the VIC type of model, and the SCS-CN method, and the framework includes a spatial description of the runoff concept of "prethreshold" and "threshold-excess" runoff [*Bartlett et al.*, 2016a]."

Bartlett et al. [2016a] did not propose a distribution function leading to the SCS-CN method. The new contribution of this paper compared with *Bartlett et al.* [2016a] is summarized as:

(Lines 297-308): "*Bartlett et al.* [2016a] developed an event-based probabilistic storage framework including a spatial description of "prethreshold" and "threshold-excess" runoff; and the framework has been utilized for unifying TOPMODEL, VIC and SCS-CN [*Bartlett et al.*, 2016b]. The extended SCS-CN method (SCS-CNx) from the probabilistic storage framework is derived given the following assumptions: 1) the spatial distribution of rainfall is exponential; 2) the spatial distribution of soil moisture deficit is uniform; and 3) the spatial distribution of storage capacity is exponential. When "prethreshold" runoff is 0 (i.e., there is only threshold-excess or saturation excess runoff), the SCS-CNx method leads to the SCS-CN method without the initial abstraction term (i.e., there is no εW term in equation (30)). In this paper, the new probability distribution function is used for storage capacity in the VIC model in which the spatial distribution of precipitation is assumed to be uniform. The obtained equation for saturation excess runoff leads to the exact SCS-CN method as shown in equation (30)."

3. I would suggest to reduce the number of equation in the text to make it more readable. The author could consider to remove or move some of the equations to the Appendix, such as euqs. 8, 11, 21, 25, 29 and 30, and some other equations. Listing some of the equations in several tables could be another alternative.

   Thank you for your suggestion. The Section 2.1 (including equations 7, 8, 9) has been moved to Appendix A. Equations 11, 25, 29, and 31 have been removed. Equation (21) has been moved to Appendix B.

4. A Table to summarize the parameters and boundary conditions of the SCS-CN, VIC-type and the unified methods should be provided.

   Thanks. The revised Table 1 summarizes the parameters and boundary conditions of the SCS-CN, VIC-type and the unified methods.

5. Line 231: "The probability density function… is represented by:". Is this function created by the author? Could you please provide some references to proof its assumptions?

   Yes. Proposing this new distribution function is the objective of the paper. When this distribution function is used for describing the spatial distribution of soil water storage capacity, we will obtain the empirical SCS-CN method. Therefore, the SCS-CN method and saturation excess runoff model (Xinanjiang/VIC type of model) are unified. The following texts describe the reasoning how this distribution function is obtained by inverse thinking.

(Lines 309-322): "This research started with the following research question: if the SCS-CN method is a saturation excess runoff generation model, what is the distribution function of soil water storage capacity? Wang and Tang (2014) showed that equation (29) is derived from the proportionality relationship of SCS-CN method, i.e., equation (30). From the comparison of boundary conditions between SCS-CN method and VIC type of model discussed in Section 4, it is observed that equation (29) does not include initial soil water storage, and the derived one from distribution function will include initial soil water storage (e.g., equation (26)). However, equation (29) can be viewed as the result of $S_0 = 0$; and W for equation (29) can be written as:

$$W = \int_0^P [1 - F(x)]dx \tag{31}$$

From equation (29), one obtains:

$$W = \frac{P+S_b-\sqrt{(S_b+P)^2-2aPS_b}}{a} \tag{32}$$

Substituting equation (32) into equation (31), one obtains:

$$\frac{P+S_b-\sqrt{(S_b+P)^2-2aPS_b}}{a} = \int_0^P [1 - F(C)]dC \tag{33}$$

Equation (20) is obtained from equation (33)."

6. Line 383: From my taste, the mentioned future works should be done in this paper. Practice application in case studies should be added to clarify the benefits of the new proposed method. It is really difficult to judge the scientific contribution of this work based on the list of 42 equations.

The SCS-CN method has been interpreted as a saturation excess runoff model in the literature. However, there is a knowledge gap for the direct linkage between the SCS-CN method and the Xinanjiang/VIC type of model. The objective of this paper is to fill this knowledge gap, i.e., discovering the distribution function for soil water storage capacity which leads to the SCS-CN method. Equation (19) shows the PDF and equation (20) shows the CDF of the proposed new distribution function. This new distribution function leads to the SCS-CN method. Since the SCS-CN method has been used in many applications in practices. In another word, the performance of this PDF has been verified by case studies using the SCS-CN method. Moreover, this PDF provides a better way to handle the initial storage compared with the SCS-CN method. The extended SCS-CN method (i.e., equation (27)), which includes initial storage explicitly, could be used for surface runoff modeling in SWAT model. But this is out of scope and purpose of this paper. This could be a future work.

In general, I think this work presents an interesting attempt to compare different calculation methods of direct surface runoff in the community of hydrological modeling. I would suggest the author to reduce the number of equation in the text to increase its readability. Moreover, some case studies should be provided to compare the performance of various methods and therefore to proof the benefits of the proposed new method.

Thank you again for your constructive comments. The manuscript has been revised based your comments. Please refer to my responses to your individual comments.

A new probability density function for spatial distribution of soil water storage capacity leads to SCS curve number method

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Abstract

Following the Budyko framework, soil wetting ratio (the ratio between soil wetting and precipitation) as a function of soil storage index (the ratio between soil wetting capacity and precipitation) is derived from the SCS-CN method and the VIC type of model. For the SCS-CN method, soil wetting ratio approaches 1 when soil storage index approaches ∞, due to the limitation of the SCS-CN method in which the initial soil moisture condition is not explicitly represented. However, for the VIC type of model, soil wetting ratio equals soil storage index when soil storage index is lower than a certain value, due to the finite upper bound of the power distribution function of storage capacity. In this paper, a new distribution function, supported on a semi-infinite interval $x \in [0, \infty)$, is proposed for describing the spatial distribution of storage capacity. From this new distribution function, an equation is derived for the relationship between soil wetting ratio and storage index. In the derived equation, soil wetting ratio approaches 0 as storage index approaches 0; when storage index tends to infinity, soil wetting ratio approaches a certain value ($\leq$1) depending on the initial storage. Moreover, the derived equation leads to the exact SCS-CN method when initial water storage is 0. Therefore, the new distribution function for soil water storage capacity explains the SCS-CN method as a saturation excess runoff model and unifies the surface runoff modeling of SCS-CN method and VIC type of model.

Keywords: SCS curve number method, VIC, Xinanjiang, saturation excess, distribution function, soil water storage capacity, soil wetting

1. Introduction

The Soil Conservation Service Curve Number (SCS-CN) method ~~[~~(*Mockus*, 1972~~]~~) has been popularly used for direct runoff estimation in engineering communities. Even though the SCS-CN method was obtained empirically (~~[~~*Ponce*, 1996; *Beven*, 2011)~~]~~, it is often interpreted as an infiltration excess runoff model (~~[~~*Bras*, 1990; *Mishra and Singh*, 1999)~~]~~. *Yu* (~~[~~1998)~~]~~ showed that partial area infiltration excess runoff generation on a statistical distribution of soil infiltration characteristics provided similar runoff generation equation as the SCS-CN method. Recently, *Hooshyar and Wang* (~~[~~2016)~~]~~ derived an analytical solution for Richards' equation for ponded infiltration into a soil column bounded by a water table; and they showed that the SCS-CN method, as an infiltration excess model, is a special case of the derived general solution. The SCS-CN method has also been interpreted as a saturation excess runoff model (~~[~~*Steenhuis et al.*, 1995; *Lyon et al.*, 2004; *Easton et al.*, 2008)~~]~~. During an interview, Mockus, who developed the proportionality relationship of the SCS-CN method, stated that "saturation overland flow was the most likely runoff mechanism to be simulated by the method" (~~[~~*Ponce*, 1996)~~]~~. Recently, *Bartlett et al.* (~~[~~2016a)~~]~~ developed a probabilistic framework, which provides a statistical justification of the SCS-CN method and extends the saturation excess interpretation of the event-based runoff of the method.

Since the 1970s, various saturation excess runoff models have been developed based on the concept of probability distribution of soil storage capacity ~~[~~(*Moore*, 1985)~~]~~. TOPMODEL is a well-known saturation excess runoff model based on spatially distributed topography (~~[~~*Beven and Kirkby*, 1979; *Sivapalan et al.*, 1987~~]~~). To quantify the dynamic change of saturation area during rainfall events, the spatial variability of soil moisture storage capacity is described by a cumulative probability distribution function in the Xinanjiang model (~~[~~*Zhao*, 1977; *Zhao et al.*, 1992~~]~~) and the Variable Infiltration Capacity (VIC) model (~~[~~*Wood et al.*, 1992; *Liang et al.*, 1994)~~]~~. The distribution of storage capacity is described by a power function in these models, which have been used for catchment scale runoff prediction and large scale land surface hydrologic simulations. *Bartlett et al.* (~~[~~2016b~~]~~) ~~unified TOPMODEL, the VIC type of model, and the SCS-CN method by~~ proposed an event-based probabilistic storage framework for unifying TOPMODEL, the VIC type of model, and the SCS-CN method, ~~which~~ and the framework includes a spatial description of the runoff concept of “prethreshold” and “threshold-excess” runoff ~~[~~(*Bartlett et al.*, 2016a)~~]~~.

Even though the SCS-CN method has been interpreted as a saturation excess runoff model in the literature, there is a knowledge gap for the direct linkage between the SCS-CN method and the Xinanjiang/VIC type of model based on a probability distribution function for the spatial variability of soil water storage capacity. If the SCS-CN method is a saturation excess runoff model, is there a distribution function for soil water storage capacity which leads to the SCS-CN method? If yes, what is the probability density function (PDF)? This is an unsolved research question. The objective of this paper is to fill this knowledge gap, i.e., discovering the distribution function for soil water storage capacity which leads to the SCS-CN method. This is a procedure of inverse modeling, i.e., identifying the distribution function of saturation excess runoff model for a known functional form of runoff generation.

Meanwhile, the identification of the new distribution function is intrigued by the linkage between the SCS-CN method and Budyko equation (*Budyko*, 1974). By applying the generalized proportionality hypothesis from the SCS-CN method to mean annual water balance, *Wang and Tang* ~~[~~(2014)~~]~~ derived a one-parameter Budyko equation ~~[*Budyko*, 1974]~~ for mean annual evaporation ratio (i.e., the ratio of evaporation to precipitation) as a function of climate aridity index (i.e., the ratio of potential evaporation to precipitation). As an analogy to the Budyko framework, the SCS-CN method and the VIC type of model at the event scale can be represented by the relationship between soil wetting ratio, defined as the ratio between soil wetting and precipitation, and soil storage index which is defined as the ratio between soil wetting capacity and precipitation. The representation of runoff generation in the Budyko-type of framework facilitates the identification of the new distribution function for soil water storage capacity leading to the SCS-CN method.

The ~~objective of this paper is to~~identified new distribution function for soil water storage capacity will unify the SCS-CN method and VIC type of model~~ by proposing a new distribution function for describing the soil water storage capacity~~. In section 2, the SCS-CN method is presented in the form of Budyko-type framework with two parameterization schemes. In section 3, the VIC type of model is presented in the form of Budyko-type framework. In section 4, the SCS-CN method is then compared with the VIC type of model from the perspectives of number of parameters and boundary conditions (i.e., the lower and upper bounds of soil storage index). In section 5, the proposed new distribution function is introduced and compared with the power distribution of VIC type of model; and a modified SCS-CN method considering initial storage explicitly is derived from the new distribution function. Conclusions are drawn in section 6.

2. SCS curve number method

In this section, the SCS-CN method is described in the form of surface runoff modeling and then is presented for infiltration modeling in the Budyko-type framework. The initial storage at the beginning of a time interval (e.g., rainfall event) is denoted by S_0 [mm], and the maximum value of average storage capacity over the catchment is denoted by S_b [mm]. The storage capacity for soil wetting for the time interval, S_p [mm], is computed by:

$$S_p = S_b - S_0 \tag{1}$$

The total rainfall during the time interval is denoted by P [mm]. Before surface runoff is generated, a portion of rainfall is intercepted by vegetation and infiltrates into the soil. This portion of rainfall is called initial abstraction or initial soil wetting denoted by W_i [mm]. The remaining rainfall ($P - W_i$) is partitioned into runoff and continuing soil wetting. This competition is captured by the proportionality relationship in the SCS-CN method:

$$\frac{W-W_i}{S_p-W_i} = \frac{Q}{P-W_i} \tag{2}$$

where W [mm] is the total soil wetting; $W - W_i$ is continuing wetting and $S_p - W_i$ is its potential value; Q [mm] is surface runoff; and $P - W_i$ is the available water and interpreted as the potential value of Q. Since rainfall is partitioned into total soil wetting and surface runoff, i.e., $P = W + Q$, surface runoff is computed by substituting $W = P - Q$ into equation (2):

$$Q = \frac{(P-W_i)^2}{P+S_p-2W_i} \tag{3}$$

This equation is used for computing direct runoff in the SCS-CN method.

The SCS-CN method can also be represented in terms of soil wetting ratio ($\frac{W}{P}$). Substituting equation (3) into $W = P - Q$ and dividing P on both sides, the soil wetting ratio equation is obtained:

$$\frac{W}{P} = \frac{\frac{S_p}{P}-\frac{W_i^2}{P^2}}{1+\frac{S_p}{P}-2\frac{W_i}{P}} \tag{4}$$

Climate aridity index is defined as the ratio between potential evaporation and precipitation. In climate aridity index, both available water supply and water demand are determined by climate.

$$\Phi_{sc} = \frac{S_p}{P} \tag{5}$$

A similar dimensionless parameter for the ratio between the maximum soil storage capacity and mean rainfall depth of rainfall events was defined in *Porporato et al.* ~~[~~(2004)~~]~~. In soil storage index, water demand is determined by soil and available water supply is determined by climate. Substituting equation (5) into equation (4), the soil wetting equation for the SCS-CN method is obtained:

$$\frac{W}{P} = \frac{\Phi_{sc} - \frac{W_i^2}{P^2}}{1 + \Phi_{sc} - 2\frac{W_i}{P}} \tag{6}$$

There are t~~T~~wo potential schemes for parameterizing the initial wetting in equation (6). ~~are discussed in the following sections.~~

~~**Parameterization scheme 1: ratio between initial wetting and storage capacity**~~

As the first scheme, t~~T~~he initial wetting is usually parameterized as the ratio between initial wetting and storage capacity in the SCS-CN method. The detail of this scheme is described in Appendix A and plotted in Figure 1. As we can see, the range of Φ_{sc} is dependent on the parameter $\lambda = \frac{W_i}{S_p - W_i}$. ~~The potential for continuing wetting is called potential maximum retention and is denoted by~~ $\cancel{S_m = S_p - W_i}$~~.~~ $\cancel{S_m}$ ~~is computed as a function of curve number which is dependent on land use/land cover and soil permeability. The ratio between~~ $\cancel{W_i}$ ~~and~~ $\cancel{S_m}$ ~~in the SCS curve number method is denoted by~~ $\cancel{\lambda = \frac{W_i}{S_p - W_i}}$~~, and then the ratio between initial soil wetting and storage capacity is computed by:~~

$$\cancel{\frac{W_i}{S_p} = \frac{\lambda}{1+\lambda}} \qquad \cancel{(7)}$$

~~The value of~~ $\cancel{\lambda}$ ~~varies in the range of~~ $\cancel{0 \le \lambda \le 0.3}$~~, and a value of 0.2 is usually used [*Ponce and Hawkins*, 1996]. Substituting equation (7) into equation (6) leads to:~~

~~$$\frac{W}{P}=\frac{1-\left(\frac{\lambda}{1+\lambda}\right)^2\Phi_{sc}}{1-\frac{2\lambda}{1+\lambda}+\Phi_{sc}{}^{-1}} \qquad (8)$$~~

~~Equation (8) is plotted in Figure 1 for $\lambda = 0.1$ and 0.3. As we can see, the range of Φ_{sc} is dependent on the parameter λ. Since $W_i \le P$, Φ_{sc} is in the range of $\left[0, 1+\frac{1}{\lambda}\right]$. Equation (8) satisfies the following boundary conditions: $\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$; and $\frac{W}{P} \to 1$ as $\Phi_{sc} \to \frac{\lambda+1}{\lambda}$. When $\lambda \to 0$, equation (8) becomes:~~

~~$$\frac{W}{P}=\frac{1}{1+\Phi_{sc}{}^{-1}} \qquad (9)$$~~

~~Equation (9) is the lower bound for $\frac{W}{P}$ based on this parameterization scheme.~~

~~**Parameterization scheme 2: ratio between initial wetting and total wetting**~~

In order to avoid the situation that the range of Φ_{sc} is dependent on the parameter λ, we can use the following parameterization scheme ~~[~~(*Chen et al.*, 2013; *Tang and Wang*, 2017)~~]~~:

$$\varepsilon = \frac{W_i}{W} \qquad (7\cancel{10})$$

Substituting equation (7~~10~~) into equation (6), we can obtain the following equation:

~~$$\frac{W}{P}=\frac{\Phi_{sc}-\varepsilon^2\frac{W^2}{P^2}}{1+\Phi_{sc}-2\varepsilon\frac{W}{P}} \qquad (11)$$~~

~~We can solve for $\frac{W}{P}$ from equation (11):~~

$$\frac{W}{P}=\frac{1+\Phi_{sc}-\sqrt{(1+\Phi_{sc})^2-4\varepsilon(2-\varepsilon)\Phi_{sc}}}{2\varepsilon(2-\varepsilon)} \qquad (\cancel{12}8)$$

Equation (8~~12~~) has the same functional form as the derived Budyko equation for long-term evaporation ratio ~~[~~(*Wang and Tang*, 2014; *Wang et al.*, 2015)~~]~~. Equation (8~~12~~) satisfies the following boundary conditions: $\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$; and $\frac{W}{P} \to 1$ as $\Phi_{sc} \to \infty$. Based on equation (7~~10~~), the range of ε is $[0, 1]$, and $\varepsilon = 1$ corresponds to the upper bound (Figure 1). Equation (8~~12~~) becomes equation (A3~~9~~) as $\varepsilon \to 0$, and it is the lower bound. Figure 1 plots equation (8~~12~~)

for ε = 0.1 and 0.3. Due to the dependence of the range of Φ_{sc} on the parameter λ in the first parameterization scheme, the second parameterization scheme is focused on in the following sections.

In the SCS-CN method, the soil wetting ratio is a function of soil storage index with a parameter for describing initial wetting. The average wetting capacity at the catchment scale is used for computing soil storage index; but the spatial variability of wetting capacity is not represented in the SCS-CN method.

3. Saturation excess runoff model

The spatial variability of soil water storage capacity is explicitly represented in the saturation excess runoff models such as VIC and Xinanjiang. In these models, the spatial variation of point-scale storage capacity (C) is represented by a power function:

$$F(C) = 1 - \left(1 - \frac{C}{C_m}\right)^{\beta} \qquad (~~13~~9)$$

where $F(C)$ is the cumulative probability, i.e., the fraction of catchment area for which the storage capacity is less than C [mm]; and C_m [mm] is the maximum value of point-scale storage capacity over the catchment. The water storage capacity includes vegetation interception, surface retention, and soil moisture capacity; β is the shape parameter of storage capacity distribution and is usually assumed to be a positive number. β ranges from 0.01 to 5.0 as suggested by *Wood et al.* ~~[~~(1992~~]~~). The storage capacity distribution curve is concave down for $0 < \beta < 1$ and concave up for $\beta > 1$. The average value of storage capacity over the catchment is equivalent to S_b in the SCS-CN method, and it is obtained by integrating the exceedance probability of storage capacity $S_b = \int_0^{C_m}(1 - F(x))\,dx$:

$$S_b = \frac{C_m}{\beta + 1} \qquad (10~~14~~)$$

Similarly, for a given C, the catchment-scale storage S [mm] can be computed (~~[~~*Moore*, 1985)~~]~~:

$$S = S_b\left[1-\left(1-\frac{C}{C_m}\right)^{\beta+1}\right]$$ (11~~5~~)

To derive wetting ratio as a function of soil storage index, the initial storage at the catchment scale is parameterized by the degree of saturation:

$$\psi = \frac{S_0}{S_b}$$ (12~~16~~)

Recalling equation (1) and the definition of soil storage index (i.e., equation (5)), we obtain:

$$\frac{S_b}{P} = \frac{\Phi_{sc}}{1-\psi}$$ (13~~17~~)

The value of C corresponding to the initial storage S_0 is denoted as C_0, and $S_0 = S_b\left[1-\left(1-\frac{C_0}{C_m}\right)^{\beta+1}\right]$ is obtained by substituting S_0 and C_0 into equation (11~~5~~). When $P + C_0 \geq C_m$, each point within the catchment is saturated and soil wetting reaches its maximum value, i.e., $W = S_p$. Substituting $C_0 = C_m - C_m\left(1-\frac{S_0}{S_b}\right)^{\frac{1}{\beta+1}}$ into $P + C_0 \geq C_m$, we obtain:

$$\Phi_{sc} \leq b \text{ where } b = (\beta+1)^{-1}(1-\psi)^{\frac{\beta}{\beta+1}}$$ (14~~8~~)

Therefore, this condition is equivalent to:

$$\frac{W}{P} = \Phi_{sc} \text{ when } \Phi_{sc} \leq b$$ (15~~9~~)

Next, we will derive $\frac{W}{P}$ for the condition of $\Phi_{sc} > b$. The storage at the end of the modeling period (e.g., rainfall-runoff event) is denoted as S_1, which is computed by:

$$S_1 = S_b\left[1-\left(1-\frac{P+C_0}{C_m}\right)^{\beta+1}\right]$$ (16~~20~~)

~~Since $W = S_1 - S_0$, wetting is computed by:~~

$$W = S_b\left[1-\left(1-\frac{P+C_0}{C_m}\right)^{\beta+1}\right] - S_0$$ ~~(21)~~

~~From equation (21), we obtain~~From equation (16) one obtains (see Appendix B~~A~~ for details):

$$\frac{W}{P} = \Phi_{sc}\left[1-\left(1-b{\Phi_{sc}}^{-1}\right)^{\beta+1}\right] \text{ when } \Phi_{sc} > b \qquad (\cancel{22}17)$$

The limit of equation (17~~22~~) for $\Phi_{sc} \to \infty$ can be obtained (see Appendix C~~B~~ for details):

$$\lim_{\Phi_{sc}\to\infty} \frac{W}{P} = (1-\psi)^{\frac{\beta}{\beta+1}} \qquad (18\cancel{23})$$

Equations (15~~9~~) and (17~~22~~) provide $\frac{W}{P}$ as a function of Φ_{sc} with two parameters (ψ and β). Figure 2 plots equations (15~~9~~) and (17~~22~~) for ψ = 0 and 0.5 when β = 0.2 and 2. As we can see, $\frac{W}{P}$ decreases as β increases for given values of ψ and Φ_{sc}; and $\frac{W}{P}$ decreases as ψ increases for given values of β and Φ_{sc}, implicating that soil wetting ratio decreases with the degree of initial saturation under a given soil storage index.

4. Comparison between SCS-CN model and VIC type of model

The SCS-CN model with the parameterization of ratio between initial wetting and total wetting is compared with the VIC type of saturation excess runoff model. In sections 2 and 3, we derived $\frac{W}{P}$ as a function of Φ_{sc} based on the SCS-CN method and the VIC type of model, which uses a power function to describe the spatial distribution of storage capacity. The SCS-CN method is a function of storage capacity S_p; but the VIC type of model is a function of storage capacity S_p and the degree of initial saturation $\frac{S_0}{S_b}$. As a result, the function of $\frac{W}{P} \sim \frac{S_p}{P}$ for the SCS-CN method has only one parameter (ε), but it has two parameters (β and ψ) for the VIC type of model.

Table 1 shows the boundary conditions for the relationships between $\frac{W}{P}$ and Φ_{sc} from the SCS-CN method and the VIC type of model. The lower boundary of the SCS-CN method with parameter ε is $\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$. However, for the VIC type of model, $\frac{W}{P} = \Phi_{sc}$ when $\Phi_{sc} \le b$. For the SCS-CN method, W reaches its maximum (S_p) when rainfall reaches infinity; while for the VIC type of model, W reaches its maximum value (S_p) when rainfall reaches a finite number ($C_m - C_0$). In other words, for the SCS-CN method, the entire catchment becomes saturated when rainfall reaches infinity; while for the VIC type model, the entire catchment becomes saturated when rainfall reaches a finite number.

As shown in Table 1, the upper boundary of the SCS-CN method (with parameter ε) is 1. However, for the VIC type of model, the upper boundary is $(1-\psi)^{\frac{\beta}{\beta+1}}$ instead of 1. This is due to the effect of initial storage in the VIC type of model. When initial storage is 0 (i.e., $\psi = 0$), the wetting ratio $\frac{W}{P}$ for the VIC type of model has the same upper boundary condition as the SCS-CN method.

5. Unification of SCS-CN method and VIC type of model

Based on the comparison between the SCS-CN method and VIC type of model, a new distribution function is proposed in this section for describing the spatial distribution of soil water storage capacity, which unifies the SCS-CN method and VIC type of model. As discussed in section 4, the upper boundary condition of the SCS-CN model (i.e., $\frac{W}{P} \to 1$ as $\Phi_{sc} \to \infty$) does not depend on the initial storage. This upper boundary condition needs to be modified by including the effect of initial storage so that the limit of $\frac{W}{P}$ as $\Phi_{sc} \to \infty$ is dependent on the degree of initial saturation like the VIC type of model. However, the lower boundary condition of the VIC model needs to be modified so that the lower boundary condition follows that $\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$ like the SCS-CN method. Through these modifications, the SCS-CN method and the VIC type of saturation excess runoff model can be unified from the functional perspective of soil wetting ratio.

Based on the comparison one may have the following questions: 1) Can the SCS-CN method be derived from the VIC type of model by setting initial storage to 0? 2) If yes, what is the distribution function for soil water storage capacity? Once we answer these questions, a modified SCS-CN method considering initial storage explicitly can be derived as a saturation excess runoff model based on a distribution function of water storage capacity, and it unifies the SCS-CN method and VIC type of model. In this section, a new distribution function is proposed for describing the spatial variability of soil water storage capacity, from which the SCS-CN method is derived as a VIC type of model.

5.1. A new distribution function

The probability density function (PDF) of the new distribution for describing the spatial distribution of water storage capacity is represented by:

$$f(C) = \frac{(2-a)\mu^2}{[(C+\mu)^2 - 2a\mu C]^{3/2}} \qquad (19\cancel{24})$$

where C is point-scale water storage capacity and supported on a positive semi-infinite interval $(C \geq 0)$; a is the shape parameter and its range is $0 < a < 2$; and μ is the mean of the distribution (i.e., the scale parameter). Figure 3a plots the PDFs for five sets of shape and scale parameters. When $a \leq 1$, the PDF monotonically decreases with the increase of C, i.e., the peak of PDF occurs at $C = 0$; while when $a > 1$, the peak of PDF occurs at $C > 0$ and the location of the peak depends on the values of a and μ. For comparison, Figure 3b plots the PDF for VIC model. $\cancel{f(C) = \frac{\beta}{C_m}\left(1 - \frac{C}{C_m}\right)^{\beta-1}}$~~:~~

$$\cancel{f(C) = \frac{\beta}{C_m}\left(1 - \frac{C}{C_m}\right)^{\beta-1} \qquad (25)}$$

As shown by the solid black curve in Figure 3b, when $0 < \beta < 1$, $f(C)$ approaches infinity as $C \rightarrow C_m$. It is a uniform distribution when $\beta = 1$. The peak of PDF occurs at $C = 0$ when $\beta > 1$. Therefore, the peak of PDF for VIC model occurs at $C = 0$ or C_m.

The cumulative distribution function (CDF) corresponding to the proposed PDF is obtained by integrating equation (19~~24~~):

$$F(C) = 1 - \frac{1}{a} + \frac{C + (1-a)\mu}{a\sqrt{(C+\mu)^2 - 2a\mu C}} \quad (\cancel{26}20)$$

Figure 4a plots the CDFs corresponding to the PDFs in Figure 3a. For comparison, Figure 4b plots the CDFs corresponding to the PDFs in Figure 3b. The storage capacity distribution curve for the proposed distribution is concave up for $a \leq 1$ and *S*-shape for $a > 1$ (Figure 4a); while the storage capacity distribution curve for VIC model is concave up for $\beta > 1$ and concave down for $0 < \beta < 1$ (Figure 4b). The S-shape of CDF (Figure 4a) is more significant with higher value of a (e.g., a=1.9). For a smaller value of a, the difference between the new PDF and VIC-type of model becomes smaller. The proposed distribution can fit the *S*-shape of cumulative distribution for storage capacity which is observed from soil data ~~[~~(*Huang et al.*, 2003)~~]~~, but the power distribution of VIC type of model is not able to fit the *S*-shape of CDF.

5.2. Deriving SCS-CN method from the proposed distribution function

The soil wetting and surface runoff can be computed when equation (20~~6~~) is used to describe the spatial distribution of soil water storage capacity in a catchment. The average value of storage capacity over the catchment is the mean of the distribution:

$$\mu = S_b \quad (21\cancel{7})$$

For a given C, the catchment-scale storage S can be computed by $S = \int_0^C [1 - F(x)]dx$ ~~[~~(*Moore*, 1985~~]~~). From equation (20~~6~~), we obtain:

$$S = \frac{C+S_b-\sqrt{(C+S_b)^2-2aS_bC}}{a} \tag{22~~8~~}$$

For a rainfall-runoff event, the average initial storage at the catchment scale is denoted as S_0 and the corresponding value of C is denoted as C_0. Substituting S_0 and C_0 into equation (22~~8~~), we obtain:

~~$S_0 = \frac{C_0+S_b-\sqrt{(C_0+S_b)^2-2aS_bC_0}}{a}$ (29)~~

~~Dividing S_b in both-hand sides of equation (29), we obtain:~~

$$m = \frac{\psi(2-a\psi)}{2(1-\psi)} \tag{23~~30~~}$$

where $\psi = \frac{S_0}{S_b}$ is defined in equation (12~~6~~), and $m = \frac{C_0}{S_b}$. ~~is defined as:~~

~~$m = \frac{C_0}{S_b}$ (31)~~

The rainfall in the catchment is assumed to be spatially uniform and the rainfall depth is denoted as P. If the spatial distribution of rainfall is not uniform, the method is applied to sub-catchments where the effect of spatial variability of rainfall is negligible. The average storage at the catchment scale after infiltration is computed by substituting $C = C_0 + P$ into equation (22~~8~~):

$$S_1 = \frac{C_0+P+S_b-\sqrt{(C_0+P+S_b)^2-2aS_b(C_0+P)}}{a} \tag{~~32~~24}$$

The soil wetting is computed as the difference between S_1 and S_0:

$$W = \frac{P+\sqrt{(C_0+S_b)^2-2aS_bC_0}-\sqrt{(C_0+P+S_b)^2-2aS_b(C_0+P)}}{a} \tag{25~~33~~}$$

Dividing P on the both-hand sides of equation (25~~33~~) and substituting $m = \frac{C_0}{S_b}$ ~~equation (31)~~, we obtain:

$$\frac{W}{P} = \frac{1+\frac{S_b}{P}\sqrt{(m+1)^2-2am}-\sqrt{\left(1+(m+1)\frac{S_b}{P}\right)^2-2am\left(\frac{S_b}{P}\right)^2-2a\frac{S_b}{P}}}{a} \tag{~~34~~26}$$

Substituting equation (13~~7~~) into equation (26~~34~~), we obtain:

$$\frac{W}{P}=\frac{1+\frac{\sqrt{(m+1)^2-2am}}{1-\psi}\Phi_{sc}-\sqrt{\left(1+\frac{m+1}{1-\psi}\Phi_{sc}\right)^2-2am\left(\frac{\Phi_{sc}}{1-\psi}\right)^2-\frac{2a}{1-\psi}\Phi_{sc}}}{a} \qquad (\cancel{35}\underline{27})$$

Figure 5 plots equation (~~35~~27) for ψ = 0, 0.4, and 0.6 when a = 0.6 and 1.8. As we can see, $\frac{W}{P}$ increases with a for given values of ψ and Φ_{sc}; and $\frac{W}{P}$ decreases with ψ for given values of a and Φ_{sc}, which is consistent with the VIC model and implicates that soil wetting ratio decreases with the degree of initial saturation under a storage index. As shown in Figure 5, equation (27~~35~~) satisfies the lower boundary of SCS-CN method and the upper boundary of the VIC model. Specifically, equation (27~~35~~) satisfies the following boundary conditions (see Appendix D~~C~~ for details) shown in Table 1:

$$\lim_{\Phi_{sc}\to 0}\frac{W}{P}=0 \qquad (\underline{28}\cancel{36}\text{-1})$$

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P}=\frac{\sqrt{(m+1)^2-2am}+a-m-1}{a\sqrt{(m+1)^2-2am}} \qquad (\underline{28}\cancel{36}\text{-2})$$

When the effect of initial storage is negligible (i.e., $\psi=0$), $\frac{S_b}{P}=\Phi_{sc}$ from equation (~~17~~13) and $m=0$ from equation (~~30~~23). Then, equation (27~~35~~) becomes:

$$\frac{W}{P}=\frac{1+\frac{S_b}{P}-\sqrt{\left(1+\frac{S_b}{P}\right)^2-2a\frac{S_b}{P}}}{a} \qquad (\cancel{37}\underline{29})$$

Equation (29~~37~~) is same as equation (~~1~~28) with $a=2\varepsilon(2-\varepsilon)$. We can obtain the following equation from equation (~~37~~29) (see Appendix E~~D~~ for detailed derivation):

$$\frac{Q}{P-\varepsilon W}=\frac{W-\varepsilon W}{S_b-\varepsilon W} \qquad (\cancel{38}\underline{30})$$

where εW is defined as initial abstraction (W_i) in the SCS-CN method. Since $S_b=S_p$ when $\psi=0$, equation (~~38~~30) is same as equation (2), i.e., the proportionality relationship of SCS-CN method.

Equation (27~~35~~) is derived from the VIC type model by using equation (~~26~~20) to describe the spatial distribution of soil water storage capacity. From this perspective, equation (~~35~~27) is a saturation excess runoff model. Since equation (27~~35~~) becomes the SCS-CN method when initial storage is negligible, equation (27~~35~~) is the modified SCS-CN method which considers the effect of initial storage on runoff generation explicitly. Therefore, the new distribution function represented by equation (20~~6~~) unifies the SCS-CN method and VIC type of model.

Bartlett et al. (~~{~~2016a)~~}~~ developed an event-based probabilistic storage framework including a spatial description of "prethreshold" and "threshold-excess" runoff; and the framework has been utilized for unifying TOPMODEL, VIC and SCS-CN (~~{~~*Bartlett et al.*, 2016b)~~}~~. The extended SCS-CN method (SCS-CNx) from the probabilistic storage framework is derived given the following assumptions: 1) the spatial distribution of rainfall is exponential; 2) the spatial distribution of soil moisture deficit is uniform; and 3) the spatial distribution of storage capacity is exponential. When "prethreshold" runoff is 0 (i.e., there is only threshold-excess or saturation excess runoff), the SCS-CNx method leads to the SCS-CN method without the initial abstraction term (i.e., there is no εW term in equation (~~38~~30)). In this paper, the new probability distribution function is used for storage capacity in the VIC model in which the spatial distribution of precipitation is assumed to be uniform. The obtained equation for saturation excess runoff leads to the exact SCS-CN method as shown in equation (~~38~~30).

This research started with the following research question: if the SCS-CN method is a saturation excess runoff generation model, what is the distribution function of soil water storage capacity? Wang and Tang (2014) showed that equation (~~37~~29) is derived from the proportionality relationship of SCS-CN method, i.e., equation (30~~8~~). From the comparison of boundary conditions between SCS-CN method and VIC type of model discussed in Section 4, it is observed that equation (~~37~~29) does not include initial soil water storage, and the derived one from distribution function will include initial soil water storage (e.g., equation (26~~34~~)). However, equation (~~37~~29) can be viewed as the result of $S_0 = 0$; and W for equation (~~37~~29) can be written as:

$$W = \int_0^P [1 - F(x)]dx$$ (~~39~~31)

From equation (~~37~~29), one obtains:

$$W = \frac{P + S_b - \sqrt{(S_b + P)^2 - 2aPS_b}}{a}$$ (~~40~~32)

Substituting equation (32~~40~~) into equation (31~~9~~), one obtains:

$$\frac{P + S_b - \sqrt{(S_b + P)^2 - 2aPS_b}}{a} = \int_0^P [1 - F(C)]dC$$ (33~~41~~)

Equation (~~26~~20) is obtained from equation (~~41~~33).

5.3. Surface runoff of unified SCS-CN and VIC model

From the unified SCS-CN and VIC model (i.e., equation (26~~34~~)), surface runoff (Q) can be computed as:

$$Q = \frac{(a-1)P - S_b\sqrt{(m+1)^2 - 2am} + \sqrt{[P + (m+1)S_b]^2 - 2amS_b{}^2 - 2aS_bP}}{a}$$ (~~42~~34)

The parameter m is computed by equation (~~30~~23) as a function of ψ and a. Equation (~~42~~34) represents surface runoff as a function of precipitation (P), average soil water storage capacity (S_b), shape parameter of storage capacity distribution (a), and initial soil moisture (ψ). Figure 6 plots equation (34~~42~~) under different values of P, S_b, a, and ψ. Figure 6a shows the effects of S_b and ψ on rainfall-runoff relationship with given shape parameter of a=1.9. The solid lines show the rainfall-runoff relations with zero initial storage (ψ=0); and the dashed lines show the rainfall-runoff relations with ψ=0.2. Given the same amount of precipitation and storage capacity, wetter soil (ψ=0.2) generates more surface runoff than drier soil (ψ=0); and the difference of runoff is higher for watersheds with larger average storage capacity. Figure 6b shows the effects of S_b and a on rainfall-runoff relationship with given initial soil moisture (ψ=0.2). The solid lines show the rainfall-runoff relations for a=1.9; and the dashed lines show the rainfall-runoff relations for a=1.2. As we can see, the shape parameter affects the runoff generation significantly for watersheds with larger average storage capacity.

In the SCS-CN method, surface runoff is computed as $Q = \frac{(P-0.2S_b)^2}{P+0.8S_b}$. The effect of initial soil moisture on runoff is considered implicitly by varying the curve number for normal, dry and wet conditions depending on the antecedent moisture condition. In the unified SCS-CN model shown in equation (34~~42~~), the effect of initial soil moisture is explicitly included through ψ, which is the ratio between average initial water storage and average storage capacity. In the SCS-CN method, the value of initial abstraction W_i is parameterized as a function of average storage capacity, i.e., $W_i = 0.2S_b$. In the unified SCS-CN model shown in equation (34~~42~~), W_i is dependent on the shape parameter a. Therefore, the unified SCS-CN model extends the original SCS-CN method for including the effect of initial soil moisture explicitly and estimating the parameter for initial abstraction.

6. Conclusions

In this paper, the SCS-CN method and the saturation excess runoff models based on distribution functions (e.g., VIC model) are presented in terms of soil wetting (i.e., infiltration). Like the Budyko framework, the relationship between soil wetting ratio and soil storage index is obtained for the SCS-CN method and the VIC type of model. It is found that the boundary conditions for the obtained functions do not fully match. For the SCS-CN method, soil wetting ratio approaches 1 when soil storage index approaches infinity, and this is due to the limitation of the SCS-CN method, i.e. the initial soil moisture condition is not explicitly represented in the proportionality relationship. However, for the VIC type of model, soil wetting ratio equals soil storage index when soil storage index is lower than a certain value, and this is due to the finite bound of the distribution function of storage capacity.

In this paper, a new distribution function, which is supported by $x \in [0, \infty)$ instead of a finite upper bound, is proposed for describing the spatial distribution of soil water storage capacity. From this new distribution function, an equation is derived for the relationship between soil wetting ratio and storage index, and this equation satisfies the following boundary conditions: when storage index approaches 0, soil wetting ratio approaches 0; when storage index approaches infinity, soil wetting ratio approaches a certain value (≤1) depending on the initial storage (e.g., at the beginning of a rainfall event, runoff is generated at the initially saturated areas~~, such as wetlands~~ ~~[~~(*Yu et al.*, 2001; *Gao et al.*, 2018)~~]~~). Meanwhile, the model becomes the exact SCS-CN method when initial storage is negligible. Therefore, the new distribution function for soil water storage capacity explains the SCS-CN method as a saturation excess runoff model, and unifies the SCS-CN method and the VIC type of model for surface runoff modeling.

Future potential work could test the performance of the proposed new distribution function for quantifying the spatial distribution of storage capacity by analyzing the spatially distributed soil data. On one hand, the distribution functions of probability distributed model ~~[~~(*Moore*, 1985)~~]~~, VIC model, and Xinanjiang model could be replaced by the new distribution function and the model performance would be further evaluated. On the other hand, the extended SCS-CN method (i.e., equation (27~~35~~)), which includes initial storage explicitly, could be used for surface runoff modeling in SWAT model, and the model performance would be evaluated.

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Appendix A

The potential for continuing wetting is called potential maximum retention and is denoted by $S_m = S_p - W_i$. S_m is computed as a function of curve number which is dependent on land use/land cover and soil permeability. The ratio between W_i and S_m in the SCS curve number method is denoted by $\lambda = \frac{W_i}{S_p - W_i}$, and then the ratio between initial soil wetting and storage capacity is computed by:

$$\frac{W_i}{S_p} = \frac{\lambda}{1+\lambda} \qquad \text{(A1~~7~~)}$$

The value of λ varies in the range of $0 \leq \lambda \leq 0.3$, and a value of 0.2 is usually used ~~[~~(*Ponce and Hawkins*, 1996)~~]~~. Substituting equation (A1~~7~~) into equation (6) leads to:

$$\frac{W}{P} = \frac{1 - \left(\frac{\lambda}{1+\lambda}\right)^2 \Phi_{sc}}{1 - \frac{2\lambda}{1+\lambda} + \Phi_{sc}{}^{-1}} \qquad \text{(A2~~8~~)}$$

Equation (A2~~8~~) is plotted in Figure 1 for $\lambda = 0.1$ and 0.3. As we can see, the range of Φ_{sc} is dependent on the parameter λ. Since $W_i \leq P$, Φ_{sc} is in the range of $\left[0, 1 + \frac{1}{\lambda}\right]$. Equation (A2~~8~~)

satisfies the following boundary conditions: $\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$; and $\frac{W}{P} \to 1$ as $\Phi_{sc} \to \frac{\lambda+1}{\lambda}$. When $\lambda \to 0$, equation (A2~~8~~) becomes:

$$\frac{W}{P} = \frac{1}{1+{\Phi_{sc}}^{-1}} \qquad (\mathrm{A3}\cancel{9})$$

Equation (A3~~9~~) is the lower bound for $\frac{W}{P}$ based on this parameterization scheme.

Appendix ~~A~~B

~~Since~~Substituting $W = S_1 - S_0$ into equation (16),~~,~~ wetting is computed by:

$$W = S_b\left[1 - \left(1 - \frac{P+C_0}{C_m}\right)^{\beta+1}\right] - S_0 \qquad (\mathrm{B1}\cancel{21})$$

~~From equation (21), we obtain~~The following equation is obtained by dividing P on both sides of equation (~~21~~B1):

$$\frac{W}{P} = \frac{S_b - S_0}{P} - \frac{S_b}{P}\left(1 - \frac{P+C_0}{C_m}\right)^{\beta+1} \qquad (\cancel{\mathrm{A1}}\mathrm{B2})$$

Substituting $\frac{C_0}{C_m} = 1 - \left(1 - \frac{S_0}{S_b}\right)^{\frac{1}{\beta+1}}$ into equation (B2~~A1~~), we obtain:

$$\frac{W}{P} = \frac{S_b - S_0}{P} - \frac{S_b}{P}\left(1 - \frac{P}{C_m} - \left[1 - \left(1 - \frac{S_0}{S_b}\right)^{\frac{1}{\beta+1}}\right]\right)^{\beta+1} \qquad (\cancel{\mathrm{A2}}\mathrm{B3})$$

Substituting equation (10~~4~~) into equation (B3~~A2~~),

$$\frac{W}{P} = \frac{S_b - S_0}{P} - \left(\left(\frac{S_b - S_0}{P}\right)^{\frac{1}{\beta+1}} - \frac{\left(\frac{S_b}{P}\right)^{-\frac{\beta}{\beta+1}}}{\beta+1}\right)^{\beta+1} \qquad (\mathrm{B4}\cancel{\mathrm{A3}})$$

Substituting equations (5) and (13~~17~~) into (B4~~A3~~), we obtain:

$$\frac{W}{P} = \Phi_{sc} - \left({\Phi_{sc}}^{\frac{1}{\beta+1}} - \frac{\left(\frac{\Phi_{sc}}{1-\psi}\right)^{-\frac{\beta}{\beta+1}}}{\beta+1}\right)^{\beta+1} \qquad (\mathrm{B5}\cancel{\mathrm{A4}})$$

which leads to:

$$\frac{W}{P} = \Phi_{sc}\left[1-\left(1-b{\Phi_{sc}}^{-1}\right)^{\beta+1}\right]$$ (B6~~A5~~)

where b is defined in equation (14~~8~~).

Appendix C~~B~~

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P} = \lim_{\Phi_{sc}\to\infty}\Phi_{sc}\left[1-\left(1-b{\Phi_{sc}}^{-1}\right)^{\beta+1}\right]$$ (~~B1~~C1)

The right hand side of equation (~~B1~~C1) is re-written as:

$$\lim_{\Phi_{sc}\to\infty}\Phi_{sc}\left[1-\left(1-b{\Phi_{sc}}^{-1}\right)^{\beta+1}\right] = \lim_{\Phi_{sc}\to\infty}\frac{1-\left(1-b{\Phi_{sc}}^{-1}\right)^{\beta+1}}{{\Phi_{sc}}^{-1}}$$ (C~~B~~2)

Since $\lim_{\Phi_{sc}\to\infty} 1-\left(1-\mathrm{b}{\Phi_{sc}}^{-1}\right)^{\beta+1} = 0$ and $\lim_{\Phi_{sc}\to\infty}{\Phi_{sc}}^{-1} = 0$, we apply the L'Hospital's Rule,

$$\lim_{\Phi_{sc}\to\infty}\frac{\left[1-\left(1-\mathrm{b}{\Phi_{sc}}^{-1}\right)^{\beta+1}\right]'}{\left({\Phi_{sc}}^{-1}\right)'} = \lim_{\Phi_{sc}\to\infty}\mathrm{b}(\beta+1)\left(1-\mathrm{b}{\Phi_{sc}}^{-1}\right)^{\beta}$$ (~~B3~~C3)

Since $\lim_{\Phi_{sc}\to\infty}\left(1-\mathrm{b}{\Phi_{sc}}^{-1}\right)^{\beta} = 1$, the limit for $\frac{W}{P}$ is obtained:

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P} = b(\beta+1)$$ (C~~B~~4)

Substituting equation (14~~8~~) into (~~B4~~C4), we obtain:

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P} = (1-\psi)^{\frac{\beta}{\beta+1}}$$ (C~~B~~5)

Appendix D~~C~~

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P} = \lim_{\Phi_{sc}\to\infty}\frac{1+\frac{\sqrt{(m+1)^2-2am}}{1-\psi}\Phi_{sc}-\sqrt{\left(1+\frac{m+1}{1-\psi}\Phi_{sc}\right)^2-2am\left(\frac{\Phi_{sc}}{1-\psi}\right)^2-\frac{2a}{1-\psi}\Phi_{sc}}}{a}$$ (~~C~~D1)

Multiplying $1+\frac{\sqrt{(m+1)^2-2am}}{1-\psi}\Phi_{sc}+\sqrt{\left(1+\frac{m+1}{1-\psi}\Phi_{sc}\right)^2-2am\left(\frac{\Phi_{sc}}{1-\psi}\right)^2-\frac{2a}{1-\psi}\Phi_{sc}}$ to the denominator and numerator of the right hand side, equation (~~C~~D1) leads to:

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P}=\frac{1}{a}\lim_{\Phi_{sc}\to\infty}\frac{\frac{2\sqrt{(m+1)^2-2am}}{1-\psi}\Phi_{sc}-\frac{2(m+1)}{1-\psi}\Phi_{sc}+\frac{2a}{1-\psi}\Phi_{sc}}{1+\frac{\sqrt{(m+1)^2-2am}}{1-\psi}\Phi_{sc}+\sqrt{\left(1+\frac{m+1}{1-\psi}\Phi_{sc}\right)^2-2am\left(\frac{\Phi_{sc}}{1-\psi}\right)^2-\frac{2a}{1-\psi}\Phi_{sc}}} \quad \text{(D~~C~~2)}$$

Dividing Φ_{sc} in the denominator and numerator, we obtain:

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P}=\frac{1}{a(1-\psi)}\lim_{\Phi_{sc}\to\infty}\frac{2\sqrt{(m+1)^2-2am}-2(m+1)+2a}{\frac{1}{\Phi_{sc}}+\frac{\sqrt{(m+1)^2-2am}}{1-\psi}+\sqrt{\left(\frac{1}{\Phi_{sc}}+\frac{m+1}{1-\psi}\right)^2-2am\left(\frac{1}{1-\psi}\right)^2-\frac{2a}{(1-\psi)\Phi_{sc}}}} \quad \text{(~~C~~D3)}$$

Therefore, the limit of $\frac{W}{P}$ as $\Phi_{sc}\to\infty$ is:

$$\lim_{\Phi_{sc}\to\infty}\frac{W}{P}=\frac{\sqrt{(m+1)^2-2am}+a-m-1}{a\sqrt{(m+1)^2-2am}} \quad \text{(~~C~~D4)}$$

Appendix E~~D~~

Substituting $a=2\varepsilon(2-\varepsilon)$ into equation (~~37~~29), one can obtain:

$$\frac{W}{P}=\frac{1+\frac{S_b}{P}-\sqrt{\left(1+\frac{S_b}{P}\right)^2-4\varepsilon(2-\varepsilon)\frac{S_b}{P}}}{2\varepsilon(2-\varepsilon)} \quad \text{(E~~D~~1)}$$

Equation (~~D1~~E1) is the solution of the following quadratic function:

$$\varepsilon(2-\varepsilon)\left(\frac{W}{P}\right)^2-\left(1+\frac{S_b}{P}\right)\frac{W}{P}+\frac{S_b}{P}=0 \quad \text{(~~D~~E2)}$$

Multiplying P^2 at the both-hand sides of equation (~~D~~E2), equation (E~~D~~2) becomes:

$$\varepsilon(2-\varepsilon)W^2-(P+S_b)W+S_bP=0 \quad \text{(~~D~~E3)}$$

Equation (~~D~~E3) can be written as the following one:

$$\frac{P-W}{P-\varepsilon W}=\frac{W-\varepsilon W}{S_b-\varepsilon W} \quad \text{(~~D~~E4)}$$

Substituting $Q=P-W$ into equation (~~D~~E4), we obtain the proportionality relationship of SCS-CN method:

$$\frac{Q}{P-\varepsilon W}=\frac{W-\varepsilon W}{S_b-\varepsilon W} \quad \text{(E~~D~~5)}$$

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Figure captions:

Figure 1: Wetting ratio $\left(\frac{W}{P}\right)$ versus soil storage index $\left(\frac{S_p}{P}\right)$ from the SCS-CN method based on two parameterization schemes: $\lambda = \frac{W_i}{S_p - W_i}$ (scheme 1) and $\varepsilon = \frac{W_i}{W}$ (scheme 2).

Figure 2: The impact of β and the degree of initial storage ($\psi = S_0/S_b$) on soil wetting ratio (W/P).

Figure 3: The probability density functions (PDF) with different parameter values: (a) the proposed PDF represented by equation (24); and (b) the power distribution of VIC model, i.e., equation (25).

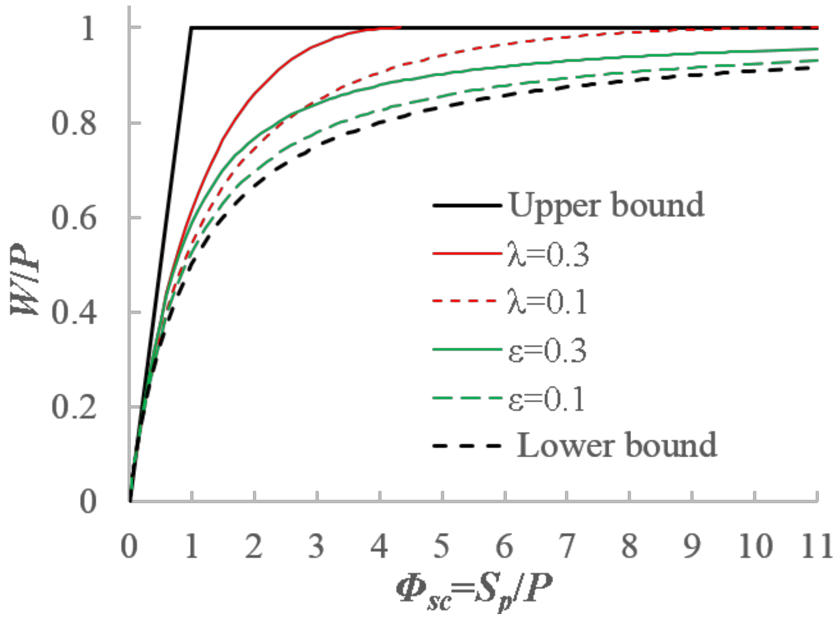
Figure 4: The cumulative distribution functions (CDF) with different parameter values: (a) the proposed distribution function represented by equation (26); and (b) the power distribution of VIC model represented by equation (13).

Figure 5: The effects of the degree of initial storage (ψ=0, 0.4, and 0.6) and shape parameter (a=0.6 and 1.8) on soil wetting in the modified SCS-CN method derived from the proposed distribution function for soil water storage capacity.

Table 1: The boundary conditions of the functions for relating wetting ratio $\left(\frac{W}{P}\right)$ to soil storage index (Φ_{sc}): 1) the SCS-CN method; 2) the VIC type of model; and 3) the modified SCS-CN method based on the proposed new distribution for VIC type of model.

~~Event Scale Model~~	~~Lower Boundary Condition~~	~~Upper Boundary Condition~~
~~SCS-CN, parameterization of initial wetting,~~ $\cancel{\varepsilon = \frac{W_t}{W}}$	$\cancel{\frac{W}{P} \to 0 \text{ as } \Phi_{sc} \to 0}$	$\cancel{\frac{W}{P} \to 1 \text{ as } \Phi_{sc} \to \infty}$
~~Power function for storage capacity distribution (VIC type of model)~~	$\cancel{\frac{W}{P} = \Phi_{sc} \text{ when } \Phi_{sc} \le a}$	$\cancel{\frac{W}{P} \to (1-\psi)^{\frac{\beta}{\beta+1}} \text{ as } \Phi_{sc} \to \infty}$
~~Modified SCS-CN method based on the proposed distribution for storage capacity~~	$\cancel{\frac{W}{P} \to 0 \text{ as } \Phi_{sc} \to 0}$	$\cancel{\frac{W}{P} \to \frac{\sqrt{(m+1)^2-2am}+a-m-1}{a\sqrt{(m+1)^2-2am}} \text{ as } \Phi_{sc} \to \infty}$

Surface Runoff Model	**Parameters**	**Lower Boundary Condition**	**Upper Boundary Condition**
SCS-CN, parameterization of initial wetting	S_p, ε	$\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$	$\frac{W}{P} \to 1$ as $\Phi_{sc} \to \infty$
Power function for storage capacity distribution (VIC type of model)	C_m, β	$\frac{W}{P} = \Phi_{sc}$ when $\Phi_{sc} \le b$	$\frac{W}{P} \to (1-\psi)^{\frac{\beta}{\beta+1}}$ as $\Phi_{sc} \to \infty$
Modified SCS-CN method based on the proposed distribution for storage capacity	S_b, a	$\frac{W}{P} \to 0$ as $\Phi_{sc} \to 0$	$\frac{W}{P} \to \frac{\sqrt{(m+1)^2-2am}+a-m-1}{a\sqrt{(m+1)^2-2am}}$ as $\Phi_{sc} \to \infty$



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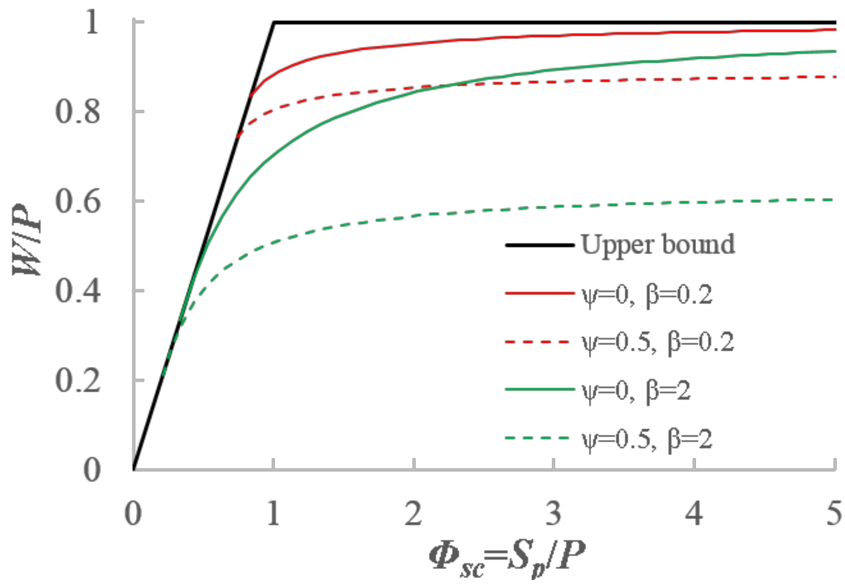
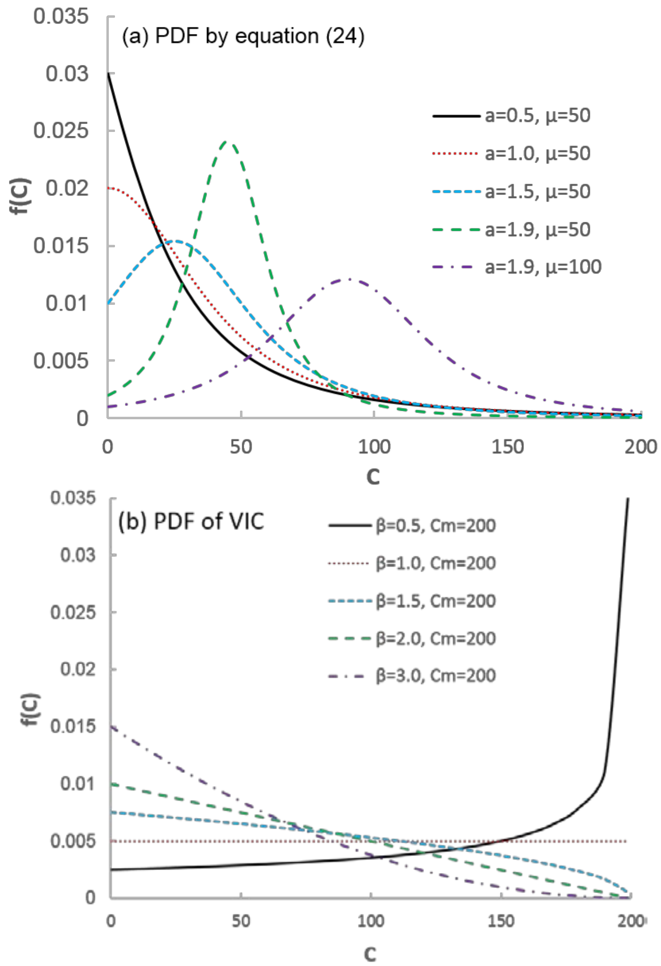


Figure 2: The impact of β and the degree of initial storage ($\psi = S_0/S_b$) on soil wetting ratio (W/P).



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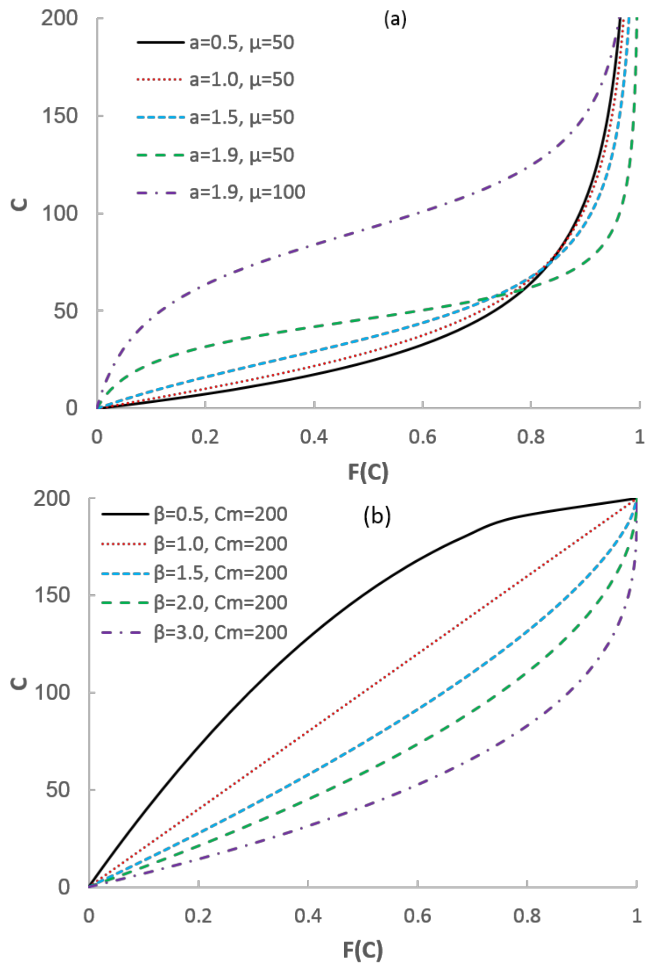

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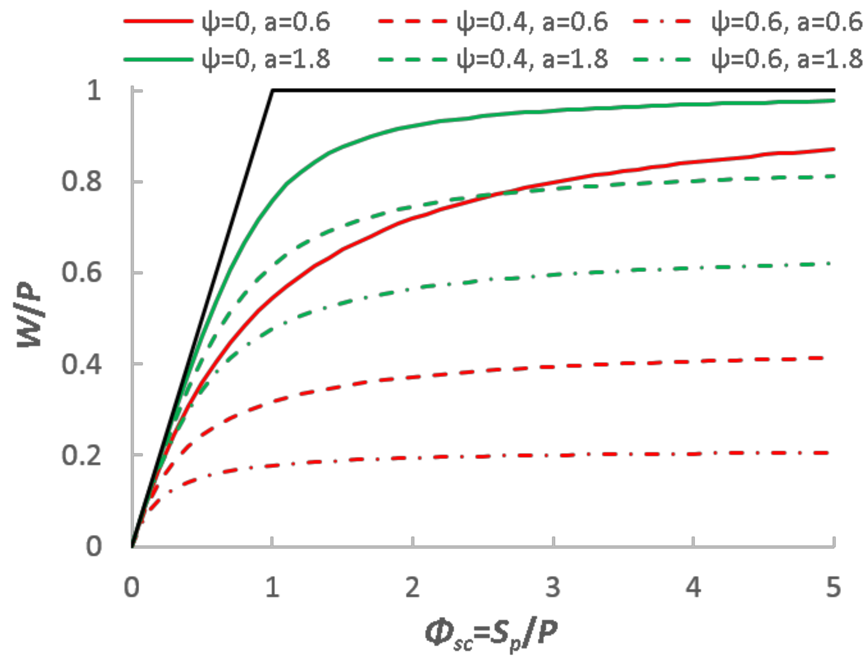

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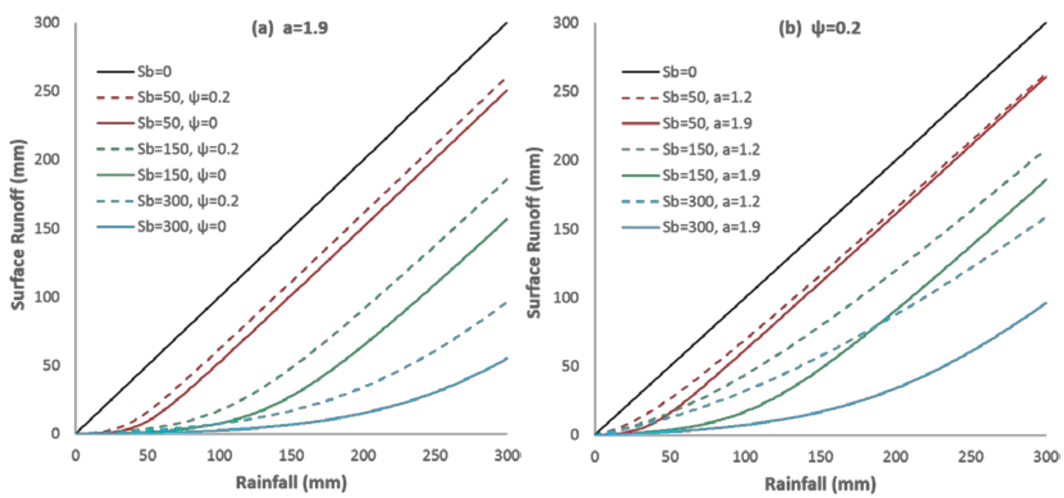

Figure 6: (a) The effects of average storage capacity and initial storage on rainfall-runoff relation; and (b) The effects of average storage capacity and shape parameter on rainfall-runoff relation.