

My thanks to Referee #4 for taking time to carefully work through the manuscript, and the many constructive comments and reference suggestions.

I do not believe I am much at odds with the comments overall, but offer some responses on specific aspects below.

... but I am not sure I really understand the contribution of the manuscript given that this issue has been recognized in the literature and even the author claims that a more correct form of the distribution may not result in different hydrological conclusions.

The Referee comment here about “recognized in the literature” is of interest. Referee #1 also notes that there is widespread awareness of $f(0)>0$ being incorrect and Referee #2 points out that $f(0) = 0$ is not really new. It should follow then that there is widespread awareness by authors of the incorrectness of any analytical derivation they might publish which leads to L-shaped transit time distributions. However, I know of no published paper which presents an L-shaped transit time distribution where the mathematical expression is followed by a statement to the effect that “this analytical solution is incorrect for small t because everyone knows that $f(0) = 0$.”

There is actually no grey area here. It is either possible or it is not possible to derive L-shaped transit time distributions via analytical argument. I would certainly hope that that HESS agrees with the latter. However, it is also not really desirable for the submitted paper to be entirely negative in tone, however correct it may be. By way of inserting a more positive note, some alternative distributions are tentatively suggested later as part of the further responses to Referee #4. Hopefully these will go some way to offset the criticisms of Referee #2 and Referee #3 about the lack of suggested alternatives.

Considering now the above Reviewer #4 comment with respect unchanged conclusions. It is entirely possible, perhaps even expected, that use of distributions of correct form will result in no change in hydrological conclusions. However, this is not an argument for accepting continued use of L-shaped distributions.

The reasons are twofold. Firstly, given equal degree of matching to data, it is better science to utilise a distribution form which is not in contradiction to what we know to be the true form of the early time portion of transit time distributions. After all, why would we knowingly choose an incorrect form? The philosophy is well summarised by J.F.C. Kingman in his comment contribution following Folks and Chhikara (1978):

“Although it is often possible to justify the use of a distribution empirically, simply because it appears to fit the data, it is more satisfactory if the structure of the distribution reflects plausible features of the underlying mechanism.”

Secondly, continued use of the L-shaped transit time distribution concept is not desirable from the hydrological viewpoint because there are some potentially useful questions which do not get to be asked. In particular, given that $f(t) = 0$, what is the lag time to first mode? (This assumes that transit time distributions are “smooth” to the extent that the first and second derivatives are continuous so modes are defined.) Is the mode always so close to zero that the lag time can be neglected, or could the lag time be of practical importance? Perhaps there is a distribution of times to first mode, reflecting the time-varying nature of transit time distributions. The mean and variance of such a distribution might be of value as descriptors of the early-time behaviour of transit time distributions from a given hydrological system. It may happen that mixing processes often allow so little distribution information to pass through the system that such questions will seldom be answered satisfactorily. However, if transit time characterisations are so poor, can we be so sure that the first mode is in fact very close to zero? Put another way, how large might a model’s lag time be before data fit starts to fall away? Perhaps looking at tracer outputs from physical catchment model results could be of help here.

Technically, the author’s primary point about the incorrectness of the early-time behavior of the transit time probability distribution is fine, especially for stationary distributions of a generally smoothed shape.

I doubt that it would ever be possible to have stationary transit time distributions in reality, but the stationary or non-stationary nature of tracer transit time distributions is of no consequence to the discussion. We can conceptualise a tracer transit time distribution being defined from an instantaneous uniform spatial scattering of tracer into a catchment, and recording the exit time of the particles as they depart. The experiment might then be repeated at any later time (including immediately after), which would result in a different tracer transit time distribution. However, both distributions would have the property $f(0) = 0$, which in both cases would negate the

possibility of L-shaped transit time distributions as defined here. “Smoothed” shapes are not a strict requirement for defining L-shaped distributions as referenced in the paper. The first derivative must be always negative for $t \geq 0$, but it could be discontinuous.

... but moreover, he fails to extend his analysis to understanding the actual shape of transit time distributions.

This is far beyond the scope of a short technical note with an argument themed against L-shaped transit time distributions. The paper makes no pretence to suggest analysis leading to deeper understanding of transit time distributions forms as we move further away from $t = 0$. The Referee is no doubt in a much better position to contribute to that.

Again, this short technical note is not incorrect, but I don't know how valuable or impactful it will be, particularly without recognizing the recent literature on dynamic transit time distributions and the fact that actual transit times may not be smooth shaped functions or parametrically characterized.

Dynamic transit time distributions are something of a distraction because whether tracer transit time distributions are dynamic or otherwise is not of relevance for an argument against L-shaped transit time distributions. All that matters is that all transit time distributions must have some form, and an L-shaped form is not one of the available options. The paper is only concerned with an argument against the use of L-shaped forms as defined in the paper. Fractal situations and the like are outside the scope of the paper.

The point is noted, however, with respect to impact and relevance. This is linked to the need to make some contribution toward alternatives. So while the main message of the paper is directed against use of L-shaped distributions, a suggestion is now made here of alternatives. That is, 1- or 2-parameter distributions of correct form with $f(0) = 0$ which might serve to replace L-shaped transit time distributions with one or two parameters. There is obviously no specific “best” replacement distribution that can be readily defended. However, it is suggested that the inverse Gaussian distribution should be given consideration as alternative for both 1- and 2-parameter L-shaped distributions, discussed further now.

The inverse Gaussian distribution has origin as a first passage time distribution and hence has $f(0) = 0$. It has some history as a transit time distribution in hydrology, dating back to the inverse Gaussian mixture model of Kirchner et al (2001), though it was not mentioned explicitly as such in that paper.

The inverse Gaussian distribution can be written in various parameterisations, the utilised form here is in terms of its mean μ and shape parameter ϕ :

$$f(t) = \left(\frac{\mu\phi}{2\pi t^3} \right)^{1/2} e^{\phi} \exp\left(\frac{1}{2}\phi \left(\frac{t}{\mu} + \frac{\mu}{t} \right) \right) \quad (1)$$

Considering first an alternative to exponential transit time distributions, a specific inverse Gaussian distribution is suggested where the shape parameter is fixed at 1. This gives the 1-parameter probability density function:

$$g(t) = \left(\frac{\mu}{2\pi t^3} \right)^{1/2} \exp\left(1 - \frac{1}{2} \left(\frac{t}{\mu} + \frac{\mu}{t} \right) \right) \quad (2)$$

This distribution has similarities to the exponential distribution except for t near zero (Fig 1). It could be useful in fact to have Fig. 1 here replace Fig. 1 in the submitted paper.

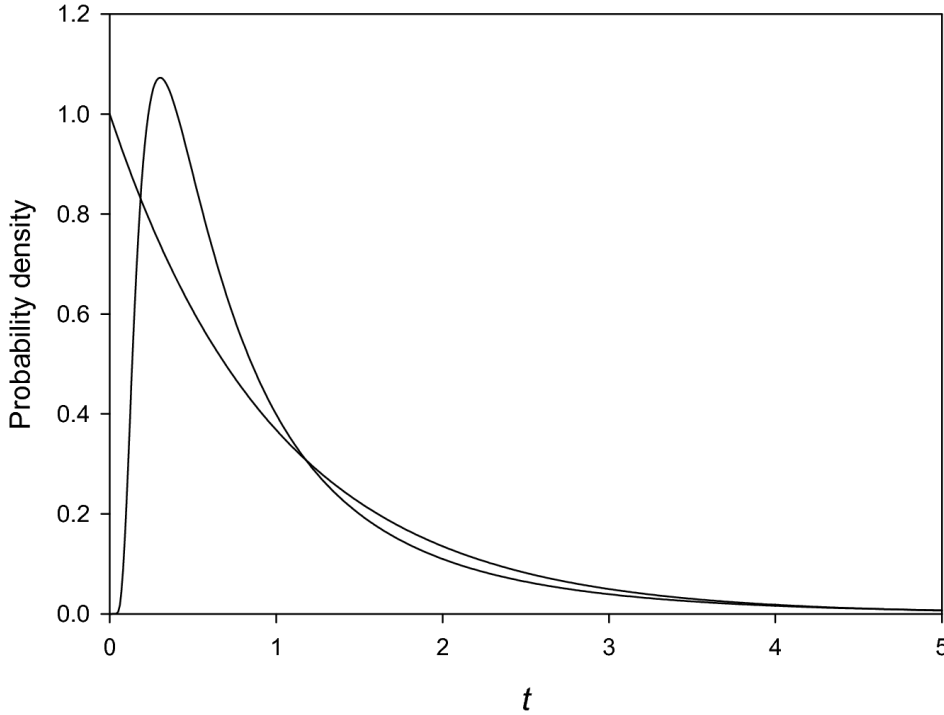


Figure 1: Comparison between the exponential distribution and the inverse Gaussian distribution as given by Eq. (2). Both distributions have mean value = 1.0.

Referee #2 suggests that when we only have confidence to estimate the mean of the distribution, then we should take that mean as being the mean of an exponential distribution. However, the exponential distribution and $g(t)$ here are both defined only by their mean values. Given the choice between single parameter distributions of correct and incorrect forms, it would seem better to select the former when other things are equal.

The other L-shaped distribution considered in the submitted paper is that given by Eq. 11 of Kirchner et al (2001). That distribution was obtained as the additive effect of a multitude of inverse Gaussian distributions with distribution origins distributed uniformly along a slope leading down to a river channel. This gives an L-shaped mixed inverse Gaussian distribution with the river acting as an absorbing barrier. This mixture distribution was referenced by Kirchner et al (2001) as giving some degree of support for L-shaped gamma transit time distributions. As it happens, Eq. 11 of Kirchner et al (2001) is an incorrect analytical expression for transit time distributions because the derivation incorporates particles already at the observation point at time zero, and are not part of the transit time distribution. Its mathematical expression cannot therefore be used as support for L-shaped gamma transit time distributions.

As noted by Reviewer # 3, a focus of the Kirchner et al (2001) paper was also for giving an explanatory model of long-tailing behaviour. The explanation was essentially topographical, with those transit time distributions originating further up the slope segment creating an extended right tail of the inverse Gaussian mixture distribution, as illustrated in Fig. 3 of that paper.

An alternative highly simplified model is proposed here, with some limited degree of hydrological connection. It enables long-tailing to be obtained via a single inverse Gaussian transit time distribution, avoiding L-shaped forms. The simple conceptual model is described below.

A tracer particle is deposited at some point on a slope segment and starts a random walk toward a river channel, which is some finite distance away and acts as an absorbing barrier. Whenever the particle is on the land surface there is a probability p that it will make an incremental movement down the slope toward the stream. There is also a probability q that it will instead make an incremental vertical movement downward from the surface into the subsurface beneath its current surface location. Whenever the particle is in the subsurface there is probability p that it will make an incremental movement vertically upward and probability q that it will make an incremental movement vertically downward. Define $p + q = 1$, so the particle is always moving.

It is recognised that this model does contain multiple hydrological disconnects: horizontal movement on the ground surface is the same as vertical movement in the subsurface, a particle can only return to the ground surface at the point where it last entered the subsurface, and no allowance is made for instantaneous multiple starting points along the slope as in the Kirchner et al (2001) model. However, for all its shortcomings the model yields topographically independent long tailing without needing L-shaped transit time forms. It is introduced here only in the spirit of suggesting an alternative to the L-shaped transit time form derived in Kirchner et al (2001). There is no suggestion that it offers in any way a better approximation to actual catchment processes. In this regard, Scher et al (2002) give a detailed discussion of a continuous time random walk model leading to long-tailing of catchment transit time distributions.

For $p > q$ the model described above is analogous to Brownian motion with drift and the transit time distribution to the channel is an inverse Gaussian distribution. However, as noted in the response to Referee #2, there is no implication that particle movement is Fickian in reality. Presumably Kirchner et al (2001) also did not envisage Fickian movement when they proposed their inverse Gaussian mixture model.

As p moves toward 0.5, implying frequent and deep particle sojourns into the subsurface, the inverse Gaussian distribution shifts toward heavy-tailed forms with small values of the shape parameter ϕ . The special case of $p = 0.5$ is the drift-free situation and can be obtained as the limit distribution $h(t)$, which arises by holding the inverse Gaussian mode constant at some value ξ and having $\phi \rightarrow 0$. This gives the 1-parameter transit time distribution:

$$h(t) = \left(\frac{3\xi}{2\pi t^3} \right)^{1/2} \exp\left(-\frac{3\xi}{2t} \right) \quad (3)$$

with cumulative distribution function:

$$H(t) = 2\Phi\left(-\left(\frac{3\xi}{t} \right)^{1/2} \right) \quad (4)$$

where Φ is the standard normal integral.

Figure 2 gives a sense of the inverse Gaussian distribution shifting its shape progressively toward the heavy-tailed $h(t)$ form as ϕ decreases.

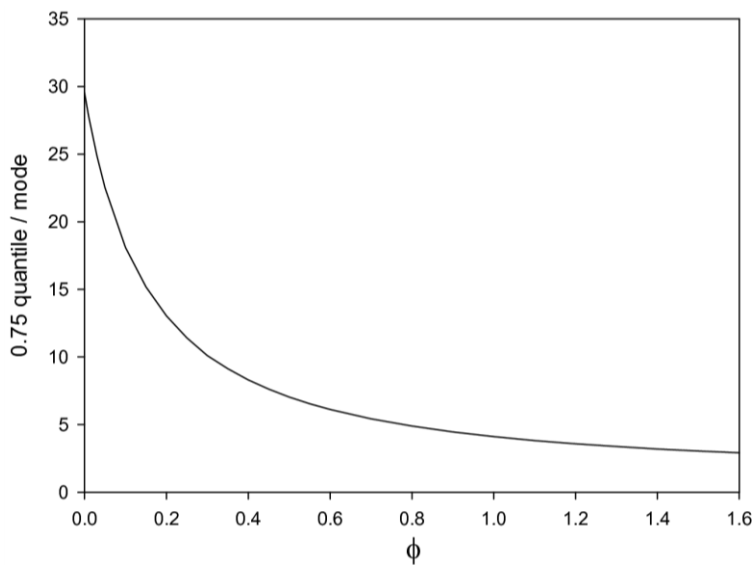


Figure 2: long-tailing behaviour of the inverse Gaussian distribution as a function of the shape parameter ϕ . The vertical axis gives the value of the distribution 0.75 quantile, expressed as multiples of the modal value. The value for $\phi = 0$ is obtained from Eq. (4).

Kingman in his comment contribution following Folks and Chhikara (1978) speculates as to possible practical application of $h(t)$. It would be interesting if after this long period of time it found some use in hydrological transit time studies, though its application might be restricted by not having defined moments. For a single point pulse of tracer its presence would be suggested by the log of tracer concentration for large t declining linearly as a function of the log of time, with gradient $-3/2$.

All the above comments associated with Eqs (1) – (4) are additional to the content of the original submitted paper. My apologies for introducing new material at this late stage in the discussion period, which the Editor may wish to extend a little to allow time for some further referee input.

I would consider revising the title

This point was also raised by Referee #1 and Referee #2. However, I wish to keep the title (if the paper is accepted) because it does describe exactly the subject matter of the paper – which will be immediately apparent in any case from the abstract. As the Referee notes, the term “L-shaped” is not unknown in the literature, along with other various letters of the alphabet for different distribution forms.

P2, L12: Yes, but at $f(0)$, the probability could also simply be low (i.e., not zero) particularly if there are some parcels of water with almost instantaneous exit like those falling on the recorder.

This comment is also reflected the Referee #2 comment that it is correct in the Kirchner (2001) model to include the river in the integral, and that rain may fall into the river.

Referee #3 is also happy with the concept that there can be tracer particles precisely at the measuring point at $t=0$ and that these particles contribute to the transit time distribution.

Referee #5 seems to concur also, with what appears to be a reference to simultaneous input and output.

And all four referees are incorrect. Rather than going through the variations on the theme one at a time, a single thought experiment is proposed which captures their arguments, following from the submitted paper. A perfect observation device records the arrival times of river-transported tracer particles which transit to the device. Of course, any tracer particles “at” the tracer device at time zero are still recorded, so the device records non-zero tracer particle frequency at exactly time zero. However, any tracer particles exactly at the observation device at exactly time zero are not part of the transit time distribution because they have not transited to the device and therefore $f(0) = 0$ because particles cannot move at infinite speed to the recorder. The particles which were “at” the device at time zero are simply measurement noise and no more part of the river system transit time distribution than tracer particles from a raindrop which happened to be sitting on the recorder at time zero. The discussion is analogous to considering the distribution of first passage times of a random walk in the presence of an absorbing barrier. Whatever else we may do, we are not permitted to initiate a random walk from within the absorbing barrier – because then the random walk could not exist. The point here is that the hydrological sciences, and other scientific disciplines, are not free to make up their own rules for the mathematical properties of first passage time distributions.

P6, L9-10: A real problem here is that the author is ignoring a significant body of work and much of the latest work on catchment transit time where the community seems to be avoiding parametric distributions in the first place. A good example is the storage selection approach (e.g., see Rinaldo et al. 2015). There are fewer and fewer papers in the literature using parametric, stationary distributions.

Not so much ignoring it perhaps but the comment should be made in the paper along the lines that when parametric distributions are being considered etc..... and recognising that their use is declining. However, the argument in the paper makes no assumption concerning whether transit time distributions happen to be stationary or not. Also, while parametric distribution use may be declining it would seem from the comments of Referee #2 that the exponential distribution at least looks to continue as the null transit time distribution with noisy data.

Finally, I have not acknowledged one at a time the many helpful comments and corrections made by the Referee. In the event of the paper advancing further they all will of course be noted and responded to with notification to the Editor.

Folks, J.L., Chhikara, R.S., 1978. The inverse Gaussian Distribution and its statistical application--A review. *Journal of the Royal Statistical Society. Series B*, 40, p.263-289.

Scher, H. et al. 2002. The dynamical foundation of fractal stream chemistry: The origin of extremely long retention times. *Geophysical Research Letters*, 29, (5), 1061, 10.1029/2001GL014123