Response to the Editor's response posted on February 2, 2018.

Editors' comment

Editor Decision: Publish subject to revisions (further review by editor and referees) *(02 Feb 2018) by Sabine Attinger*

Comments to the Author:
Dear authors,

Please address the comments the reviewer gave and then reviewer 1 and I will have another look to the revised paper.

Best regards
Sabine

Authors' reply

We would like to thank the Editor for her response. We provide below our point-by-point responses to all the referees and we specify the specific changes made as well as their page and line numbers based on the revised manuscript; these page and line numbers will be shown in an orange font color. We also provide a marked-up version of our manuscript at the end of this document showing all the changes made in the revised document.

Response to the comments of Anonymous Referee #1 published on August 31, 2017 concerning the manuscript with reference number: hess-2017-394.

We would like to thank Referee #1 for his/her insightful feedback. Our responses to the specific points raised by the referee are provided below. Please note that the referee's comments will be presented in italics, preceded by a "**C**", while the corresponding authors' responses will be presented in normal typeface with a blue font, preceded by an "**R**". For some responses, the text which was changed or added to the manuscript (based on suggestions from the referee comments) is quoted and placed under "Specific author changes". Please note that the pages and line numbers provided in this document in an orange font color are from the revised version of the manuscript.

General Comments

C1: *The paper compares the numerical results of PDFs of the state variables of an unsteady open-channel flow to Monte Carlo reference simulations. The PDF equations are derived in a companion paper. The mean discharge, the mean flow velocity, the mean flow depth, and the PDFs of the discharge are compared in detail.*
The comparison of the numerical results of the method derived in the companion paper is of interest, but some important points are missing, which will be addressed in the comments below.

R1: We would like to thank the referee for their review, and we will be addressing the specific points in our responses below.

Specific Comments

C2: *Please shorten your manuscript. It has too many repetitions and some statements are obvious.*

R2: Following the referee's suggestion, we have worked on adjusting the manuscript to remove any repetitions and obvious statements. On the other hand, we would like to note that we have added some text in several places of the manuscript as a response to some of the comments from the two Referee Comments and the Short Comment posted for this manuscript.

C3: *In the abstract, on l. 1, you state that you use a "newly proposed Fokker-Planck Equation (FPE) methodology", whereas on p. 2, l. 32, you state that such a methodology has been applied many times. Please be more specific about what exactly is new about your methodology.*

R3: After rereading the abstract and the introduction with the referee's point in mind, we understand how the writing may have been unclear concerning the novelty of the methodology being used. In fact, what was meant by the "methodology" mentioned on Page 2: Line 32 (of the original manuscript) was the "technique" developed by Kavvas (2003). This technique has been applied to other processes with different governing equations where FPEs specific to those processes were obtained and applied successfully. However, to the authors' knowledge, the technique had never been applied to the Saint-Venant equations to tackle the unsteady open-channel flow problem. As such, the novelty of the proposed FPE methodology that was developed in the companion paper was to figure out how to apply the Kavvas (2003) technique to the Saint-Venant equations, and then to go forth with developing the

FPE that is specifically for the stochastic unsteady open-channel flow process, which has not been developed before. As such, changes have been made to the abstract as well as to the second half of the introduction in order to clarify this matter to the reader.

Specific author changes

In the abstract, the second sentence now reads as follows (Page 1: Lines 9-10)

> *"This methodology computes the ensemble behavior and variability of the unsteady open-channel flow process by directly solving for its time-space evolutionary probability distribution."*

The second half of the introduction has been considerably adjusted, and now reads as follows (Page 2: Line 19 to Page 3: Line 7)

> *" In order to circumvent having to solve the Saint-Venant equations repeatedly for a large number of times, this study uses a new methodology that solves for the time-space evolutionary probability distribution of the unsteady open-channel flow process in only one simulation. From this probabilistic solution, one can then obtain the ensemble mean and ensemble variance of the process as they evolve in time and space. This new methodology is proposed, explained, and derived in the companion paper by Dib and Kavvas (2017), which makes use of the ensemble averaging technique developed in Kavvas (2003) to obtain a Fokker–Planck Equation (FPE) that specifically describes an unsteady open-channel flow process. Some other hydrologic processes have been successfully simulated by following a similar procedure, which involved applying the ensemble averaging technique of Kavvas (2003) to their corresponding governing equations and obtaining the FPEs specific to their case. These include: unsaturated water flow (Kim et al., 2005a), root-water uptake (Kim et al., 2005b), solute transport (Liang and Kavvas, 2008), snow accumulation and melt (Ohara et al., 2008), unconfined groundwater flow (Cayar and Kavvas, 2009a, b), as well as kinematic open-channel flow (Ercan and Kavvas, 2012a, b).*
>
> *Note that in addition to producing the statistical properties in a computationally efficient manner through one simulation, the FPE methodology developed for the unsteady open-channel flow process directly solves for, and is linear in, the probability density of the dependent variables. Moreover, while this methodology assumes a finite correlation time for the considered process, it does not make any linearization assumptions and it does not have limitations on the working range of the parameter space.*
>
> *Therefore, in the wake of the preceding discussions, the first objective of this study is to use the FPE methodology derived in the companion paper for the unsteady open-channel flow process (Dib and Kavvas, 2017) and to apply it to a representative stochastic unsteady open-channel flow problem in order to solve for the probability density of the state variables of the flow process and to provide a quantitative description of the expected behavior and variability of the system in one single simulation. The second objective is to evaluate the performance of the proposed methodology and to validate its results by comparing the statistical properties of the flow variables computed by the FPE methodology against those calculated by the MC approach."*

C4: *Please provide more information about the numerical setup of the MC simulations, like details about the finite-difference scheme, time step, grid size, parallelisation, ...*

R4: Following the referee's suggestion, additional information has been provided regarding the MC simulations.

Specific author changes

Technical information has been added directly before Section 3.1 (Page 6: Lines 9-11):

> *"All simulations for the MC approach and the FPE methodology were run on a computer having 16 GB of RAM and an Intel i7 processor with four cores, each core having a base frequency of 2.40 GHz and a maximum frequency of 3.40 GHz."*

The below information has been added (Page 7: Lines 1-8):

> *"For this study, the characteristic form of the Saint-Venant equations was discretized in an explicit manner by substituting the time derivatives with their first order finite-difference forms, as detailed in several references, e.g., Viessman et al. (1977) and Sturm (2001). The values of the dependent variables at the new time steps were computed at the points of intersection of the positive and negative characteristic curves, which rendered the final solution on an irregular x–t grid. The solution was then interpolated onto a rectangular grid, with a Δx of 75 m and a Δt of 3 min. The simulations were performed using a parallelized process which optimized the computational time by running the simulations over the total number of available cores (with no hyperthreading)."*

C5: *The paper will benefit from a plot of the standard deviation of the discharge dependent on the position and the time, analogue to figure 1. Please add such a plot*

R5: The referee's suggestion was followed as detailed below.

Specific author changes

The requested plot has been added to the manuscript and denoted as Figure 8, and it is as follows:

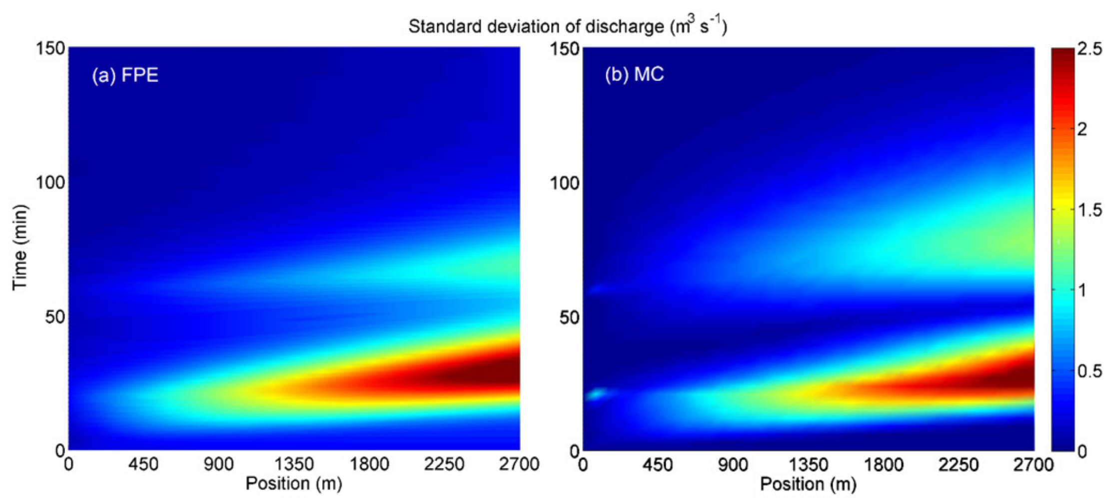

Figure 1. Standard deviation of flow discharge over channel position and time obtained by (a) the FPE methodology and (b) the MC approach.

Moreover, a discussion regarding this figure has been added in the Discussion section (Page 10: Lines 10-17):

> *"Figure 8 shows a comparison of the standard deviation of the flow discharge over space and time as computed by the FPE methodology and by the MC simulations. Both plots of this figure reveal that the standard deviation experiences two triangular areas of high values, the earlier in time being generally higher than the later, and both areas showing a stronger intensity further downstream. While the general resemblance of the FPE plot to the MC plot is good, the second area of elevated standard deviations is more compact in the FPE results and shows slightly lower values than those of the MC results. In an attempt to study such differences more closely, cross sections of both plots from Fig. 8 at specific times and at specific channel locations were compared individually (Figs. 9 and 10), in a manner similar to the ensemble average plots discussed previously."*

C6: *Your choice for 1000 Monte Carlo realisations seems arbitrary. This is especially problematic when comparing the computational times of the MC simulations and the FPE simulations. Please add a comparison of MC simulations with fewer realisations, resulting in about 7 hours of computational time, like the FPE simulations, to the MC results with 1000 realisations. If one is only interested in the ensemble averages, it is very likely that fewer ensemble members will be sufficient for accurate results. This implies that you compare the results for the ensemble averages and also for the standard deviations with 1000 and fewer realisations. Maybe also add the results for 500 realisations. Furthermore, MC simulations are predestined for parallelisation. Neither did you write if the MC simulations where computed in parallel, nor did you incorporate this in your comparison of the computational times between the two different approaches. How difficult would it be to parallelise the numerical scheme of the FPE method?*

R6: While it is true that a lower number of Monte Carlo (MC) realizations may be sufficient for accurately calculating the ensemble average of the process, this study is concerned with comparing not only the ensemble averages but also the standard deviations in order to determine the variability of the system considered. As such, the number of MC realizations must be large enough to numerically approximate, with sufficient accuracy, the second moment of the stochastic quantities for the problem at hand.

To discuss this issue, and following the referee's comment above, we ran the MC simulations with 50, 100, 200, and 500 realizations and we compared the flow discharge ensemble averages and standard deviations against those obtained from our 1000-realization run used in the manuscript. We plotted the percent relative differences of these simulations as compared to the results obtained from the 1000 realizations (Figure 2 and Figure 3 below).

From Figure 2, it is clear that the number of MC realizations affects the accuracy of the computed ensemble average flow discharge, but not too significantly (among the realizations shown). While increasing the number of realizations provides greater agreement with the results of the 1000-realization run, the relative difference was still low, even for the 50-realization run (a maximum of around 2.8%). Therefore, the number of realizations required for comparing the ensemble averages can be much smaller than 1000, with not much loss of accuracy.

However, the same cannot be said regarding the standard deviations of the flow discharge. In fact, it is very clear in Figure 3 that the number of realizations is extremely important in determining the accuracy of the standard deviation results. When compared to the 1000-realization run, the standard deviations show absolute differences that reach or exceed 65% for 50 realizations, 35% for 100 realizations, 20% for 200 realizations, and 15% for 500 realizations. Therefore, while 1000 MC realizations may seem like a large number at first, the comparisons here show that this number of realizations is necessary to produce a sufficiently accurate computation of the standard deviation for the problem at hand.

As for the referee's comment regarding parallelization, we would like to note that the 1000-realization MC run simulated for this study was parallelized and run over four cores (with no hyperthreading), thus noticeably reducing the computational time as compared to an un-parallelized run. With such parallelization, the timing of the MC run with 1000 realizations was almost around 2.5 days, as stated in the manuscript. Using the same parallelization, we found that a MC run with 100 realizations took around 7 hours, which is similar to the time taken by the FPE methodology. However, as seen in the previous discussion, 100 realizations are not enough to provide sufficiently accurate results for the standard deviations, and thus we believe it would not be of much benefit to add these results to the comparison against the FPE methodology. Moreover, including in the manuscript such a comparison, as well as a comparison against 500 realizations, may cause a digression from the main idea of the manuscript, which is to gauge the performance of the FPE methodology. Therefore, we believe it would be preferable not to add such comparisons; however, we added some text informing the reader of the parallelization of the MC simulations, as we detailed in our reply **R4** above, as well as to clarify our choice of 1000 simulations in the manuscript (the added paragraph is quoted at the end of this reply).

Finally, concerning parallelizing the numerical scheme of the FPE methodology, the following may be said. If we observe the computational times of the implicit numerical solution of the FPE methodology, the portion of the code requiring the largest time turns out to be the filling out of the coefficient matrix, especially when the discretizations for α and β are small. Therefore, parallelizing this portion of the code may allow one to reduce the computational time of this method. We believe that the difficulty in such parallelization is not too high. After deciding on the number of cores over which the parallelization will occur, one can adjust the code portion containing the loops that fill out the coefficient matrix in a way that divides the loops onto the different number of cores. As such, the coefficient matrix will be filled out in a shorter period of time.

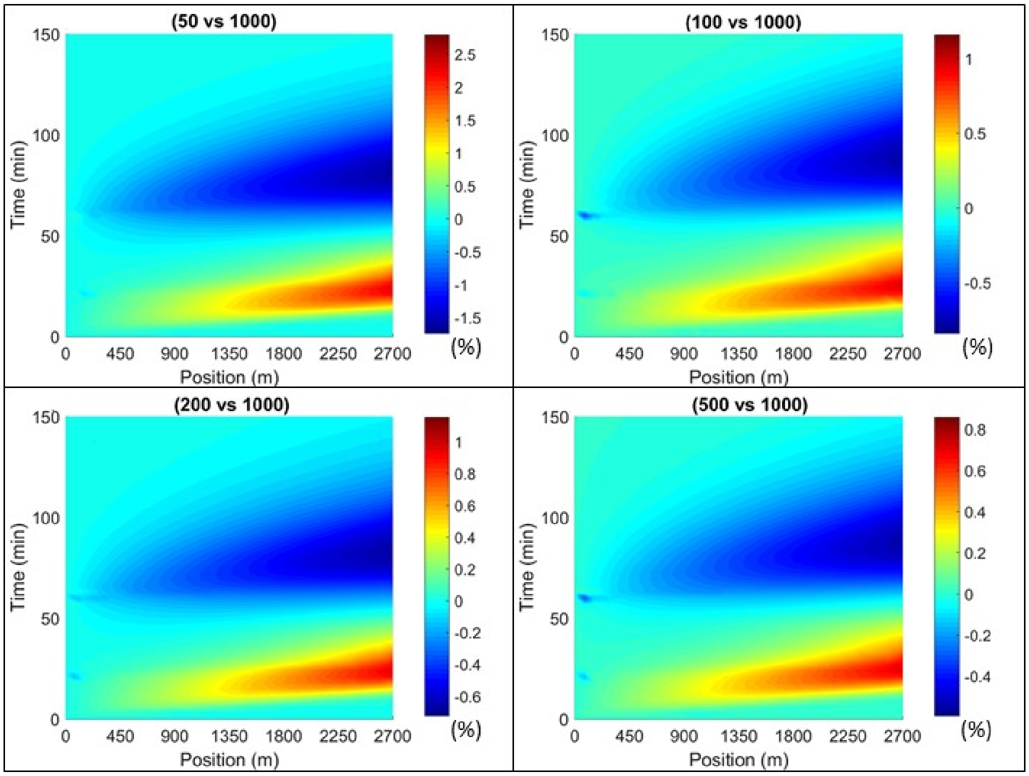

Figure 2. Percent relative difference for the flow discharge ensemble average obtained from MC simulations with 50, 100, 200, and 500 realizations, when compared to the results obtained from 1000 realizations.

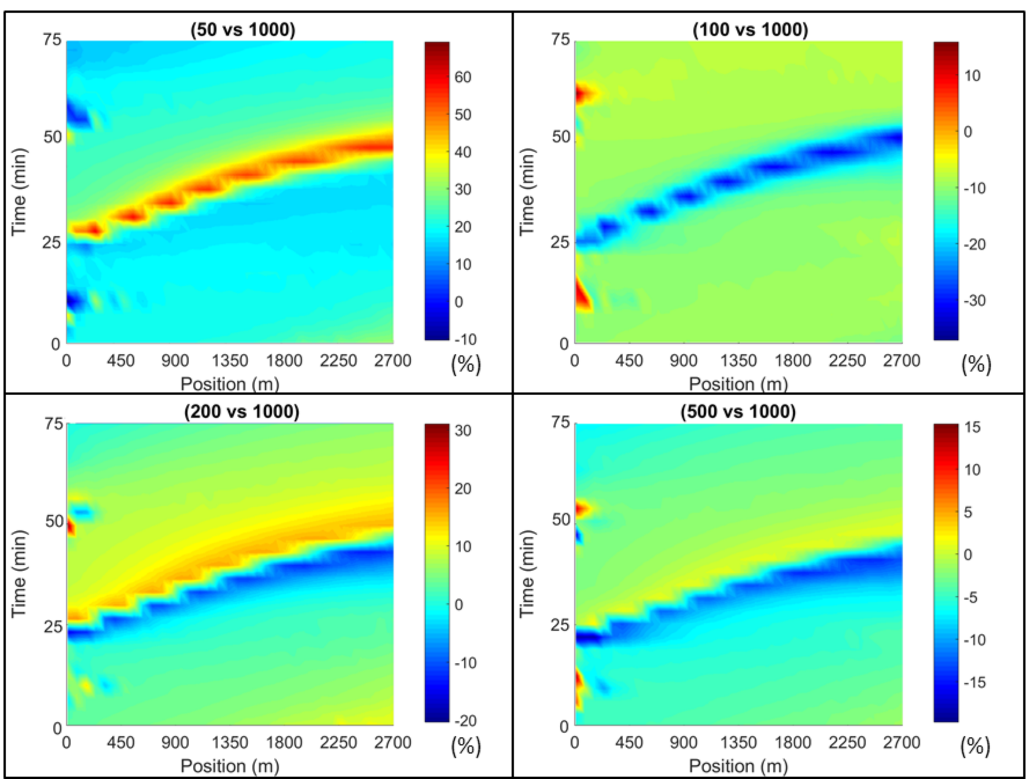

Figure 3. Percent relative difference for the flow discharge standard deviation obtained from MC simulations with 50, 100, 200, and 500 realizations, when compared to the results obtained from 1000 realizations.

Specific author changes

Text was added to clarify further the choice of 1000 MC simulations by adjusting the paragraph on Page 7: Lines 18-25 to read as follows:

> *" Following the preceding discussion, the Saint-Venant equations were deterministically solved for a total of 1000 times, each time using a different realization of n that was generated based on the PDF chosen in Sect. 3.1. While a lower number of realizations may have been sufficient for accurate computations of the first moment of the flow variables, it would not have been sufficient for the accurate computations of the second moment. In fact, the standard deviation of the flow discharge computed using 50, 100, 200, and 500 realizations showed absolute relative differences that reached 65%, 35%, 20%, and 15%, respectively, when compared to the results of the 1000-realization run. Therefore, the number of realizations in this study was selected to be large enough to numerically approximate, with sufficient accuracy, both the first and the second moments of the stochastic quantities of the problem at hand."*

C7: *On p. 11, l. 20 you write that the numerical errors caused by the spatial and temporal discretisation could lead to discrepancies in comparison to the MC results. Please check this by performing simulations with higher and lower resolutions of the spatial and temporal discretisation.*

R7: With the Courant condition being used to limit the time step in the FPE runs, the temporal discretization becomes variable during a simulation, while also being affected by the α-β discretization. So there is no user-decided constant time step for the FPE simulation. However, the main effect that was alluded to by the phrase quoted by the referee is the one occurring due to the discretization in the α and β dimensions. This is because using relatively large α and β discretizations may cause discrepancies in the FPE results. Therefore, following the referee's suggestion, we ran the FPE methodology using several discretizations to test what differences may incur. The discretization in the α-β plane used for the results in the manuscripts was $\Delta\alpha = \Delta\beta = 0.5$, and will be presented as 0.5x0.5 ($\Delta\alpha x \Delta\beta$). The additional FPE runs were done with several discretizations, some with higher and some with lower resolutions. Sample results of these are shown for the discharge and its standard deviation in Figure 4 and Figure 5, respectively.

In both of these figures, the subplots on the left show the plots of the discharge or standard deviation of the discharge of all the different discretizations attempted, along with the MC results (thick red line). Whereas, the subplots on the right show only the few high-resolution discretizations that are close to the one used in the manuscript.

Looking at both figures, one can notice a clear difference in the results when the FPE methodology is run with lower resolutions. In fact, such a difference is clear in the discharge results (Figure 4, left subplots) and is even more prominent in the standard deviation results (Figure 5, left subplots). However, the subplots on the right show that the higher resolutions provide plots that converge to the same locations, and that are the closest to the MC results. These resolutions included the one used in the manuscript (0.5x0.5) as well as 0.25x0.25, 0.33x0.33 and 1x1, all of which provide similar results. As such, from these figures, it is clear that the resolution used in this study was high enough not to incur any major discrepancies in the FPE results, but that there is a high possibility for such discrepancies to occur if lower resolutions are used.

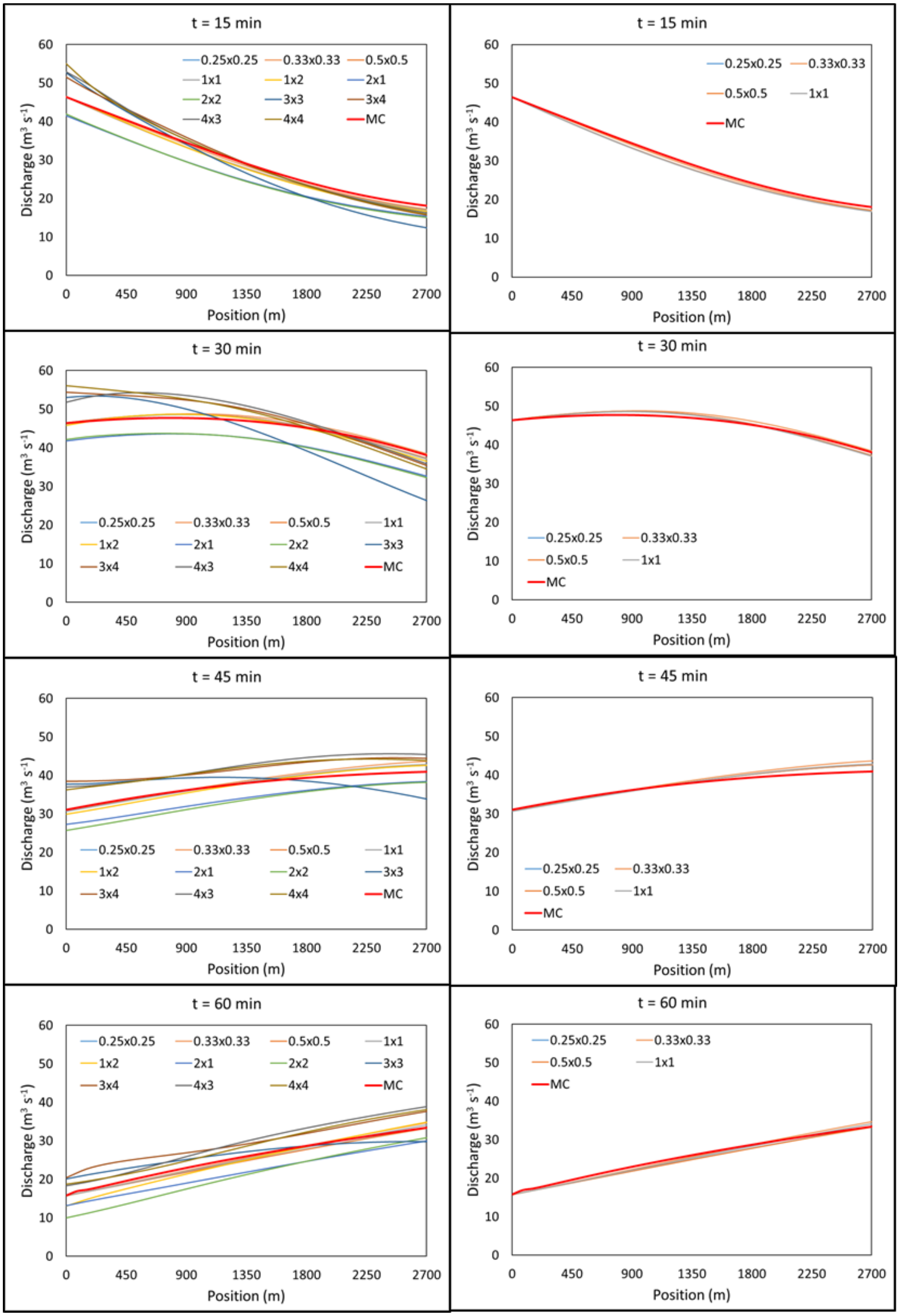

Figure 4. Plots of discharge results obtained from the FPE methodology solved using different α-β discretizations. The results of the Monte Carlo (MC) simulations are also plotted for comparison (thick red line). The figures on the left show all the different discretizations attempted; the figures on the right show only the few discretizations closest to the one used in the manuscript (0.5x0.5).

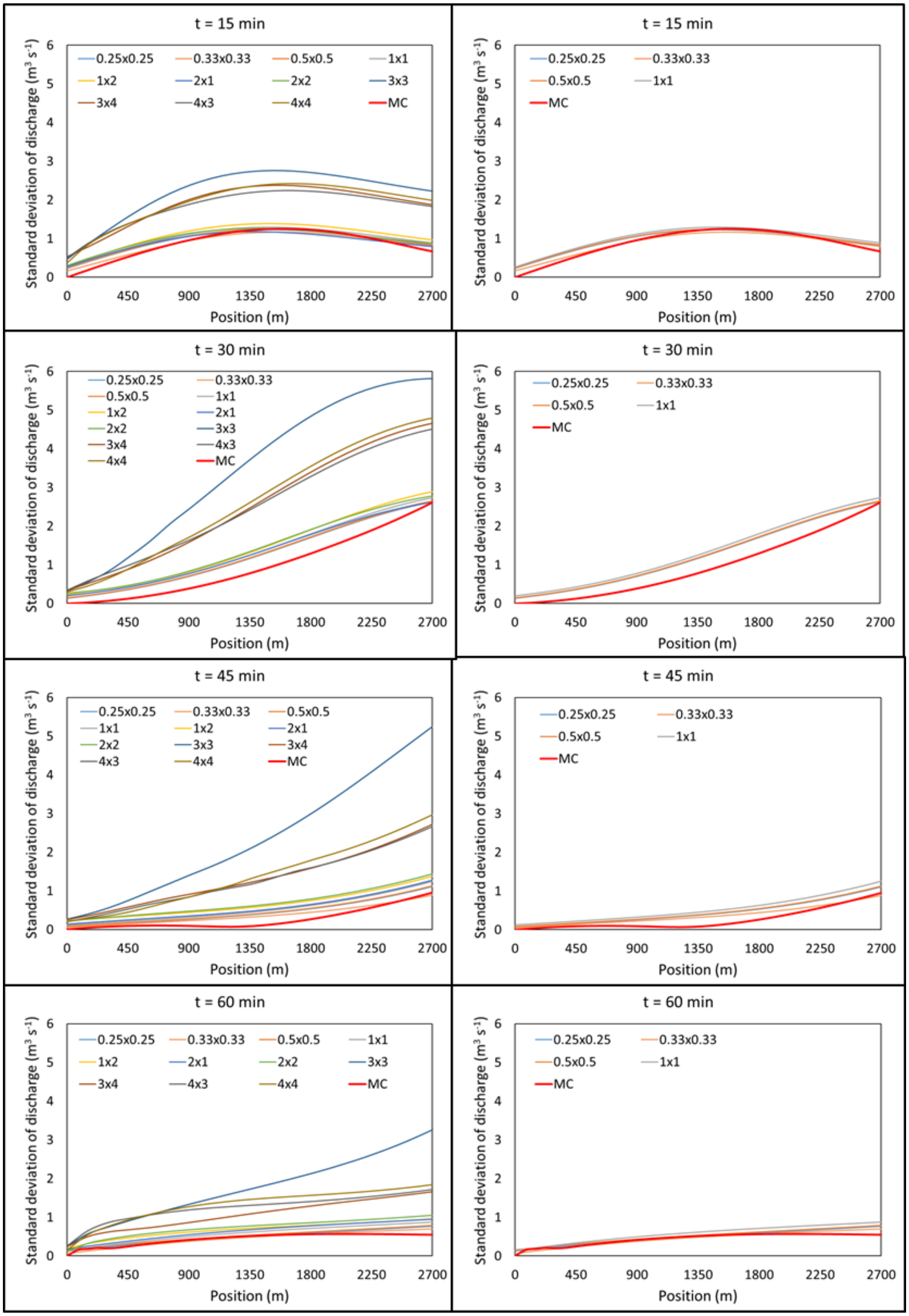

Figure 5. Plots of the standard deviation of discharge results obtained from the FPE methodology solved using different α-β discretizations. The results of the Monte Carlo (MC) simulations are also plotted for comparison (thick red line). The figures on the left show all the different discretizations attempted; the figures on the right show only the few discretizations closest to the one used in the manuscript (0.5x0.5).

C8: *Please comment on the implications for applications, like flood forecasting, of the errors made by the FPE approach.*

R8: Following the referee's comment, we comment on this issue as detailed below.

Specific author changes
Text was added to Page 11: Lines 20-24 as follows:

> *"Such discrepancies may be faced when using the FPE methodology in engineering flow applications, such as flood forecasting and flood control. The variability of the flow in flood forecasting applications, for example, may be underestimated at the downstream end of the reach, specifically during the later time periods. This would impact the range of flows that are forecasted to occur at the downstream end."*

C9: *On p. 13, l. 12-13 you state that it is an advantage that the simulation can be performed in only one run, but you do not motivate why this is an advantage. For parallelisation, it is even a disadvantage.*

R9: As described in the manuscript, one of the advantages of having the simulation performed in only one run is the reduced computational time and expense required as compared to the MC simulations. However, while such an advantage may not seem immense when only one uncertain parameter is involved (especially with the possibility of parallelizing the MC simulations), the advantage becomes much more prominent when the governing equations involve a larger number of uncertain parameters and boundary conditions. In this case, the computational expense of MC simulations would exponentially increase due to the higher number of simulations needed to maintain the desired accuracy in the results, thus significantly increasing the computational time regardless of parallelization. On the other hand, the FPE methodology would only require simple adjustments of the FPE in order to account for the additional uncertainties. After that, the FPE would be solved in the same way as was done for this study, with barely any implications on the computational expense (more details regarding possible changes to the FPE in such a situation are provided in our reply to **C4** of Referee #1 for the companion paper). The advantage is made clearer to the reader in the Conclusion as detailed below.

Specific author changes
The second half of the second paragraph of the Conclusion section (Page 13: Line 29 to Page 14: Line 9) was adjusted as follows:

> *"Furthermore, with the FPE approach, the computational time was significantly less than the time taken by the MC approach, and the FPE methodology results were obtained by running only one simulation, as opposed to the large number of simulations performed by the MC approach. Such an advantage becomes prominent with a greater number of uncertain parameters and boundary conditions, in which case the computational expense of the MC simulations that is needed to preserve the desired accuracy would exponentially increase. On the other hand, only simple adjustments would be required for the FPE, which could then be solved as was done in this study, with minor implications on its computational expense. Therefore, the results obtained in this study indicate that the proposed FPE methodology may be a powerful and time-efficient approach for predicting the ensemble average and variance behavior, in both space and time, for the unsteady open-channel flow process under an uncertain roughness coefficient, hence being an approach that would be essential for engineering flow problems."*

C10: *The discussion of the results lacks some points or is partly in contrast to what the figures show. Figures 3c,d show a slight decrease of discharge at early times with the FPE method, how do you explain that behaviour? Please explain the offset of the standard deviation at time t=0 for the FPE results. In figure 8c, the FPE result does not reproduce the decrease of the standard deviation. How do you explain the variation of the standard deviation at early times from the MC simulations in figure 8d? Figures 9 and 10 show more of a qualitative match of the results, than a quantitative match.*

R10: The referee's questions are addressed by adding some discussion and adjusting some discussion within Section 4. Small changes have been made in several locations, but the main addition is detailed below.

Specific author changes

Discussion has been added on Page 10: Line 31 to Page 11: Line 5; the added text provides the following explanation:

> *"However, the standard deviation of the FPE methodology shows an offset at time t = 0 when compared to the MC simulations, which can also be noticed at the upstream positions of Fig. 9. Recall that the initial and upstream flow discharge is assumed to be known. Nonetheless, a single known value of the flow discharge, when joined with a spread of roughness coefficients due to the uncertainty involved, leads to an unavoidable spread in the velocity and depth values. Since α and β are functions of the velocity and celerity (and in turn, depth), this spread is translated onto the α–β plane of the FPE methodology. As a result, with an uncertain roughness coefficient, the only way to numerically represent a deterministic discharge in the α–β plane is to have a spread of probability mass over the values involved. The existence of this spread on a non-continuous, discretized α–β plane may have had the most contribution to the offset of the standard deviation at the initial times and positions of Figs. 9 and 10."*

C11: *Your outlook on p. 13, l. 16-20 rather belongs to the companion paper. What about faster or more accurate numerical schemes? How could the discrepancies be reduced?*

R11: The outlook mentioned by the referee has been removed from this paper and moved to the companion paper, along with additional details elaborating on the expansion of the methodology to problems with different sources of uncertainty, as recommended by Referee #1 in **C9** for the companion paper. Moreover, a new outlook was added to this paper following the suggestion of Referee #1 above; it is detailed below.

Specific author changes

The final paragraph of the conclusion has been changed to read as follows (Page 14: Lines 10-16):

> *" While the FPE methodology satisfactorily described the ensemble average and variability of the open-channel flow system in this study, this methodology is open to improvements especially with regard to reducing any discrepancies in its numerical results. Running a more comprehensive version of the FPE methodology, by including only some of the simplifying assumptions used in this study, may be one option. Another option may involve using a higher-order and more accurate numerical scheme for the discretization of the multidimensional FPE. As such, numerous opportunities present themselves for future research within this topic, all of which would be of great benefit in the further improvement of the proposed methodology."*

Technical Corrections

C12: *The tense of your abstract makes it read like a summary, please change the tense accordingly.*

R12: As per the referee's suggestion, we adjusted the tense of the abstract by changing it from the past to the present tense.

C13: *In the abstract you write that the total simulation period of the FPE method is smaller than that of the MC approach. You certainly mean computational time.*

R13: It is true that we meant the computational time of the FPE method is smaller than that of the MC approach, and we thank the referee for his/her input regarding this point. We made the necessary adjustment accordingly.

C14: *Check the indentations at the beginning of chapters and after equations, please delete them.*

R14: Indentations have been deleted from beginning of chapters and after equations.

C15: *On p. 2, l. 31 you write that you do not limit the working space of the parameter space. What about parameter combinations, where the neglected cross-covariance terms become large or the system shows a memory?*

R15: As may be seen from Kavvas (2003), the uncertainty in multiple parameters (for example, besides the roughness parameter, also an uncertainty in bedslope, etc.) is accounted for within the resulting Lagrangian–Eulerian Fokker–Planck Equation (LEFPE) for the targeted stochastic process in terms of each parameter's variance and in terms of the cross-covariances of the uncertain parameters. Due to the underlying cumulant expansion theory that was used in the derivation of the particular form of the LEFPE for the stochastic Saint-Venant open channel flow, there is no limitation on the size of the variances and cross-covariances of the uncertain parameters. The only limitation for the use of the LEFPE is that the covariance times of the uncertain quantities in the modeling system must be finite.

C16: *You write about the conservation of particles, although no particles where introduced in your paper, please reformulate.*

R16: We thank the referee for their input regarding this point, since it is true that no specific discussion about particles occurs in this manuscript. Noting that the FPE can be considered as the conservation equation for probability mass, we reformulated the phrase in the manuscript that mentions "particles" and changed it from "to prevent any particles from leaving the domain" to "to prevent any probability mass from leaving the domain".

C17: *On p. 2, l. 6 you write that parameters become random through uncertainties. You should write that the parameters are formulated as random functions in order to capture the uncertainties.*

R17: The phrasing has been adjusted following the referee's comment.

C18: *How did you ensure that Manning's coefficient never fell below 0.01, as described on p.6, l. 27?*

R18: We included the possibility of specifying a cutoff value in our code when generating random values of Manning's roughness coefficient (*RN*), and so, ideally a cutoff value of 0.01 can be easily specified to truncate any generated *RN* values that are under 0.01. However, the mean (μ_{RN} = 0.035) and standard deviation (σ_{RN} = 0.005) that we used for *RN* in this study allowed us to generate *RN* values that remained quite far from the 0.01 cutoff value. In fact, from the 68-98-99.7 rule, we find that 99.7% of the values would lie between *RN* values of 0.02 and 0.05 (i.e., $\mu_{RN} \pm 3\sigma_{RN}$). Therefore, while generating *RN* values for this study, we never had the need to truncate or discard any of the generated *RN* values because none of them ever fell below 0.01.

Specific author changes

The corresponding sentence has been adjusted as follows (Page 6: Lines 23-26):

> *"Moreover, the selected mean and standard deviation allowed the generation of n values which never fell below 0.01, thus complying with the fact that the roughness coefficients for flows in natural streams and excavated channels are always greater than 0.01 (Chow, 1959)."*

C19: *The arguments of a PDF are usually separated by a semi-colon into arguments for which the PDF is a density and normal arguments, e.g. Pope (1985).*

R19: This has been corrected in all the respective equations following the referee's comment.

C20: *The notation of the m-dimensonal delta function is confusing, drop the exponent m.*

R20: The expression of Eq. (9) has been adjusted following the referee's comment.

C21: *P. 9, l. 11: What other form of energy, besides kinetic energy is dissipated due to shear stresses?*

R21: Besides kinetic energy, the other form of useful fluid energy which is dissipated due to shear stresses is potential energy. It has been added to the manuscript.

C22: *P. 9, l. 18: change "very minimal"*

R22: The phrase containing "very minimal" was deleted, following the suggestion of Referee #2 in **C11**.

C23: *P. 13, l. 9: I do not think that the results for the PDFs are satisfactory in general. In some cases they are, in others they are not.*

R23: We understand the referee's point concerning the PDFs. However, we reiterate the novelty of the technique used in this study and the equations derived to tackle the unsteady open-channel flow problem in a purely probabilistic manner, which has not been previously done for this hydrologic process. As such, the PDF results, while not as good as may be desired, may still be considered acceptable results, considering that the method is capable of providing the user with not only the ensemble average, but also the general behavior of the system variability in an efficient amount of time. Also, with such a promising methodology, there is room for improvement especially concerning the numerical method used.

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Viessman, W., Knapp, J. W., Lewis, G. L., and Harbaugh, T. E.: Introduction to hydrology, 2nd ed., Series in civil engineering, IEP-Dun-Donnelley, Harper & Row, New York, 704 pp., 1977.

Response to the comments of Anonymous Referee #2 published on January 5, 2018 concerning the manuscript with reference number: hess-2017-394.

We would like to thank Referee #2 for his/her insightful feedback. Our responses to the specific points raised by the referee are provided below. Please note that the referee's comments will be presented in italics, preceded by a "**C**", while the corresponding authors' responses will be presented in normal typeface with a blue font, preceded by an "**R**". For some responses, text which was changed or added to the manuscript (based on suggestions from the referee comments) is quoted and placed under "Specific author changes". Please note that the pages and line numbers provided in this document in an orange font color are from the revised version of the manuscript.

C1: *The authors present a novel way of using the Fokker–Planck Equation (FPE) to derive directly (one simulation) the probability distributions of velocity and depth resulting from uncertain roughness in a hypothetical unsteady open-channel flow problem. Although the efficiency gain over Monte Carlo simulation for the particular case presented seems limited, improving direct ways for probabilistic modelling is a relevant contribution.*

R1: We would like to thank the referee for this review.

C2: *The paper is well-written and well-structured, with sometimes a bit too many reminders of the story-line and mentioning in an early stage already the main conclusions (e.g. P.2 l.28).*

R2: We thank the referee for the positive comment. We would like to note that we worked on adjusting the manuscript to remove any repetitions and obvious statements, including those specified by the referee in the Detailed Comments below.

Main Comments

C3: *The title is confusing because of the "Ensemble modeling", whereas the main objective of the authors is to present a single simulation solution for providing a pdf. I suggest changing the title of this paper and the companion paper, taking out the term "Ensemble". (e.g. into something like "Fokker-Planck modelling of stochastic open-channel flow in term..", or "Deterministic modelling of..")*

R3: We thank the referee for his suggestion and we understand his point. We have considered the referee's suggestion; however, we certainly do prefer preserving the original title.

C4: *I would suggest to continue reporting and discussing the results for velocity and depth also in the latter part of the Results section (even if only in text, because with figures it would become too long), rather than only discussing discharge results. For velocity and depth, differences are likely to be larger and may lead to better understanding of what are the causes, because in discharge differences in velocity and depth may be cancelled out.*

R4: We thank the referee for his suggestion. Following the referee's recommendation in the above comment, as well as in **C14** and **C16** below, we have added a paragraph to the manuscript to briefly discuss the results of the standard deviation for the velocity and depth. We do not include their figures as to not lengthen the manuscript as mentioned by the referee, but we describe their behavior and provide ranges for their values and some relative differences as suggested by the referee in **C16**.

Specific author changes

A paragraph that addresses the referee's comment is added on Page 11: Line 31 to Page 12: Line 6, and reads as shown below. Note that due to an additional figure, which became denoted as Figure 8 (see Referee #1 **C5**), all figure numbers starting from 8 and above (in the original manuscript) have been increased by one for the revised manuscript. The numbers in the below paragraph refer to the new figure numbers.

> *" Concerning the standard deviations of the velocity and depth, it is important to note that their behavior over position is somewhat different from that of the flow discharge. In fact, the standard deviations of the velocity at the same four time positions of Fig. 9 seem to be relatively constant at each time position, having a value between 0.015 and 0.02 $m^3\ s^{-1}$. On the other hand, the standard deviation of the depth showed a greater range of values at each time position, as a function of location, with values ranging between 0.15 and 0.5 m. When looking at the standard deviations as a function of time at the same four locations of Fig. 10, both standard deviations seem to show that their values increase to reach a maximum and then decrease to levels similar to original levels, not unlike their corresponding ensemble average plots over time. Again, the range of change in the standard deviation of the velocity is much smaller (0.015 to 0.02 $m^3\ s^{-1}$) than that of the depth (0.15 to 0.5 m). Note that the relative differences of the FPE results when compared to the MC results reach up to 23% and 29% for velocity and depth, respectively."*

C5: *Please include a sensitivity analysis of the MC results with respect to the number of iterations. It would be interesting to check if with more simulations the results go nearer to the FPE result or further away (or no difference), and if with fewer simulations the same result is achieved. This is relevant for the claim on computational efficiency, as also pointed out by Referee#1 (fifth specific comment).*

R5: We kindly refer the referee to our reply to **C6** of Referee #1. In our reply, we present a sensitivity analysis showing how the number of realizations of the MC simulations affects the results of the ensemble average and standard deviation of the flow discharge, thus explaining in greater detail the choice of 1000 simulations in this study. We also mention that we believe that including such a long discussion and sensitivity analysis in the manuscript may cause a digression from the main idea of the manuscript, which is mainly to gauge the performance of the FPE methodology. Therefore, we believe that it would be preferable not to add such an analysis to the manuscript. Nonetheless, we include some text in the manuscript to clarify our choice of 1000 simulations to the reader, and we briefly mention the lower accuracy occurring at lower MC realizations (please see "Specific author changes" in our response to **C6** of Referee #1).

C6: *The analysis and discussion on computational time needs to be more detailed (including computational times, hardware used, etc.) and expanded. In particular with whether the FPE approach is suitable for parallelisation, if not, the MC analysis, for the case study presented, can be easily made more efficient. The authors could perhaps also include their expectations on the applicability and computational efficiency of their FPE method for larger systems. Would the gain with respect to MC increase or not?*

R6: More details have been given in the revised manuscript for the discussion of computational time needs as suggested by the referee. Moreover, a short discussion regarding the parallelization of the FPE as well as the greater advantage of the computational efficiency of the FPE methodology for systems with greater numbers of uncertainties is also added to the manuscript. These additions will be detailed below. The referee is also kindly referred to our replies to Referee #1, especially regarding **C6** (last two paragraphs) and **C9.**

<u>Specific author changes</u>
Technical information has been added directly before Section 3.1 (Page 6: Lines 9-11):

> *"All simulations for the MC approach and the FPE methodology were run on a computer having 16 GB of RAM and an Intel i7 processor with four cores, each core having a base frequency of 2.40 GHz and a maximum frequency of 3.40 GHz."*

Two paragraphs were added to the end of the Discussion section (Page 12: Line 31 to Page 13: Line 15), and read as follows:

> *" It should be noted that these FPE results required a significantly less amount of time for computation as opposed to the MC results. Recall that the 1000 MC simulations were parallelized and run over all four cores (with no hyperthreading), thus noticeably reducing the computational time as compared to an un-parallelized run. With such parallelization, the MC simulations ran for over 2 days. On the other hand, the results of the FPE methodology, which was not parallelized, were obtained in about 7 hours.*
>
> *If we observe the computational times of the implicit numerical solution of the FPE methodology, the portion of the simulation requiring the greatest time is filling out the coefficient matrix, especially for small α and β discretizations. Parallelizing this portion over the four cores would allow one to considerably reduce the time to fill out the coefficient matrix, thus reducing the total computational time of this method. Without the parallelization of the FPE methodology, its one simulation may still not seem to provide an immense advantage when only one uncertain parameter is involved, especially with the possibility of parallelizing the MC simulations among a much larger number of cores. Nonetheless, when the problem being solved involves a greater number of uncertain parameters and boundary conditions, or even a larger system, such an advantage may prove to be crucial. In fact, the computational expense of the MC simulations for such a case would be expected to increase exponentially due to the higher number of simulations needed to maintain the desired accuracy in the results, thus significantly increasing the computational time regardless of parallelization. On the other hand, such additional uncertainties can be easily implemented into the FPE methodology by making simple changes and additions that will be reflected in Eq. (5), after which the FPE would be solved following similar steps as discussed for this study, with minimal implications on the computational expense."*

C7: *The gain in computational efficiency, as presently described, seems limited. Hence, the claimed contribution there, in Abstract and Conclusions, should be down-sized or contextualised.*

R7: The referee's point is acknowledged, and adjustments have been made to the corresponding text in the Abstract and the Conclusion as detailed below.

Specific author changes

In the Abstract, Page 1: Line 16-18 was adjusted as follows:

> *"Moreover, the total computational time of the FPE methodology is smaller than that of the MC approach, which could prove to be a particularly crucial advantage in systems with a large number of uncertain parameters."*

In the Conclusion, Page 14: Lines 1-6 now read as follows:

> *"…, and the FPE methodology results were obtained by running only one simulation, as opposed to the large number of simulations performed by the MC approach. Such an advantage becomes prominent with a greater number of uncertain parameters and boundary conditions, in which case the computational expense of the MC simulations that is needed to preserve the desired accuracy would exponentially increase. On the other hand, only simple adjustments would be required for the FPE, which could then be solved as was done in this study, with minor implications on its computational expense."*

Detailed Comments

C8: *P.2 l.28: "..producing the complete ensemble model results.." is not correct, because, if I understand correctly, the method does not reproduce the individual traces (ensemble members). Therefore, this should be something like "..producing the statistical properties.."*

R8: The sentence has been adjusted following the referee's comment.

C9: *P.7 l.14-15: Explain the choice of 1000 simulations. Report the sensitivity of the statistical characteristics to the number of simulations in the MC.*

R9: We kindly refer the referee to our reply to **C6** of Referee #1 which expands on this topic and explains in greater detail the choice of 1000 simulations.

C10: *P.8 l.27-28: Repetition. There is already a sentence connecting Sections 3 and 4 in lines 23-25. Consider leaving out one of the two.*

R10: Following the referee's comment, the sentence at the end of Section 3 has been removed to reduce the repetition.

C11: *P.9 l.18: Repetition. Delete ".., with very minimal differences among the two"*

R11: As suggested by the referee, the phrase "with very minimal difference among the two" was deleted.

C12: *P.9 l.32-33: However, ... ,but... Consider reformulating.*

R12: As suggested by the referee, the sentence was reformulated as detailed below.

Specific author changes

The reformulated sentence now reads as follows (Page 9: Lines 30-31):

> *"Similarly to Fig. 4, a slight overestimation can be noticed from the FPE methodology especially around the peak depths, but with a maximum relative difference of only around 7.5%."*

C13: *P.10 l.13-17: Reformulate removing redundancies. (Or consider leaving out, because it reads perhaps too much as general conclusions, while this is in the middle of presenting and discussing results)*

R13: The sentences noted by the referee have been left out.

C14: *P.10 l.20: Why do the authors continue only with Discharge? Differences in velocity and depth may be cancelling each other in the resulting discharge. Also when thinking of flood risk management applications, it may be more interesting to look at velocity and depth variance.*

R14: As suggested previously by the referee, we provide additional text in the manuscript to briefly discuss the results for the standard deviations of the velocity and depth. We kindly refer the referee to our reply to **C4** for our full response.

C15: *P.10 l.18-23: Too much repetition. Suggest to shorten and merge with next paragraph where actually the presentation of variability results starts.*

R15: As suggested by the referee, the noted paragraph was shortened to one and merged with the following paragraph as follows:

> *"In a similar manner to the ensemble averages, the relative performance of the FPE methodology in predicting the system's variability was examined, this time by checking the standard deviations."*

C16: *P.10 l.33: "relatively small" Suggest to add some of the differences in %. Also provide differences in standard deviation for velocity and depth. P.11 l.15-21: The results for velocity and depth may help in understanding the causes of differences in variability.*

R16: The phrase structure for the sentence of Page 10: Line 33 has been changed due to changes from other comments. The percent relative differences have been added for the standard deviation for velocity and depth as suggested by the referee. Again, we kindly refer the referee to our reply to **C4** for our full response.

C17: *P.11 l.27-31: As described in main comments above, please expand the analysis and discussion of computational efficiency, and make it a separate paragraph.*

R17: We kindly refer the referee to our reply to **C6** above for the full response regarding this matter.

C18: *P.12 l.18: General sentence. Consider deleting.*

R18: As suggested by the referee, the sentence has been deleted.

Response to the comments of Dr. Alberto Bellin from the Short Comment published on September 12, 2017 concerning the manuscript with reference number: hess-2017-394.

We would like to thank Dr. Bellin for his insightful comments. Our responses to the specific points raised by him are provided below. Please note that his comments will be presented in italics, preceded by a "**C**", while our responses will be presented in normal typeface with a blue font, preceded by an "**R**". For some responses, text which was changed or added to the manuscript (based on suggestions from the comments) is quoted and placed under "Specific author changes". Please note that the pages and line numbers provided in this document in an orange font color are from the revised version of the manuscript.

C1: *In this manuscript the authors present an illustrative example of the Fokker Planck equation associated to the De Saint Venant equation for a spatially uniform but random roughness coefficient.*

R1: We thank Dr. Bellin for his review of this companion paper.

C2: *As I already commented in my review of the previous manuscript I think that the material is not enough to justify a two-paper series. Therefore, my suggestion is for merging the two manuscripts into one. The first part (up to page 5) of the second manuscript summaries what already presented in the first one and is not longer needed in the merged manuscript, while the second part can be easily merged with the first manuscript.*

R2: We again thank Dr. Bellin for his comment and concern regarding this matter. However, as we have mentioned in our reply to comment **C2** from his review of our companion paper (Referee #2 for hess-2017-393), we still believe in the ability for both manuscripts to remain as two standalone, companion papers. We would also like to note that neither Referee #1 of the companion paper, nor Referees #1 and #2 of this manuscript have presented any concerns regarding having two standalone manuscripts, nor have they presented the desire to join the two manuscripts into one. Below is our original reply to Dr. Bellin's comment **C2** from our companion paper regarding the same matter:

> *"We thank the referee for the comment. The ensemble-averaging technique used in this study was indeed developed by Kavvas (2003) and this technique has been applied to other processes with different governing equations where Fokker–Planck Equations (FPEs) specific to those processes were obtained and applied successfully. However, this technique had never been applied to the Saint-Venant equations to tackle the stochastic unsteady open-channel flow problem. As such, the novelty of the proposed FPE methodology that was developed in this manuscript was, firstly, to figure out how to apply the Kavvas (2003) technique to the Saint-Venant equations especially through the transformations that provided us with the state variables α and β and that allowed us to write the Saint-Venant equations as four Ordinary Difference Equations (ODEs) in the specific forms of Eqs. (15) to (18) (for this was not a straightforward process), and secondly, to go forth with developing the FPE that is specifically for the stochastic unsteady open-channel flow process, an equation which has not been developed before. Hence, this study clearly derives and presents an entirely new FPE that can be used to solve for the probability density of the state variables of a stochastic open-channel system, which is not found elsewhere in the literature. And while the numerical discretization was made following the Chang and Cooper (1970) scheme, the scheme was generalized from its original one-dimensional form and adapted to the four-dimensional FPE that*

was being solved in this study. Therefore, joining this manuscript and the companion manuscript into one manuscript, while placing a large portion of this first manuscript in the appendix, would mostly take away from the importance of these equations and from the work that was done in arriving to those equations. As such, we believe in the novelty of the equations derived in this manuscript, and thus we believe in its ability to stand as its own manuscript. We would also like to note that Referee #1, who has read and reviewed both manuscripts, has not mentioned any desire for the joining of the two papers into one, in which case we would assume that Referee #1 may not have seen any concern with them being two standalone, companion papers."

C3: *The boundary condition used at the lower end of the channel in the solution of the de Saint Venant Equation in a Monte Carlo framework are unclear and should be better specified and justified. At page 7 line 12 I read "... while at the downstream end, the channel was assumed to be hydraulically long so that the flow can be taken as normal flow, thus satisfying the Manning's equation". This sentence is unclear: what is normal flow? Mathematically the downstream condition can be of imposed water depth y, with the velocity V obtained through the Manning's equation, or imposed velocity (or water discharge) and again the Manning's equation provides the water depth. The imposed condition can be either stationary, i.e. constant, or transient. It seems from the results that the authors choose the second option (Figures 2, 3 4 and 5 shows that the ensemble mean of both y and Q changes with time at the lower end of the channel), but no details on the specific boundary conditions are provided in the text. Assuming that one of the above two boundary conditions have been selected at the lower end of the channel, this choice should reflect to the boundary condition for the stochastic variables in the FPE equation. For instance, if y is imposed its PDF is a Dirac delta, while the pdf of the velocity is related to the pdf of the roughness coefficient through the Manning's equation. In turn, the PDFs of α and* β *depends from the PDFs of y and V through equations (13) and (14) of the first manuscript. Similar arguments can be applied if the BC is of imposed water discharge (i.e. velocity) and the water depth is computed through the Manning's equation. What puzzles me is that the authors impose a reflection boundary at the end of the channel (line 15, page 8), which is apparently not compatible with the previous conditions and with those imposed in the MC simulations. Since Figures from 2 to 5 show clearly that the water wave interacted with the downstream boundary, I am expecting that the boundary condition here should have an impact on the solution. This reinforce the need to select compatible boundary conditions in both models.*

R3: We thank Dr. Bellin for this comprehensive and crucial comment regarding the downstream boundary condition (BC). We certainly agree with his final sentiment regarding the importance of selecting a compatible downstream BC for both the MC and FPE models. Regarding the first point Dr. Bellin raises, it is true that the downstream BC may be imposed as water depth (*y*) or as velocity (*V*). However, to add to this, we provide a quote from the book *Open Channel Hydraulics* by Sturm (2001) specifying that "the downstream boundary condition in a flood routing problem also might be a stage or discharge hydrograph, but in some cases, it could be a depth-discharge relationship." Hence, concerning the last part of Page 7: Line 12 which Dr. Bellin indicated, what we meant by normal flow was that the Manning's equation is assumed valid at this downstream boundary, and that our downstream BC is not a specified/imposed depth or velocity, but instead it is the depth-discharge relationship represented by the Manning's equation. The way we computed the depth and velocity downstream will be explained in the following paragraph.

Recall that the manner with which we solve our MC simulation is through the method of characteristics, which is explained in detail in Viessman et al. (1977) and Sturm (2001). We describe here, in short, the steps used in order to compute the values of the flow variables at the downstream boundary. When using the method of characteristics, the solution is usually found at the intersection of the forward (C_1) and backward (C_2) characteristic curves. The point of intersection is denoted by *P*. However, when calculating the values for the downstream boundary condition, only C_1 is used (Eq. (1)), along with its compatibility equation (Eq. (2)). Now, for the downstream BC, the unknown value is at the intersection of C_1 with the downstream boundary (again this point is denoted by *P*). C_1 originates from a point from the previous time step, and this point (denoted by *L*) is upstream of point *P*. We use the known flow variables at point *L* (V_L and y_L) from the previous known time step (t_L) located at a known location (x_L) in order to compute t_P from the equation of C_1 (Eq. (1)). Then, from the compatibility equation along C_1 (Eq. (2)), we get a nonlinear expression in depth (y_P) which results from the substitution of V_P with its expression from Manning's equation as a function of y_P. As such, this nonlinear expression is solved to find y_P which is then used to compute V_P. Therefore, while no imposed values for y_P and V_P are set at the downstream boundary, the expression of the Manning's equation in combination with the equations corresponding to the forward characteristic curve C_1 are used to determine these variables.

Since none of the flow variables (*y* or *V*) are imposed at the downstream boundary conditions, we do not need to specify any Dirac delta function for these variables at the downstream boundary. Moreover, we would like to mention that the reflecting boundary conditions for the FPE method were used to describe the boundaries of the α and β dimensions, as mentioned in the original manuscript on Page 8: Lines 17-19. However, for the upstream boundary condition, the probability densities were known. Finally, the downstream boundary condition for FPE was formulated in a way that would replicate the downstream BC of the MC model with its depth-discharge relationship (Manning's). Moreover, in the manuscript we note that we also extend the downstream boundary condition much further than the required reach length in the downstream direction in order to eliminate any of its effects on the numerical solution.

Specific author changes

The manuscript text describing the downstream boundary conditions for the MC and FPE simulations have been adjusted, and the new versions are shown below.

The text of for MC now reads as follows (Page 7: Lines 13-17):

> *"As for the boundary conditions, the flow hydrograph at the channel entrance was given, while at the downstream end, the channel was assumed to be hydraulically long so that the flow can be taken as normal flow, thus satisfying Manning's equation. As such, the downstream boundary condition was chosen as the depth-discharge relationship represented by Manning's equation. This equation, along with the equations corresponding to the positive characteristic (i.e., Eqs. (1) and (2)), was used to compute the flow variables at the downstream boundary."*

The text of for FPE now reads as follows (Page 8: Lines 20-24):

> *"As for the upstream boundary, recall that the discharge hydrograph was assumed to be known upstream, in which case the probability densities at the upstream boundary of the x–dimension were known for all t > 0. Finally, the downstream boundary in the x–dimension was formulated to replicate that of the MC model, and it was extended downstream much further than the required length of the reach so as to eliminate any of its effect on the numerical solution."*

C4: *In addition, given the boundary condition of the FPE at the initial section I was expecting here the standard deviation of the water discharge equal to zero, as in the MC simulations. However, this is not the case, as shown in Figures 8 and 9. This unexpected result needs justification.*

R4: We kindly refer Dr. Bellin to our reply to **C10** of Referee #1 of this manuscript, which includes our full response regarding the same matter.

References

Sturm, T. W.: Open channel hydraulics, McGraw-Hill series in water resources and environmental engineering, McGraw-Hill, Boston, 493 pp., 2001.

Viessman, W., Knapp, J. W., Lewis, G. L., and Harbaugh, T. E.: Introduction to hydrology, 2nd ed., Series in civil engineering, IEP-Dun-Donnelley, Harper & Row, New York, 704 pp., 1977.

Ensemble modeling of stochastic unsteady open-channel flow in terms of its time-space evolutionary probability distribution: numerical application

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Abstract. The characteristic form of the Saint-Venant equations ~~was~~ is solved in a stochastic setting by using a newly proposed Fokker–Planck Equation (FPE) methodology. This methodology computes the ensemble behavior and variability of ~~a~~ the unsteady flow in open channels ~~system~~ by directly solving for ~~its~~ the flow variables' time-space evolutionary probability distribution. The new methodology ~~was~~ is tested on a stochastic unsteady open-channel flow problem, with an uncertainty arising from the channel's roughness coefficient. The computed statistical descriptions of the flow variables ~~were~~ are compared to the results obtained through Monte Carlo (MC) simulations in order to evaluate the performance of the FPE methodology. The comparisons show~~ed~~ that the proposed methodology can adequately predict the results of the considered stochastic flow problem, including the ensemble averages, variances, and probability density functions in time and space. ~~However, u~~Unlike the large number of simulations performed by the MC approach, only one simulation ~~was~~ is required by the FPE methodology. Moreover, the total ~~simulation period~~computational time of the ~~-~~FPE methodology ~~was~~ is ~~significantly~~ smaller than that of the MC approach, which could prove to be a particularly crucial advantage in systems with a large number of uncertain parameters. As such, the results obtained in this study indicate that the proposed FPE methodology is a powerful and time-efficient approach for predicting the ensemble average and variance behavior, in both space and time, for an open-channel flow process under an uncertain roughness coefficient.

1 Introduction

One of the most important types of unsteady open-channel flow problems is that of flood routing (Scharffenberg and Kavvas, 2011). This problem considers an initially uniform flow rate through an open channel, after which a flood wave enters upstream of the channel reach, and is translated downstream. The routing process, which predicts the spatial shape and temporal development of this flood wave as it traverses downstream (Viessman et al., 1977), involves solving for two dependent flow variables as a function of time and space through the river reach: velocity and depth, or discharge and depth. Solving for these two unknowns by means of the hydraulic routing technique involves using two governing equations

(Chow, 1959), the continuity equation and the momentum equation, which are jointly known as the Saint-Venant equations (Sturm, 2001).

While it may be possible to assume that the flow and channel parameters are deterministic so as to obtain a deterministic solution to the Saint-Venant equations, such parameters usually exhibit high uncertainty in the real world (Gates and AlZahrani, 1996; Ercan and Kavvas, 2012a). In fact, uncertainties in these parameters may originate from several factors, including a channel's physical conditions and geometric parameters, its upstream boundary and initial conditions, as well as any lateral inflows (Chow, 1959; Sturm, 2001; Liang and Kavvas, 2008; Ercan and Kavvas, 2012a). As such, ~~With such uncertainties,~~ the parameters may be formulated as random functions, thus becoming~~e~~ spatially and/or temporally random, in which case the system can no longer be assumed deterministic. This necessitates the solution of the governing equations within a stochastic framework, from which a quantitative description of the ensemble behavior and variability of the process is obtained. In this manner, the two dependent flow variables are solved for their statistical properties, not for their deterministic values, at designated time-space positions.

Among the available approaches that can be used to solve the Saint-Venant equations within a stochastic framework, the Monte Carlo (MC) approach is one of the most well-known due to its abundant use in simulating differential equations with stochastic parameters (Freeze, 1975; Smith and Freeze, 1979; Bellin et al., 1992). It is also generally accepted as the most robust approach for uncertainty evaluation, as well as the benchmark for comparing other new methods (Scharffenberg and Kavvas, 2011). However, one of the main disadvantages of the MC approach is its computational expense, which results from the large number of simulations that it usually involves.

In order to circumvent having to solve the ~~governing~~ Saint-Venant equations repeatedly for a large number of times, this study uses a new methodology that solves for the time-space evolutionary probability distribution of the unsteady open-channel flow ~~system~~ process in only one simulation. From this probabilistic solution, one can then obtain the ensemble mean and ensemble variance of the process as they evolve in time and space. This new methodology, which is proposed, explained, and derived in the companion paper by Dib and Kavvas (2017), makes use of the ~~. In short, the~~ ensemble averaging technique developed in Kavvas (2003) ~~Kavvas (2003)~~to obtain a Fokker–Planck Equation (FPE) that specifically describes an unsteady open-channel flow process. ~~is used to determine the deterministic equation for the evolutionary probability distribution of the governing stochastic differential equations of the flow process, thus providing their Lagrangian-Eulerian form of the Fokker-Planck equation (LEFPE). The LEFPE is then simplified to a classical Fokker-Planck Equation (FPE), which deterministically describes the temporal and spatial evolution of the probability density of the dependent variables of the flow process. Through an implicit discretization, the obtained FPE is solved numerically in order to compute the statistical properties of the dependent variables, thus describing the ensemble behavior and variability of the system being considered.~~Some other hydrologic processes have been successfully simulated by following a similar procedure, which involved applying the ensemble averaging technique of Kavvas (2003) to their corresponding governing equations and obtaining the FPEs specific to their case. These include: unsaturated water flow (Kim et al., 2005a), root-water uptake (Kim et al., 2005b), solute transport (Liang and Kavvas, 2008), snow accumulation and melt (Ohara et al., 2008), unconfined groundwater flow (Cayar and Kavvas, 2009a, b), as well as kinematic open-channel flow (Ercan and Kavvas, 2012a, b).

Note that ~~I~~in addition to producing the ~~complete ensemble model results~~statistical properties in a computationally efficient manner ~~by~~ through~~using~~ one simulation, the ~~proposed~~ FPE methodology developed for the unsteady open-channel flow process directly solves for, and is linear in, the probability density of the dependent variables. Moreover, ~~Note that~~ ~~w~~while this methodology assumes a finite correlation time for the considered process, it does not make any linearization assumptions and it does not have limitations on the working range of the parameter space ~~(Ercan and Kavvas, 2012a)~~. ~~Many hydrologic processes have been successfully simulated byKavvas (2003) using such a methodology(Kavvas, 2003) including unsaturated water flow (Kim et al., 2005a), root-water uptake (Kim et al., 2005b), solute transport (Liang and Kavvas, 2008), snow accumulation and melt (Ohara et al., 2008), unconfined groundwater flow (Cayar and Kavvas, 2009a, b), as well as kinematic open-channel flow (Ercan and Kavvas, 2012a, b).~~

Therefore, in the wake of the preceding discussions, the first objective of this study is to ~~apply~~ use the ~~proposed~~ FPE methodology derived in the companion paper for the unsteady open-channel flow process (Dib and Kavvas, 2017) and to apply it to a representative stochastic unsteady open-channel flow problem in order to solve for the probability density of the state variables of the flow process~~,~~ and to provide a quantitative description of the expected behavior and variability of the system in one single simulation. The second objective is to evaluate the performance of the proposed methodology and to validate its results by comparing the statistical properties of the flow variables computed by the FPE methodology against those calculated by the MC approach.

2 Saint-Venant equations: characteristic form and ensemble-averaged form

The Saint-Venant equations solved in this study are written for the unsteady open-channel flow of an incompressible fluid in a rectangular, prismatic channel with no lateral inflow. The method of characteristics is used to transform these equations from two partial differential equations into a system of four ordinary differential equations (ODEs). These four ODEs include two characteristic equations describing the two characteristic paths, and two compatibility equations which must be satisfied along their corresponding characteristic path. The characteristic form of the Saint-Venant equations is shown below (Sturm, 2001):

Positive characteristic curve (C_1)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = V + c \tag{1}$$

Flow process condition to be satisfied along C_1

$$\left(\frac{\mathrm{d}(V+2c)}{\mathrm{d}t}\right)_1 = \mathrm{g}\left(S_{0,1} - S_{\mathrm{f},1}\right) \tag{2}$$

Negative characteristic curve (C_2)

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = V - c \tag{3}$$

Flow process condition to be satisfied along C_2
$$\left(\frac{\mathrm{d}(V-2c)}{\mathrm{d}t}\right)_2 = \mathrm{g}\left(S_{0,2}-S_{\mathrm{f},2}\right) \tag{4}$$

where V is the average flow velocity, c is the wave celerity which is equal to $\sqrt{gy}$ for a rectangular channel where y is the flow depth, x is the position, t is the time, S_0 is the slope of the channel bottom, S_f is the friction slope, and g is the acceleration of gravity. Note that $S_{0,\mathrm{i}}$ denotes $S_0(x_\mathrm{i},t)$; the same applies to S_f. The positive and negative characteristic curves are defined as C_1 and C_2, respectively, and the variables or derivatives corresponding to C_1 and C_2 are denoted by the subscripts 1 and 2, respectively.

The Lagrangian–Eulerian form of the Fokker–Planck equation (LEFPE), corresponding to the Saint-Venant equations, that can solve for the multivariate probability density function (PDF) of the hydrologic state variables of an unsteady open-channel flow problem was obtained through the following steps: (i) using the two substitutions $V+2c=\alpha$ and $V-2c=\beta$ on Eqs. (1) to (4), (ii) applying the technique in Kavvas (2003) while assuming the uncertainty arising from the Manning's roughness coefficient, and (iii) performing several simplifying assumptions. The result is shown in Eq. (5), while its detailed derivation can be found in the companion paper by Dib and Kavvas (2017).

$$\begin{aligned}
&\frac{\partial P(x_1,x_2,\alpha,\beta;\cancel{,}t)}{\partial t} = \\
&-\frac{\partial}{\partial x_1}\left\{P\left[\frac{3}{4}\langle\alpha(x_1,t)\rangle+\frac{1}{4}\langle\beta(x_1,t)\rangle\right]\right\} \\
&-\frac{\partial}{\partial x_2}\left\{P\left[\frac{1}{4}\langle\alpha(x_2,t)\rangle+\frac{3}{4}\langle\beta(x_2,t)\rangle\right]\right\} \\
&-\frac{\partial}{\partial \alpha}\left\{P\left[\mathrm{g}\,S_0-\frac{\mathrm{g}}{4\mathrm{k}^2}\left(\frac{2}{b}\right)^{4/3}\left\langle n^2(x_1,t)\cdot[\alpha(x_1,t)+\beta(x_1,t)]^{\,2}\cdot\left\{\frac{8\mathrm{g}b}{[\alpha(x_1,t)-\beta(x_1,t)]^{\,2}}+1\right\}^{4/3}\right\rangle\right]\right\} \\
&-\frac{\partial}{\partial \beta}\left\{P\left[\mathrm{g}\,S_0-\frac{\mathrm{g}}{4\mathrm{k}^2}\left(\frac{2}{b}\right)^{4/3}\left\langle n^2(x_2,t)\cdot[\alpha(x_2,t)+\beta(x_2,t)]^{\,2}\cdot\left\{\frac{8\mathrm{g}b}{[\alpha(x_2,t)-\beta(x_2,t)]^{\,2}}+1\right\}^{4/3}\right\rangle\right]\right\} \\
&+\frac{\partial^2}{\partial x_1^2}\left\{P\left[\left(\frac{9}{16}\right)\mathrm{Var}[\alpha(x_1,t)]+\left(\frac{1}{16}\right)\mathrm{Var}[\beta(x_1,t)]+\left(\frac{3}{8}\right)\mathrm{Cov}[\alpha(x_1,t),\beta(x_1,t)]\right]\right\} \\
&+\frac{\partial^2}{\partial x_2^2}\left\{P\left[\left(\frac{1}{16}\right)\mathrm{Var}[\alpha(x_2,t)]+\left(\frac{9}{16}\right)\mathrm{Var}[\beta(x_2,t)]+\left(\frac{3}{8}\right)\mathrm{Cov}[\alpha(x_2,t),\beta(x_2,t)]\right]\right\} \\
&+\frac{\partial^2}{\partial \alpha^2}\left\{P\left[\frac{\mathrm{g}^2}{16\mathrm{k}^4}\left(\frac{2}{b}\right)^{8/3}\mathrm{Var}\left[n^2(x_1,t)\cdot[\alpha(x_1,t)+\beta(x_1,t)]^{\,2}\cdot\left\{\frac{8\mathrm{g}b}{[\alpha(x_1,t)-\beta(x_1,t)]^{\,2}}+1\right\}^{4/3}\right]\right]\right\} \\
&+\frac{\partial^2}{\partial \beta^2}\left\{P\left[\frac{\mathrm{g}^2}{16\mathrm{k}^4}\left(\frac{2}{b}\right)^{8/3}\mathrm{Var}\left[n^2(x_2,t)\cdot[\alpha(x_2,t)+\beta(x_2,t)]^{\,2}\cdot\left\{\frac{8\mathrm{g}b}{[\alpha(x_2,t)-\beta(x_2,t)]^{\,2}}+1\right\}^{4/3}\right]\right]\right\}
\end{aligned} \tag{5}$$

In Eq. (5), b denotes the width of the channel, n denotes Manning's roughness coefficient, and k denotes the conversion factor between SI and US units for Manning's formula. The remaining variables are as defined previously. Note that in Eq. (5), and other later equations, $P(x_1, x_2, \alpha, \beta, t)$ is sometimes substituted by P for simplicity. Moreover, note that Eq. (5)

resembles an advection-diffusion equation, in which the first four bracketed terms multiplied by P on their left-hand-sides resemble the advection coefficients, and the last four bracketed terms resemble the diffusion coefficients. Hence, after denoting the advection coefficients by F and the diffusion coefficients by D, Eq. (5) was simplified into Eq. (6) below, which is the final analytical form of the proposed FPE methodology~~ for the stochastic solution of the Saint-Venant equations in one simulation~~.

$$\begin{aligned}\frac{\partial P(x_1, x_2, \alpha, \beta; \cancel{,} t)}{\partial t} = &-\frac{\partial}{\partial x_1}F_1 P - \frac{\partial}{\partial x_2}F_2 P - \frac{\partial}{\partial \alpha}F_\alpha P - \frac{\partial}{\partial \beta}F_\beta P \\ &+\frac{\partial^2}{\partial x_1^2}D_1 P + \frac{\partial^2}{\partial x_2^2}D_2 P + \frac{\partial^2}{\partial \alpha^2}D_\alpha P + \frac{\partial^2}{\partial \beta^2}D_\beta P\end{aligned} \tag{6}$$

An appropriate numerical scheme to solve Eq. (6) was derived following Chang and Cooper (1970), as discussed in detail in the companion paper. This scheme involves implicitly discretizing the equation, while noting that the dependent variables are to be solved at the intersection of the characteristic curves C_1 and C_2, which renders $x_1 = x_2 = x$. As such, the discretized version of Eq. (6) was derived in the companion paper to be as follows:

$$\begin{aligned}
P^{\mathrm{n}}_{\mathrm{h,k,l}} = &\left\{\begin{aligned}
&1 + \frac{\Delta t}{\Delta x}\delta^{\mathrm{n+1}}_{1;\,\mathrm{h}}F^{\mathrm{n}}_{1;\,\mathrm{h}+\frac{1}{2}} + \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{1;\,\mathrm{h}+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\left(1-\delta^{\mathrm{n+1}}_{1;\,\mathrm{h-1}}\right)F^{\mathrm{n}}_{1;\,\mathrm{h}-\frac{1}{2}} + \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{1;\,\mathrm{h}-\frac{1}{2}}\\
&+ \frac{\Delta t}{\Delta x}\delta^{\mathrm{n+1}}_{2;\,\mathrm{h}}F^{\mathrm{n}}_{2;\,\mathrm{h}+\frac{1}{2}} + \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{2;\,\mathrm{h}+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\left(1-\delta^{\mathrm{n+1}}_{2;\,\mathrm{h-1}}\right)F^{\mathrm{n}}_{2;\,\mathrm{h}-\frac{1}{2}} + \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{2;\,\mathrm{h}-\frac{1}{2}}\\
&+ \frac{\Delta t}{\Delta \alpha}\delta^{\mathrm{n+1}}_{\alpha;\,\mathrm{k}}F^{\mathrm{n}}_{\alpha;\,\mathrm{k}+\frac{1}{2}} + \frac{\Delta t}{(\Delta \alpha)^2}D^{\mathrm{n}}_{\alpha;\,\mathrm{k}+\frac{1}{2}} - \frac{\Delta t}{\Delta \alpha}\left(1-\delta^{\mathrm{n+1}}_{\alpha;\,\mathrm{k-1}}\right)F^{\mathrm{n}}_{\alpha;\,\mathrm{k}-\frac{1}{2}} + \frac{\Delta t}{(\Delta \alpha)^2}D^{\mathrm{n}}_{\alpha;\,\mathrm{k}-\frac{1}{2}}\\
&+ \frac{\Delta t}{\Delta \beta}\delta^{\mathrm{n+1}}_{\beta;\,\mathrm{l}}F^{\mathrm{n}}_{\beta;\,\mathrm{l}+\frac{1}{2}} + \frac{\Delta t}{(\Delta \beta)^2}D^{\mathrm{n}}_{\beta;\,\mathrm{l}+\frac{1}{2}} - \frac{\Delta t}{\Delta \beta}\left(1-\delta^{\mathrm{n+1}}_{\beta;\,\mathrm{l-1}}\right)F^{\mathrm{n}}_{\beta;\,\mathrm{l}-\frac{1}{2}} + \frac{\Delta t}{(\Delta \beta)^2}D^{\mathrm{n}}_{\beta;\,\mathrm{l}-\frac{1}{2}}
\end{aligned}\right\} P^{\mathrm{n+1}}_{\mathrm{h,k,l}}\\
&+\left\{\begin{aligned}
&\frac{\Delta t}{\Delta x}\left(1-\delta^{\mathrm{n+1}}_{1;\,\mathrm{h}}\right)F^{\mathrm{n}}_{1;\,\mathrm{h}+\frac{1}{2}} - \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{1;\,\mathrm{h}+\frac{1}{2}}\\
&+\frac{\Delta t}{\Delta x}\left(1-\delta^{\mathrm{n+1}}_{2;\,\mathrm{h}}\right)F^{\mathrm{n}}_{2;\,\mathrm{h}+\frac{1}{2}} - \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{2;\,\mathrm{h}+\frac{1}{2}}
\end{aligned}\right\} P^{\mathrm{n+1}}_{\mathrm{h+1,k,l}}\\
&+\left[\frac{\Delta t}{\Delta \alpha}\left(1-\delta^{\mathrm{n+1}}_{\alpha;\,\mathrm{k}}\right)F^{\mathrm{n}}_{\alpha;\,\mathrm{k}+\frac{1}{2}} - \frac{\Delta t}{(\Delta \alpha)^2}D^{\mathrm{n}}_{\alpha;\,\mathrm{k}+\frac{1}{2}}\right] P^{\mathrm{n+1}}_{\mathrm{h,k+1,l}}\\
&+\left[\frac{\Delta t}{\Delta \beta}\left(1-\delta^{\mathrm{n+1}}_{\beta;\,\mathrm{l}}\right)F^{\mathrm{n}}_{\beta;\,\mathrm{l}+\frac{1}{2}} - \frac{\Delta t}{(\Delta \beta)^2}D^{\mathrm{n}}_{\beta;\,\mathrm{l}+\frac{1}{2}}\right] P^{\mathrm{n+1}}_{\mathrm{h,k,l+1}}\\
&+\left\{\begin{aligned}
&-\frac{\Delta t}{\Delta x}\delta^{\mathrm{n+1}}_{1;\,\mathrm{h-1}}F^{\mathrm{n}}_{1;\,\mathrm{h}-\frac{1}{2}} - \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{1;\,\mathrm{h}-\frac{1}{2}}\\
&-\frac{\Delta t}{\Delta x}\delta^{\mathrm{n+1}}_{2;\,\mathrm{h-1}}F^{\mathrm{n}}_{2;\,\mathrm{h}-\frac{1}{2}} - \frac{\Delta t}{(\Delta x)^2}D^{\mathrm{n}}_{2;\,\mathrm{h}-\frac{1}{2}}
\end{aligned}\right\} P^{\mathrm{n+1}}_{\mathrm{h-1,k,l}}\\
&+\left[-\frac{\Delta t}{\Delta \alpha}\delta^{\mathrm{n+1}}_{\alpha;\,\mathrm{k-1}}F^{\mathrm{n}}_{\alpha;\,\mathrm{k}-\frac{1}{2}} - \frac{\Delta t}{(\Delta \alpha)^2}D^{\mathrm{n}}_{\alpha;\,\mathrm{k}-\frac{1}{2}}\right] P^{\mathrm{n+1}}_{\mathrm{h,k-1,l}}\\
&+\left[-\frac{\Delta t}{\Delta \beta}\delta^{\mathrm{n+1}}_{\beta;\,\mathrm{l-1}}F^{\mathrm{n}}_{\beta;\,\mathrm{l}-\frac{1}{2}} - \frac{\Delta t}{(\Delta \beta)^2}D^{\mathrm{n}}_{\beta;\,\mathrm{l}-\frac{1}{2}}\right] P^{\mathrm{n+1}}_{\mathrm{h,k,l-1}}
\end{aligned} \tag{7}$$

where h: 0, 1, 2, …, N_{H} denotes the domain of x; k: 0, 1, 2, …, N_{K} denotes the domain of α; l: 0, 1, 2, …, N_{L} denotes the domain of β; and n: 0, 1, 2, … denotes the domain of time t. The expression for the δ_{α} parameter is given in Eq. (8) below, while the expressions for the other δ parameters can be written in a similar manner.

$$\delta^{\mathrm{n+1}}_{\alpha;\,\mathrm{k}} = \frac{D^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}} - \left(D^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}} - \Delta\alpha F^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}\right)\exp\left[\Delta\alpha \dfrac{F^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}}{D^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}}\right]}{\Delta\alpha F^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}\left\{\exp\left[\Delta\alpha \dfrac{F^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}}{D^{\mathrm{n}}_{\alpha;\,\mathrm{i,j,k+\frac{1}{2},l}}}\right] - 1\right\}} \tag{8}$$

~~Eq. (7) is, therefore, the discretized version of the FPE that was obtained from the proposed methodology. This equation can be used to solve the stochastic Saint-Venant equations with uncertain parameters. Implicitly solving~~ Equation~~.~~ (7) can be solved implicitly ~~involves~~to comput~~ing~~e the joint PDF of the state variables within the x–α–β domain. From this, ~~, from which~~ a quantitative probabilistic description of the ensemble behavior and variability can be determined for the hydrologic system represented by the stochastic Saint-Venant equations with uncertain parameters. ~~In the following sections, Eq. (7) will be applied to a hypothetical application problem, and its performance will be measured by comparing its results against the results obtained by using the MC approach.~~

3 Application of the Monte Carlo approach and the new Fokker–Planck Equation methodology to a hydraulic routing problem

A hypothetical hydraulic routing problem will be the illustrative example which the new proposed methodology will be tested on. The hypothetical problem involves a river reach that is 2.70 km long, sloping at 0.0015 throughout the whole reach, having a rectangular cross section with a constant width of 6.1 m, and having no lateral inflow. Initially (at $t = 0$), the river is assumed to have a steady uniform flow of 15.5 $\mathrm{m^3\,s^{-1}}$ throughout the reach. At t > 0, the upstream flow increases linearly to reach 56 $\mathrm{m^3\,s^{-1}}$ at $t = 20$ min, then decreases linearly back to reach the initial flow value at $t = 60$ min, after which it remains at the same constant flow value for $t > 60$ min. ~~If all parameters are assumed to be known, the described problem will be deterministic in nature. However, f~~For the purpose of this study, the problem is made stochastic through the uncertainty in the Manning's roughness coefficient, whose statistical characteristics must be estimated. All simulations for the MC approach and the FPE methodology were run on a computer having 16 GB of RAM and an Intel i7 processor with four cores, each core having a base frequency of 2.40 GHz and a maximum frequency of 3.40 GHz.

3.1 Estimating the statistical characteristics of Manning's roughness coefficient

Manning's roughness coefficient (n) represents the resistance of the bed of a channel to the flow of water in it (Chow, 1959). Generally, a higher n value defines a greater resistance to the flow. The value of this coefficient depends on several factors, including: surface roughness, vegetation, obstructions, as well as channel irregularity and alignment (Chow, 1959), all of which may exhibit considerable variability along the length of an open channel. It is, therefore, crucial to account for such variability in order to better represent the behavior of the unsteady open-channel flow system being solved.

The published and technical studies with sizeable datasets to address the variability of n in such open channels are few. However, some studies have provided median and range values, while others have attempted to fit different probability distributions to the data (Gates and AlZahrani, 1996). Using such information, Manning's n was assumed for this study to be a random variable with a normal distribution having a mean of 0.035 and a standard deviation of 0.005. The mean and standard deviation were chosen in a way that provides realistic values and distributions of n that fall within the ranges and statistics provided by Gates and AlZahrani (1996) and that conform to the table of typical values in Chow (1959). Moreover, the selected mean and standard deviation allowed the generation of ~~the generated~~ n values ~~from the assumed statistics~~which never fell below 0.01, thus complying with the fact that the roughness coefficients for flows in natural streams and excavated channels are always greater than 0.01 (Chow, 1959). As such, no truncation or discarding of any generated n values was required. The chosen probability density function (PDF) for the roughness coefficient was used by the MC approach as well as by the new FPE methodology when solving for the ensemble behavior and variability of the hypothetical flow problem.

3.2 Application of the Monte Carlo approach

For the hypothetical routing problem of this study, the MC approach requires repeatedly solving the Saint-Venant equations in a deterministic manner for a large number of different roughness coefficient (n) realizations. To deterministically solve the Saint-Venant equations in their full form, several numerical techniques have been developed because their analytical solution has not been possible due to the presence of nonlinear terms (Sturm, 2001; Chaudhry, 2008). For this study~~As such~~, the characteristic form of the Saint-Venant equations ~~Eqs. (1) to (4) were~~was discretized in an explicit manner by substituting the time derivatives with their first order ~~solved numerically through a~~ finite-difference forms~~discretization~~, ~~which is~~as detailed in several references, e.g., Viessman et al. (1977) and Sturm (2001). The values of the dependent variables at the new time steps were computed at the points of intersection of the positive and negative characteristic curves, which rendered the final solution on an irregular x–t grid. The solution was then interpolated onto a rectangular grid, with a Δx of 75 m and a Δt of 3 min. The simulations were performed ~~, by~~ using a parallelized process which optimized the computational time by running the simulations over ~~four cores, with no hyperthreading~~the total number of available cores (with no hyperthreading).

~~The Courant condition was used in order to ensure t~~The stability of the numerical method being used was ensured by checking the Courant condition ~~-~~(Sturm, 2001). Furthermore, two boundary conditions, one at each end of the reach, and two initial conditions were defined in this study, since the problem deals with the subcritical unsteady non-uniform flow case (Sturm, 2001). As initial conditions, the discharge at every location along the river was provided (taken as 15.5 $m^3 s^{-1}$, as explained in the problem description), and the flow was assumed to be initially uniform and steady. As for the boundary conditions, the flow hydrograph at the channel entrance was given, while at the downstream end, the channel was assumed to be hydraulically long so that the flow can be taken as normal flow, thus satisfying Manning's equation. As such, the downstream boundary condition was chosen as the depth-discharge relationship represented by Manning's equation. This equation, along with the equations corresponding to the positive characteristic (i.e., Eqs. (1) and (2)), was used to compute the flow variables at the downstream boundary.

Following the preceding discussion, the Saint-Venant equations were deterministically solved for a total of 1000 times, each time using a different realization of n that was generated based on the PDF chosen in Sect. 3.1. While a lower number of realizations may have been sufficient for accurate computations of the first moment of the flow variables, it would not have been sufficient for the accurate computations of the second moment. In fact, the standard deviation of the flow discharge computed using 50, 100, 200, and 500 realizations showed absolute relative differences that reached 65%, 35%, 20%, and 15%, respectively, when compared to the results of the 1000-realization run. Therefore, ~~T~~ the number of realizations in this study was ~~chosen~~selected to be large enough to numerically approximate, with sufficient accuracy, both the first and the second moments of the stochastic quantities of the problem at hand. ~~This MC approach provided a large number of solution realizations that formed an ensemble for each of the dependent variables, which were then analyzed in order to deduce their statistical properties, e.g., means and variances. The statistical results of the dependent variables are provided in Sect. 4, where they will be compared to the results of the new proposed FPE methodology.~~

3.3 Application of the proposed Fokker–Planck Equation methodology

Unlike the MC approach, the FPE methodology aims at solving for the ensemble behavior and variability of the stochastic unsteady open-channel flow system in one shot by ~~. The FPE methodology solves~~computing ~~for~~ the probability density function of its ~~of the unsteady open-channel flow~~ dependent variables over time and space. ~~, thus providing the ability to statistically describe the system in one simulation run.~~

Solving the hypothetical routing problem following the FPE methodology involved solving Eq. (7). Since this equation is the result of implicit discretization, it is unconditionally stable and requires no constraint on the size of the time step for its stability. However, to obtain sufficiently accurate solutions, the time step must still be limited with the Courant condition (Zheng and Bennett, 2002). Therefore, at every time position, the multidimensional Courant condition was checked in order to determine the appropriate size of the next time step.

Furthermore, to correctly solve the hydraulic routing problem, both the initial and boundary conditions of the problem in the physical space must be accurately represented in the probability space for use in the FPE methodology. Note that when the initial condition in the physical space is taken as deterministic, the Dirac delta function ($\delta(s)$) can be used to represent it in the probability space. ~~For the multidimensional case, the Dirac delta function can be written as a product of one-dimensional Dirac delta functions.~~ Assuming that $\boldsymbol{H}$ is a vector of m state variables (m-dimensional), and $\boldsymbol{H_0}$ is the corresponding vector of initial conditions, the initial condition PDF of $\boldsymbol{H}$ can be written as

$$P(\boldsymbol{H},t) = P(\boldsymbol{H},0) = \delta\cancel{\delta^{m}}(\boldsymbol{H}-\boldsymbol{H}_0) \tag{9}$$

~~where m is the number of dimensions.~~ For the purpose of this study, $\boldsymbol{H}$ corresponds to a point in the x–α–β domain (3-dimensional), and $\boldsymbol{H_0}$ represents the initial condition that defines the deterministic flow at time $t = 0$ in the x–α–β domain~~,~~

~~and m is equal to 3~~. However, since it is not possible to numerically represent the Dirac delta function as a function with infinite value at only one position, it was estimated in the probability domain as a function with a very high value over a small bottom width, while preserving an area of unity.

Concerning the boundary conditions, they are usually divided for FPEs into two categories: accessible and inaccessible. Inaccessible boundaries are defined as those boundaries that could never be reached if the process starts from any interior point of the domain. On the other hand, for accessible boundaries, there is a positive probability that these boundaries will be reached from the interior of the domain within a finite amount of time (Feller, 1954). Accessible boundaries can be further subdivided into absorbing and reflecting, or no-flux, boundaries.

Note that since the FPE can be considered as the conservation equation for probability mass, a probability mass of unity needs to be conserved in the probability domain of the system. Therefore, using a reflecting (no-flux) boundary condition would be the most suitable choice for this study in order to ensure the completeness of the probability domain and to prevent any ~~“particles”~~probability mass from leaving the domain (Gardiner, 1985). Such a condition was used to describe the boundaries of both the α and β dimensions, noting that the minimum and maximum boundaries of both α and β were chosen to be far enough so that they would encompass all possibilities that could occur for the considered routing problem. ~~Moreover~~As for the ~~, recall that the~~ upstream boundary, recall that the discharge hydrograph was assumed to be known upstream, in which case ~~. As a result,~~ the probability densities at the upstream boundary of the x–dimension were known for all $t > 0$. Finally, ~~, whereas~~ the downstream boundary in the x–dimension was formulated to replicate that of the MC model, and it was extended downstream much further than the required length of the reach~~enough~~ so as to eliminate any of its effect on the numerical solution.

~~Therefore, following the application steps just described, the FPE methodology was applied to obtain the solution of the hypothetical stochastic open-channel flow problem and to determine the ensemble behavior of the system and the statistical distributions of the flow variables, as will be discussed in Sect. 4.~~

4 Numerical results and discussion

Both the MC approach and the proposed FPE methodology were used to solve the stochastic Saint-Venant equations of the hypothetical hydraulic routing problem presented in Sect. 3, in order to determine the ensemble behavior of the system and the statistical distributions of the flow variables. While the state variables directly solved for were the velocity and depth (or celerity), the discharge was easily computed from these two variables.

A plot of the ensemble average discharge over time and space computed by the FPE methodology can be seen in Fig. 1 alongside a plot of the same results that were obtained by the MC simulations. From this figure, it is clear that the ensemble average discharge computed by the FPE methodology resembles the one obtained from the MC simulations quite well, while showing the same behavior and evolution of the mean discharge in both time and space as a result of the applied upstream wave. From the average behavior of the system, both plots show that the wave that was initiated upstream is observed to be transmitted downstream through the reach. In this case, the discharge at every location is seen to increase from the initial value of 15.5 $m^3 s^{-1}$ to some peak value, after which it decreases back again to the initial flow value. However, it is noticeable in both plots that the average peak discharge becomes lower at locations further downstream. This decrease comes as a result of the dissipation of energy through viscous effects as the water flows along the reach (Jeppson, 2011). In fact, the shearing stresses due to the vertical velocity gradients within the water lead to a loss of the fluid energy (~~mainly~~ kinetic and potential) as non-recoverable energy (e.g., increase in temperature), causing the peak velocity and discharge to decrease along the reach.

For a clearer comparison between the FPE and MC mean discharge results, cross sections of both plots from Fig. 1 at specific times (Fig. 2) and at specific channel locations (Fig. 3) were compared individually. ~~The mean discharge was plotted as a function of location at different times (Fig. 2) and as a function of time at different locations (Fig. 3).~~ From Fig. 2, it is clear that the change in the ensemble average discharge as a function of position (location within the channel reach along the flow direction) as computed by the FPE methodology almost perfectly replicates the corresponding results obtained by the MC simulations~~, with very minimal differences among the two~~. In both methods, the effect of the wave is clear as it causes an increase in the discharge first upstream, and then throughout the reach over time. Comparing the mean discharge at specific positions over time (Fig. 3) reveals that the FPE methodology also predicts the temporal change in the mean discharge very well at different channel locations. While a slight decrease in the discharge can be seen at early times of Figs. 3c and 3d, which may be attributed to the numerical discretization, ~~In fact,~~ the timing and positions of the peak discharges predicted by the FPE methodology are very similar to those of the MC approach in all plots of Fig. 3, with a maximum relative difference of only around 6%.~~ Therefore, it may be inferred from these results that the ensemble average discharge is predicted quite well by the FPE methodology when compared against the results obtained from the MC simulations.~~

Similar results were also plotted for the ensemble average flow depth. Figure 4 shows the mean river channel flow profile at different times as computed by the FPE and the MC methods. This figure shows how the applied upstream wave affects the depth profile of the river at different times, which is similarly predicted in both the FPE and MC simulations. Compared to the MC results, the ensemble average depth is predicted quite well by the FPE methodology, which only slightly overestimates the mean depth values at times between t = 30 min and t = 45 min. A comparison of the changes in depth as a function of time (Fig. 5) reveals that the FPE methodology provides a good match to the MC results while showing the same peaking pattern as the wave reaches a specific channel location. The timings of the peaks of the FPE methodology greatly match those of the MC simulations.~~ However,~~ ~~s~~Similarly to Fig. 4, a slight overestimation can be noticed from the FPE methodology especially around the peak depths, but with a ~~the~~ maximum relative difference ~~between both methods was~~of only around 7.5%. ~~Hence, Figs. 4 and 5 both reveal that the FPE methodology can represent well the temporal and spatial evolution in the ensemble average flow depth due to the applied upstream wave, when compared to the MC results.~~

As for the velocity, Figs. 6 and 7 show the same two kinds of plots as before by comparing the mean ensemble average velocity results of the FPE methodology to those of the MC approach. Figure 6 shows that the mean velocity computed by the FPE methodology provides a good match to the MC results, with a slight underestimation. The pattern of change in the mean velocity as a function of position that is presented by the FPE methodology is very similar to that shown by the MC simulations, which demonstrates the effects of the applied upstream wave as it travels downstream with time. Figure 7 also reveals a good match between the two modeling approaches, with the FPE methodology providing the same pattern of velocity change over time as computed by the MC simulations. Similarly, a slight underestimation of the FPE mean velocity is also seen in Fig. 7, but the maximum absolute relative difference between both methods was only around 11%. Therefore, Figs. 1 to 7 show the capability of the FPE methodology in predicting the ensemble averages of the flow variables quite well over space and time, which in turn provides ~~Hence, in both figures, the FPE methodology seems to maintain its ability to provide a good match to the mean velocity values and patterns obtained by the MC simulations.~~support and validation to the simplifications and assumptions applied to the FPE methodology.

~~Therefore, from the preceding discussion, it may be inferred that the FPE methodology can predict the ensemble average discharge, depth, and velocity well over time and space even with the simplifications and assumptions applied when deriving the methodology. Since the ensemble average results generally match between the FPE and MC methods, and since the differences between them are mostly minute, these results provide support and validation to the simplifications and assumptions applied to the FPE methodology that was used to estimate the ensemble average behavior of the system.~~

~~Although checking the ensemble average of the flow variables provides an estimate of the mean behavior of the system, it is also important to check the variability of the system in order to have an idea about the range of possible results that may be expected. The standard deviation may be considered a good measure for such variability. Focusing on the flow discharge for the rest of the discussion, the standard deviation for the discharge computed by the FPE methodology was compared against the MC results to see the relative performance of the FPE methodology in predicting the system's variability.~~

In a similar manner to the ensemble averages, the relative performance of the FPE methodology in predicting the system's variability was examined, this time by checking the standard deviations. Figure 8 shows a comparison of the standard deviation of the flow discharge over space and time as computed by the FPE methodology and by the MC simulations. Both plots of this figure reveal that the standard deviation experiences two triangular areas of high values, the earlier in time being generally higher than the later, and both areas showing a stronger intensity further downstream. While the general resemblance of the FPE plot to the MC plot is good, the second area of elevated standard deviations is more compact in the FPE results and shows slightly lower values than those of the MC results. In an attempt to study such differences more closely, cross sections of both plots from Fig. 8 at specific times and at specific channel locations were compared individually (Figs. 9 and 10), in a manner similar to the ensemble average plots discussed previously~~Figures 8 and 9 compare plots of the standard deviation of the flow discharge as computed by the FPE and the MC methods. Similar to the~~

~~ensemble average plots discussed previously, the former figure presents the results as a function of channel location at different times, while the latter presents them as a function of time at different channel locations~~.

Looking at the change in the standard deviation of the discharge over position, Fig.~~ure~~ 9~~8~~ reveals how the movement of the wave in the downstream direction causes an increase in the standard deviation of the discharge, thus increasing the variability in the discharge. After the whole channel is affected by the moving wave, the variability is seen to be highest at the downstream end of the channel (e.g., Figs. 9~~8~~b, 9~~8~~c, and 9~~8~~d). As the wave subsides, the standard deviation starts to decrease (Figs. 9~~8~~c and 9~~8~~d). Comparing the results of the FPE methodology to those of the MC approach shows that the FPE results correctly predict the patterns of change in the standard deviation as a function of position at each of the different times (Fig. 9~~8~~). While the values of the standard deviations are slightly overestimated in some of the FPE results, especially at t = 30 min, ~~the differences in the results are relatively small~~the FPE results are still quite close to the MC results. ~~Therefore, it may be inferred that the FPE methodology provides an effective representation of the spatial changes in the standard deviation of the discharge, when compared to the results of the MC simulations.~~

Concerning the changes in the standard deviation of the discharge over time, Fig. 10~~9~~ shows that the standard deviation forms an M-shaped pattern at all the plotted locations, which can also be deduced from Fig. 8. For the MC results, the magnitudes of the standard deviation forming this M-shaped pattern are observed to be generally increasing as a function of the distance away from the upstream boundary. It is noticeable that the results of the FPE methodology also predict an M-shaped pattern and an increase in the magnitude of the standard deviation of the discharge as a function of location. However, the standard deviation of the FPE methodology shows an offset at time t = 0 when compared to the MC simulations, which can also be noticed at the upstream positions of Fig. 9. ~~While~~Recall that the initial and upstream flow discharge is assumed to be known~~, it was not possible to numerically transfer such information onto one single position in the α–β plane of the FPE methodology~~. ~~In fact~~Nonetheless, a single known value of the flow discharge, when joined with a spread of roughness coefficients due to the uncertainty involved, leads to an unavoidable spread in the velocity and depth values. Since α and β are functions of the velocity and celerity (and in turn, depth), this spread ~~was~~is translated onto the α–β plane of the FPE methodology. As a result, with an uncertain roughness coefficient, the only way to numerically represent a deterministic discharge in the α–β plane ~~was~~is to have a spread of probability mass over the values involved. The existence of this spread on a non-continuous, discretized α–β plane~~This spread~~ ~~may~~may have had the most contribution to the offset of the standard deviation at the initial times and positions of Figs. 9 and 10.

Nonetheless, ~~-~~when looking at the standard deviation after the initial times, ~~T~~one can see that the timings, shapes, and magnitudes of the first peak for the MC results are generally matched well by the FPE results as shown by the plots of Fig. 10~~9~~. Moreover, the locations and magnitudes of the second peak are well-represented by the FPE results in Figs. 10~~9~~a and 10~~9~~b. However, differences arise in the position and magnitude of the second peak for locations further downstream (Figs. 10~~9~~c and 10~~9~~d). In fact, while moving further downstream, the second peak is observed to be underestimated by the FPE results and slightly shifted back in time when compared to the MC simulations. Moreover, the standard deviation following the second peak shows a faster decline in time for the FPE methodology than for the MC approach, while the local minimum between the two peaks is seen to be overestimated by the FPE results.

Several factors may have caused the discrepancies in the standard deviation of the discharge as computed by the FPE methodology. Among those factors may be the approximations and assumptions that were applied in order to simplify the FPE methodology to an easily solvable degree. These approximations have simplified the computations of the drift and diffusion coefficients of the equation, thus causing some discrepancies in the movement and the spread of the probability mass within the domain. Other factors may include the numerical method selected for computing the FPE methodology, as well as the associated spatial and temporal discretizations that were used. All of these may have had an effect on the representation of the probability flow within the domain, thus leading to some discrepancies in the variability results. Such discrepancies may be faced when using the FPE methodology in engineering flow applications, such as flood forecasting and flood control. The variability of the flow in flood forecasting applications, for example, may be underestimated at the downstream end of the reach, specifically during the later time periods. This would impact the range of flows that are forecasted to occur at the downstream end.

Nonetheless, it may still be inferred that the FPE methodology performs satisfactorily in predicting the variability of the discharge in both space and time. In fact, the problem being solved is highly nonlinear, and involves a large uncertainty in one of its parameters. As such, estimations in the mean and variance are even more difficult to accurately quantify. The FPE methodology was capable of not only correctly matching the general patterns of the spatial and temporal changes in the discharge standard deviation, but also correctly providing the standard deviation values within a range that is very similar to the range computed by the MC simulations.

Concerning the standard deviations of the velocity and depth, it is important to note that their behavior over position is somewhat different from that of the flow discharge. In fact, the standard deviations of the velocity at the same four time positions of Fig. 9 seem to be relatively constant at each time position, having a value between 0.015 and 0.02 $m^3\ s^{-1}$. On the other hand, the standard deviation of the depth showed a greater range of values at each time position, as a function of location, with values ranging between 0.15 and 0.5 m. When looking at the standard deviations as a function of time at the same four locations of Fig. 10, both standard deviations seem to show that their values increase to reach a maximum and then decrease to levels similar to original levels, not unlike their corresponding ensemble average plots over time. Again, the range of change in the standard deviation of the velocity is much smaller (0.015 to 0.02 $m^3\ s^{-1}$) than that of the depth (0.15 to 0.5 m). Note that the relative differences of the FPE results when compared to the MC results reach up to 23% and 29% for velocity and depth, respectively.

~~Moreover, these FPE results required a significantly less amount of time for computation as opposed to the MC results. In fact, the 1000 MC simulations ran for over 2 days, whereas the results of the FPE methodology were obtained in about 7 hours. Therefore, despite the uncertainty introduced by the roughness coefficient and despite the approximations~~

~~assumed for the FPE, the FPE methodology was able to provide an effective and satisfactory representation of both the spatial and temporal variability in the discharge within a much shorter simulation time period.~~

A final comparison between the FPE methodology and the MC approach is to compare the probability density functions (PDFs) of the flow discharge at different times and locations. The large number of simulations (1000) done using the MC approach provided an equal number of flow discharge results at each specific x–t position in the space–time plane. These values were then used in order to estimate the PDF of the discharge at that specific x–t position. The FPE methodology, on the other hand, directly solved for the evolution of the PDFs of the state variables through time and space. However, recall that the FPE was solved in the x–α–β domain. With the discharge being a function of both α and β, the PDF of the discharge at each specific x–t position was deduced from the x–α–β PDF provided by the FPE methodology.

For comparison, the PDFs of the flow discharge obtained from the FPE and MC methods were plotted at three different channel locations (x = 900 m, 2250 m, and 2700 m) and at four different times for each location (t = 15 min, 30 min, 45 min, and 60 min), as shown in Fig. 11~~10~~. For most of the plots, the peak values of the PDFs predicted by the FPE methodology are similar to those computed by the MC approach. However, ~~it may be noted that as the spread of the PDF becomes greater along the x-axis, the peak values become smaller in order to preserve an area of unity under the graph. Therefore, a larger variability leads to lower PDF values. Such an example can be seen~~taking ~~in~~ Fig. 11~~10~~g as an example,~~,~~ ~~where a major difference is noticed between the peaks of the PDFs resulting from the MC and FPE methods. While~~ while the peak of the MC PDF has a value of around 0.35, that of the FPE methodology has a value of around 0.065. This is because ~~T~~the PDF of the FPE methodology has a larger spread, which causes the reduction in its PDF values to preserve an area of unity. It may be noted that this plot corresponds to a point close to the local minimum of the plot in Fig. 10~~9~~b, in which the FPE methodology is seen to predict a larger standard deviation than the MC approach. Therefore, while both of the PDFs, corresponding to the FPE and MC approaches, seem to be located within the same range and~~,~~ thus providing similar expected values of the discharge, the greater variability of the FPE PDF causes its lower peak values.

~~The prediction of the evolution of the PDFs for the flow discharge in space and time is a rather difficult task for the problem considered in this study. A lot of~~Many different factors ~~are involved that would~~may have affected ~~their~~ the evolution and calculation of the PDFs for the flow discharge in space and time. Such factors may even include~~, including~~ the step sizes in the α and β directions whose values directly affect the computation of the discharge PDFs. However, the PDFs predicted by the FPE methodology are generally seen to be following similar trends to those computed by the MC approach (Fig. 11~~10~~) while also satisfactorily predicting the ranges and locations of the PDFs (along the x-axis). Thus, with all the variability encompassing the routing problem considered, and with all the assumptions and approximations used during the application of the FPE methodology, the probability densities predicted by the proposed FPE methodology are considered to be rather encouraging.

It should be noted that these FPE results required a significantly less amount of time for computation as opposed to the MC results. Recall that the 1000 MC simulations were parallelized and run over all four cores (with no hyperthreading), thus noticeably reducing the computational time as compared to an un-parallelized run. With such parallelization, the MC simulations ran for over 2 days. On the other hand, the results of the FPE methodology, which was not parallelized, were obtained in about 7 hours.

If we observe the computational times of the implicit numerical solution of the FPE methodology, the portion of the simulation requiring the greatest time is filling out the coefficient matrix, especially for small α and β discretizations. Parallelizing this portion over the four cores would allow one to considerably reduce the time to fill out the coefficient matrix, thus reducing the total computational time of this method. Without the parallelization of the FPE methodology, its one simulation may still not seem to provide an immense advantage when only one uncertain parameter is involved, especially with the possibility of parallelizing the MC simulations among a much larger number of cores. Nonetheless, when the problem being solved involves a greater number of uncertain parameters and boundary conditions, or even a larger system, such an advantage may prove to be crucial. In fact, the computational expense of the MC simulations for such a case would be expected to increase exponentially due to the higher number of simulations needed to maintain the desired accuracy in the results, thus significantly increasing the computational time regardless of parallelization. On the other hand, such additional uncertainties can be easily implemented into the FPE methodology by making simple changes and additions that will be reflected in Eq. (5), after which the FPE would be solved following similar steps as discussed for this study, with minimal implications on the computational expense.

5 Summary and conclusions

This study applied the proposed FPE methodology derived in the companion paper by Dib and Kavvas (2017) to a stochastic unsteady open-channel flow problem, with an uncertain roughness coefficient. The equations used to describe the open-channel flow problem were the Saint-Venant equations, transformed into their characteristic form by the method of characteristics. The proposed FPE methodology was applied in order to solve for the probability density of the flow state variables (velocity and depth/celerity, as well as discharge) and to provide a quantitative description of the expected behavior and variability of the stochastic system in one single simulation, as opposed to the large number of simulations usually performed by the MC approach. The unsteady open-channel flow problem was also solved using the MC approach in order to use its results for evaluating the performance of the proposed FPE methodology.

Comparisons of the FPE results to those of the MC simulations revealed the effectiveness of the proposed FPE methodology in describing the ensemble behavior and variability of the ~~stochastic Saint-Venant open-channel flow problem, with an uncertainty in the roughness coefficient~~stochastic flow problem. In fact, the FPE methodology was found to replicate the ensemble average discharge of the MC simulations quite well in both space and time. In addition, it was also capable of effectively representing well the temporal and spatial change in the ensemble average depth and the ensemble average velocity. Furthermore, this method provided a good representation of the patterns and ranges of the standard deviation of the discharge over time and space, and showed a satisfactory prediction of the spatial and temporal trends and ranges of the flow discharge PDFs. These encouraging results were obtained despite simplifications applied to the FPE methodology.

~~Furthermore,~~ Furthermore, ~~w~~with the FPE approach, the ~~simulation~~ computational time ~~period~~ was significantly less than the time taken by the MC approach~~. Moreover~~, and the FPE methodology results were obtained by running only one simulation, as opposed to the large number of simulations performed by the MC approach. Such an advantage becomes even more prominent with a greater number of uncertain parameters and boundary conditions, in which case the computational expense of the MC simulations that is needed to preserve the desired accuracy would exponentially increase. On the other hand, only simple adjustments would be required for the FPE, which could then be solved as was done in this study, with minor implications on its computational expense. Therefore, the results obtained in this study indicate that the proposed FPE methodology may be a powerful and time-efficient approach for predicting the ensemble average and variance behavior, in both space and time, for the unsteady open-channel flow process under an uncertain roughness coefficient, hence being an approach that ~~.~~would be essential for engineering flow problems.

While the FPE methodology satisfactorily described the ensemble average and variability of the open-channel flow system in this study, this methodology is ~~undoubtedly~~ open to improvements especially ~~regarding~~with regard to reducing any discrepancies in its numerical results. Running a more comprehensive version of the FPE methodology, by including only some of the simplifying assumptions used in this study, may be one option. Another option may involve using a higher-order and more accurate numerical scheme for the discretization of the multidimensional FPE. As such, numerous opportunities present themselves for future research within this topic, all of which would be of great benefit in the further improvement of the proposed methodology.

~~While this study considered the uncertainty in the system to be originating from the roughness coefficient, uncertainties may arise from other sources as well. Hence, future research could entail investigating the uncertainties due to the channel slope, channel cross section, lateral inflows, as well as initial and boundary conditions. Moreover, applying the proposed FPE methodology to systems which include more than one source of uncertainty could be a further extension of the methodology in its attempt to effectively and efficiently describe such highly nonlinear and stochastic systems.~~

6 Data availability

This study involved the application of a new proposed methodology for the stochastic solution of a hypothetical unsteady open-channel flow problem. All the required parameters of this flow problem are provided in the study, and thus may be easily used for replicating the solution, if needed.

Competing interests. The authors declare that they have no conflict of interest.

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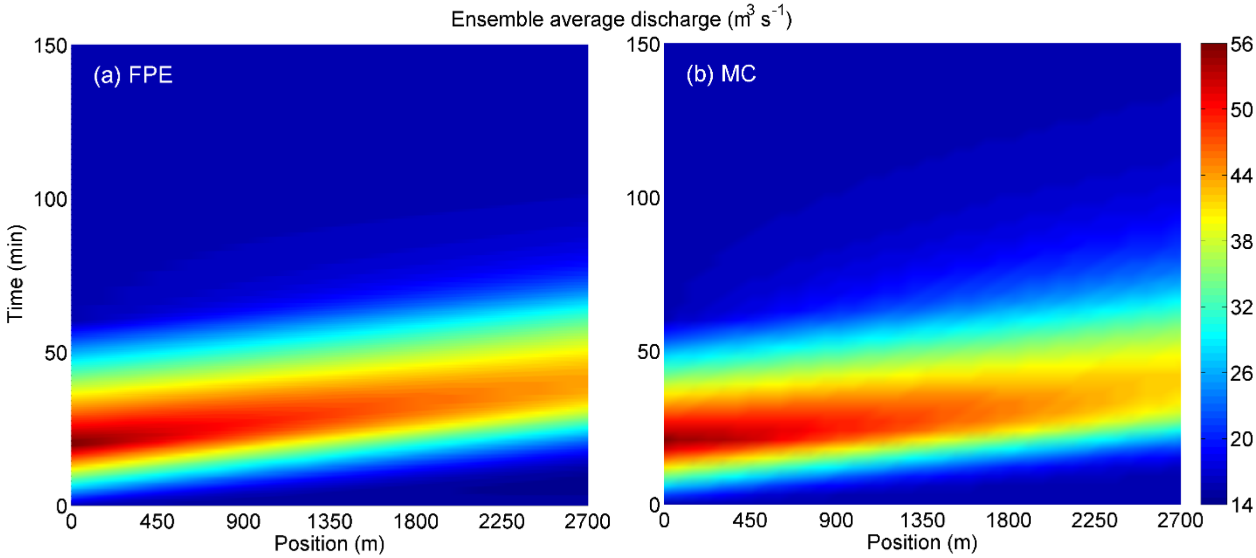

Figure 1: Ensemble average discharge over channel position and time obtained by (a) the FPE methodology and (b) the MC approach.

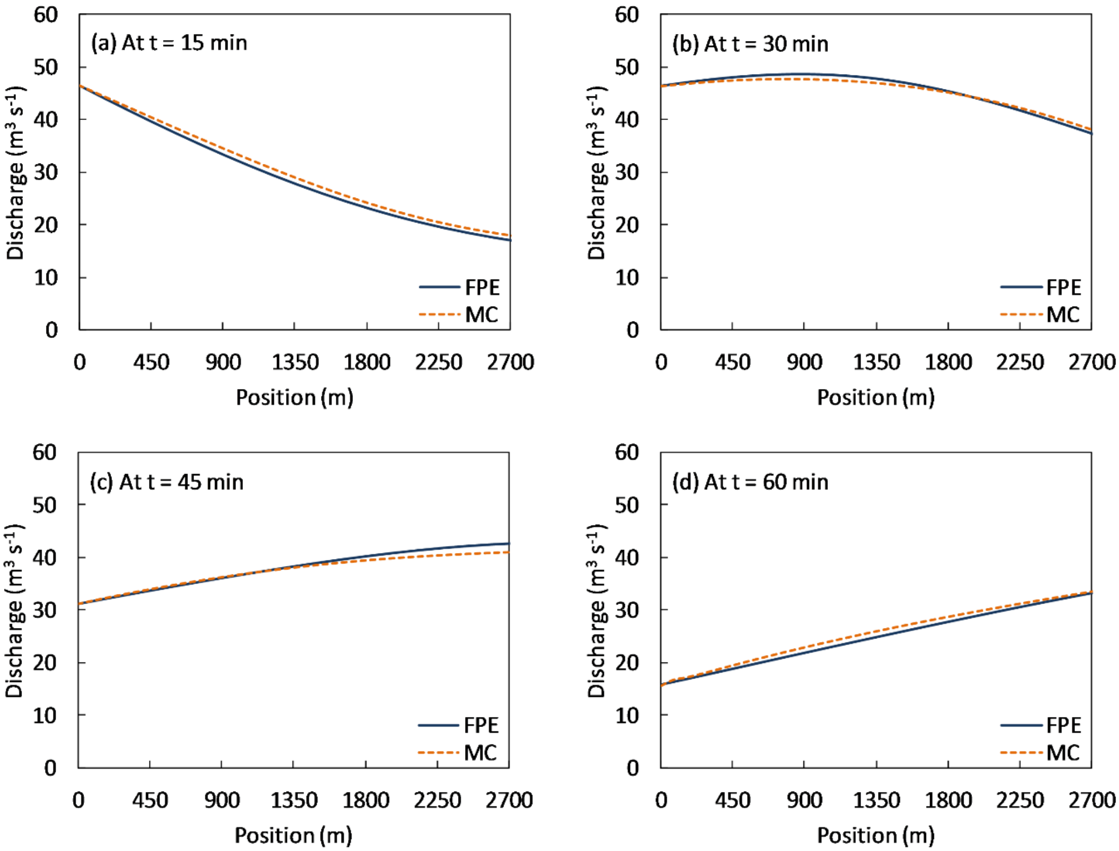

Figure 2: Comparison of the ensemble average discharge obtained by the FPE methodology and the MC approach as a function of channel location (Position), at different times.

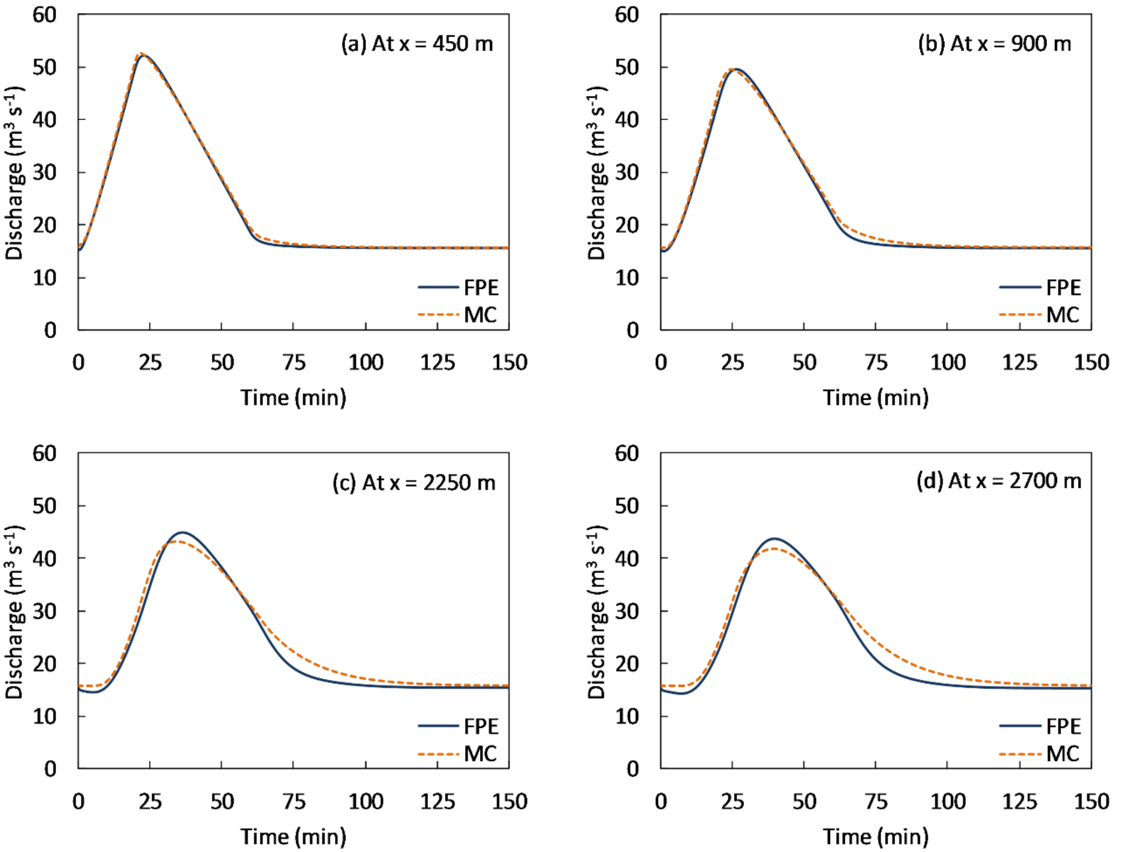

Figure 3: Comparison of the ensemble average discharge obtained by the FPE methodology and the MC approach as a function of time, at different channel locations.

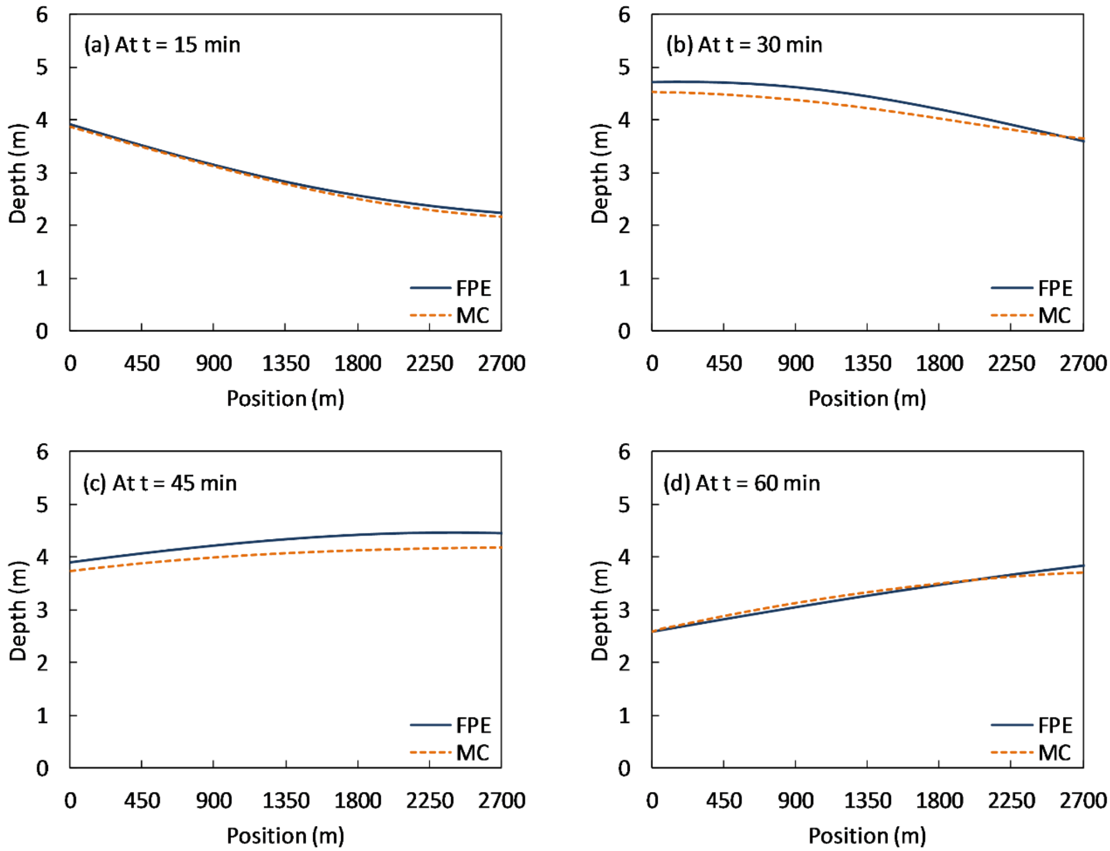

Figure 4: Comparison of the ensemble average flow depth obtained by the FPE methodology and the MC approach as a function of channel location, at different times.

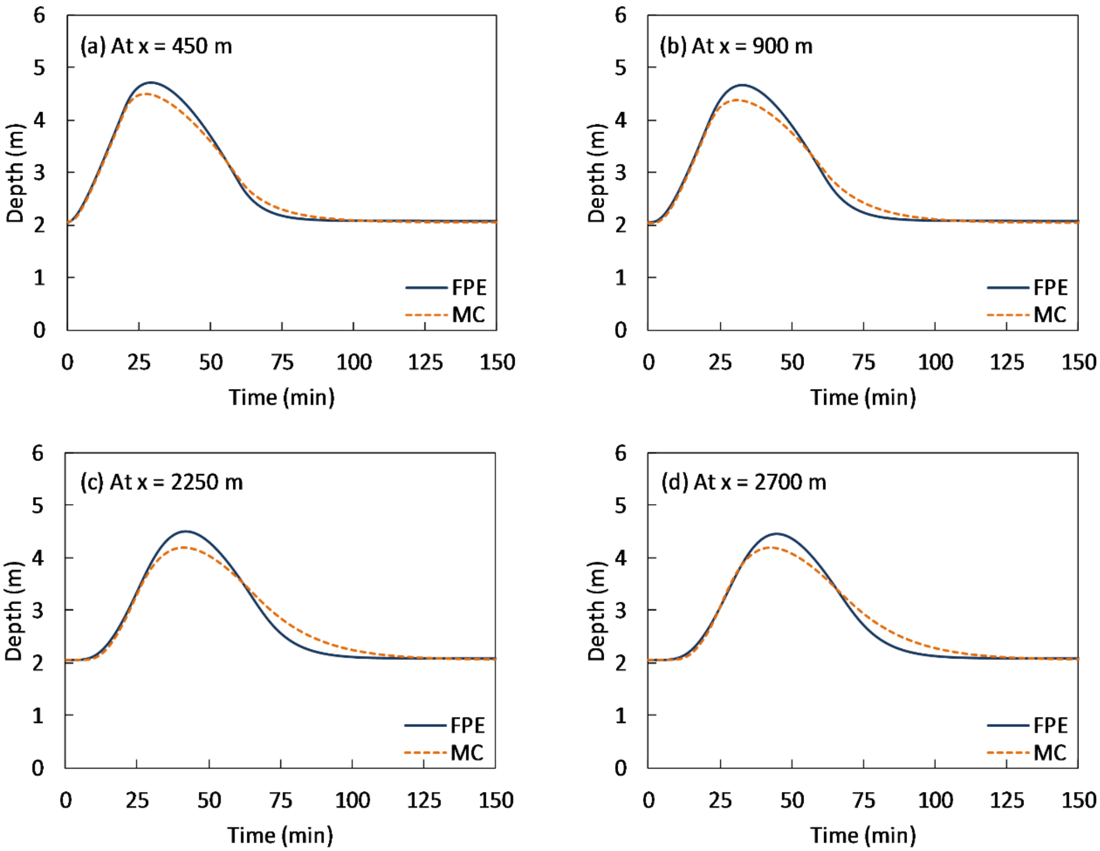

Figure 5: Comparison of the ensemble average flow depth obtained by the FPE methodology and the MC approach as a function of time, at different channel locations.

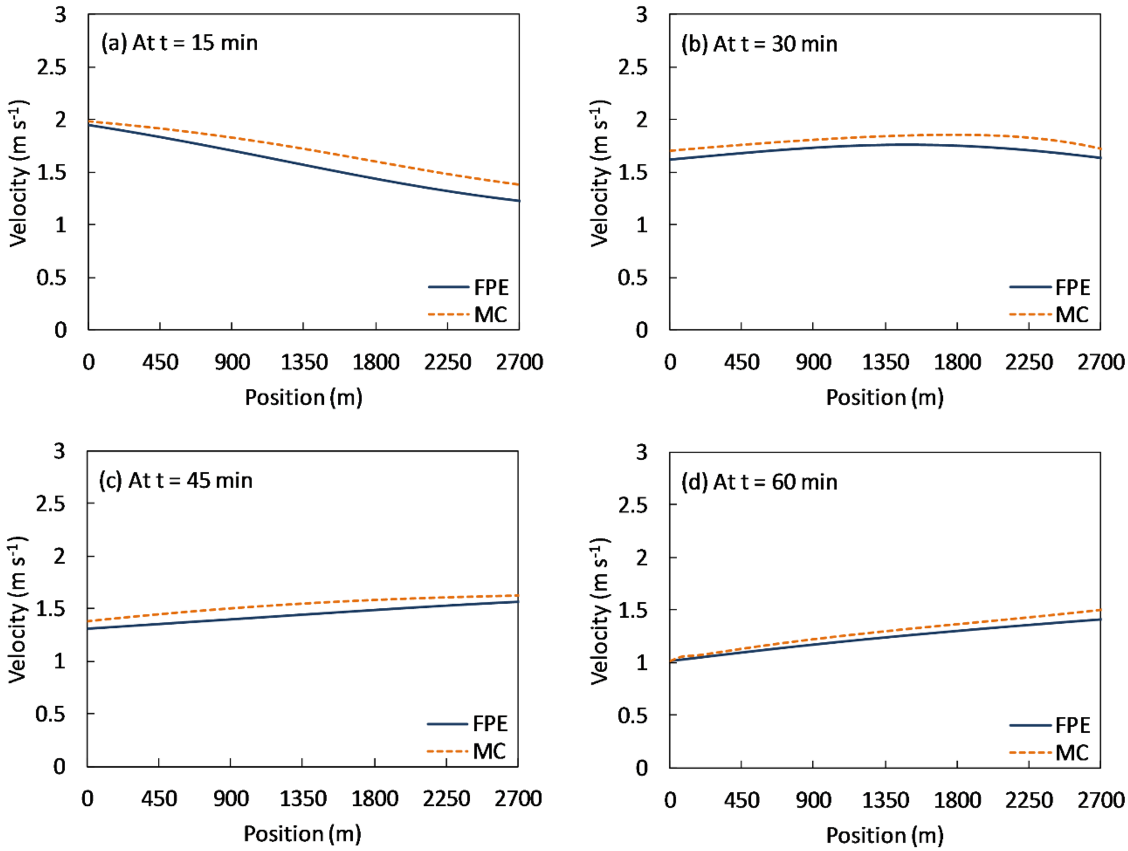

Figure 6: Comparison of the ensemble average velocity obtained by the FPE methodology and the MC approach as a function of channel location, at different times.

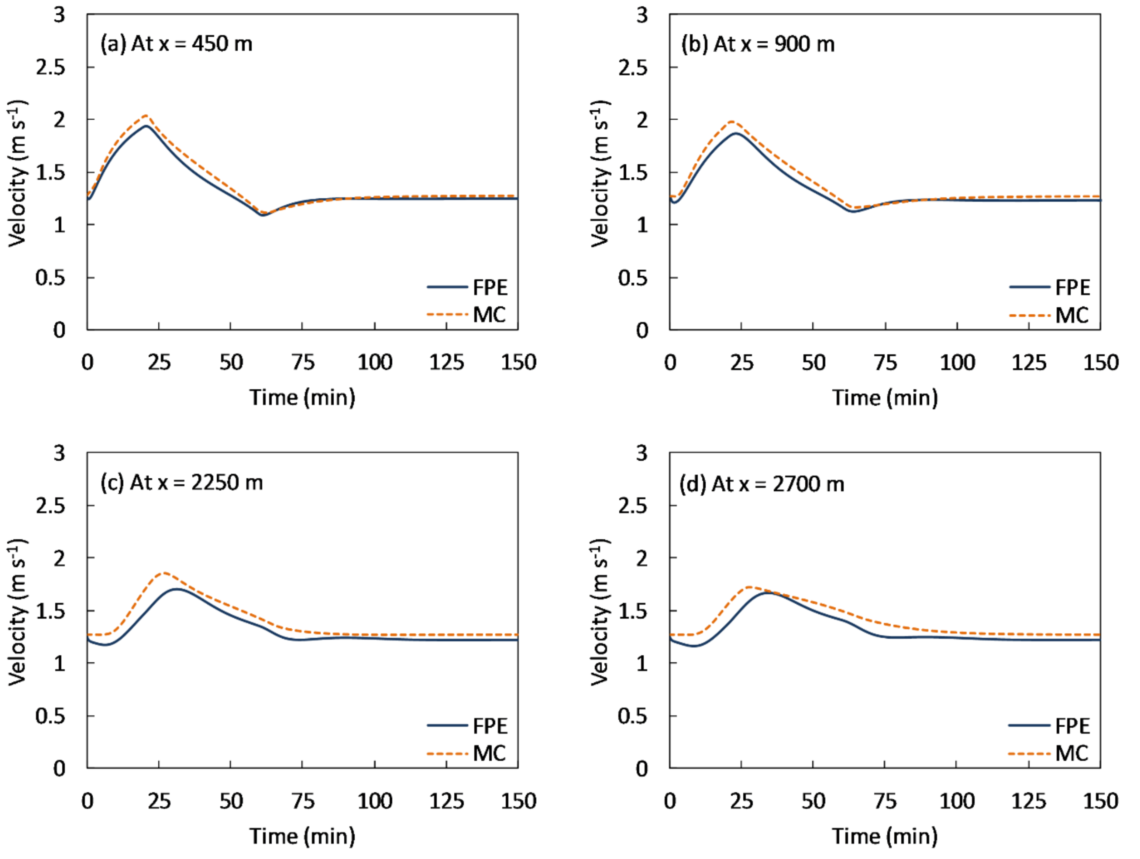

Figure 7: Comparison of the ensemble average velocity obtained by the FPE methodology and the MC approach as a function of time, at different channel locations.

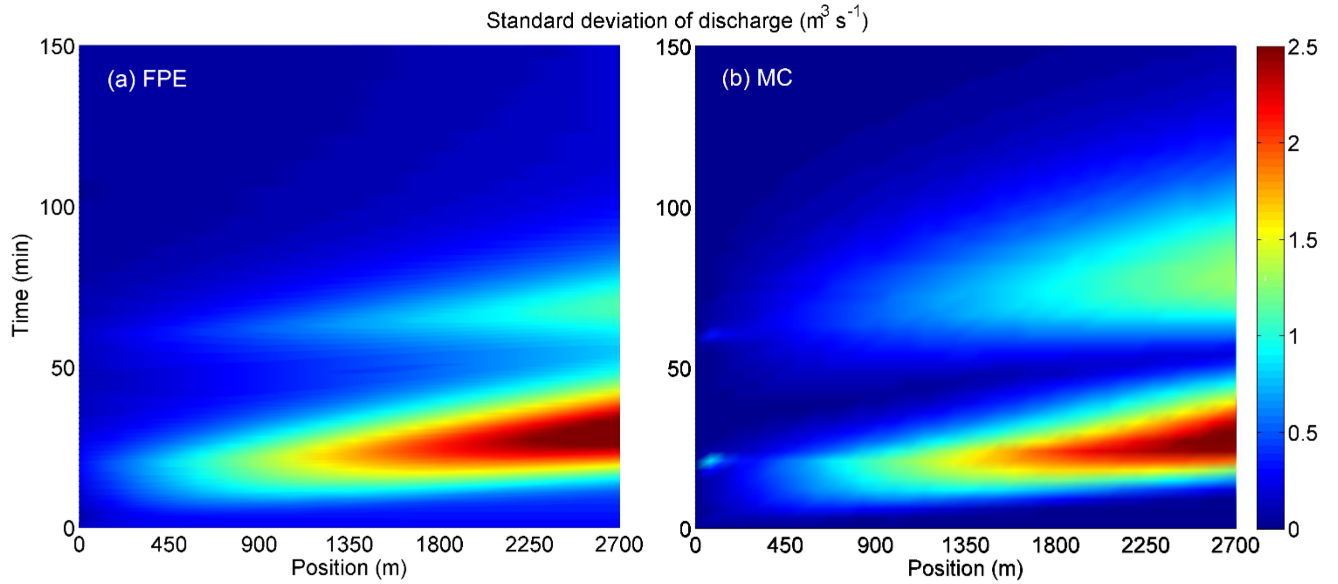

Figure 8: Standard deviation of flow discharge over channel position and time obtained by (a) the FPE methodology and (b) the MC approach.

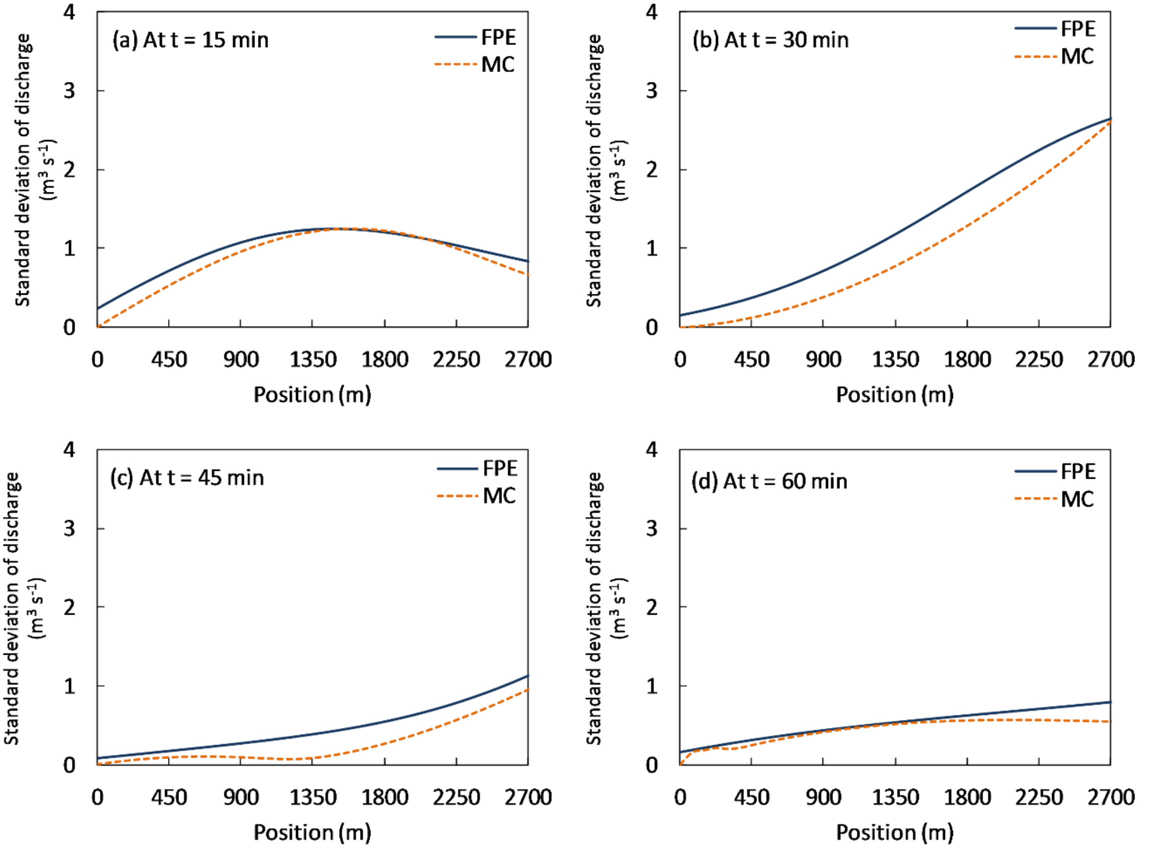

Figure 9: Comparison of the standard deviation of the flow discharge obtained by the FPE methodology and the MC approach as a function of channel location, at different times.

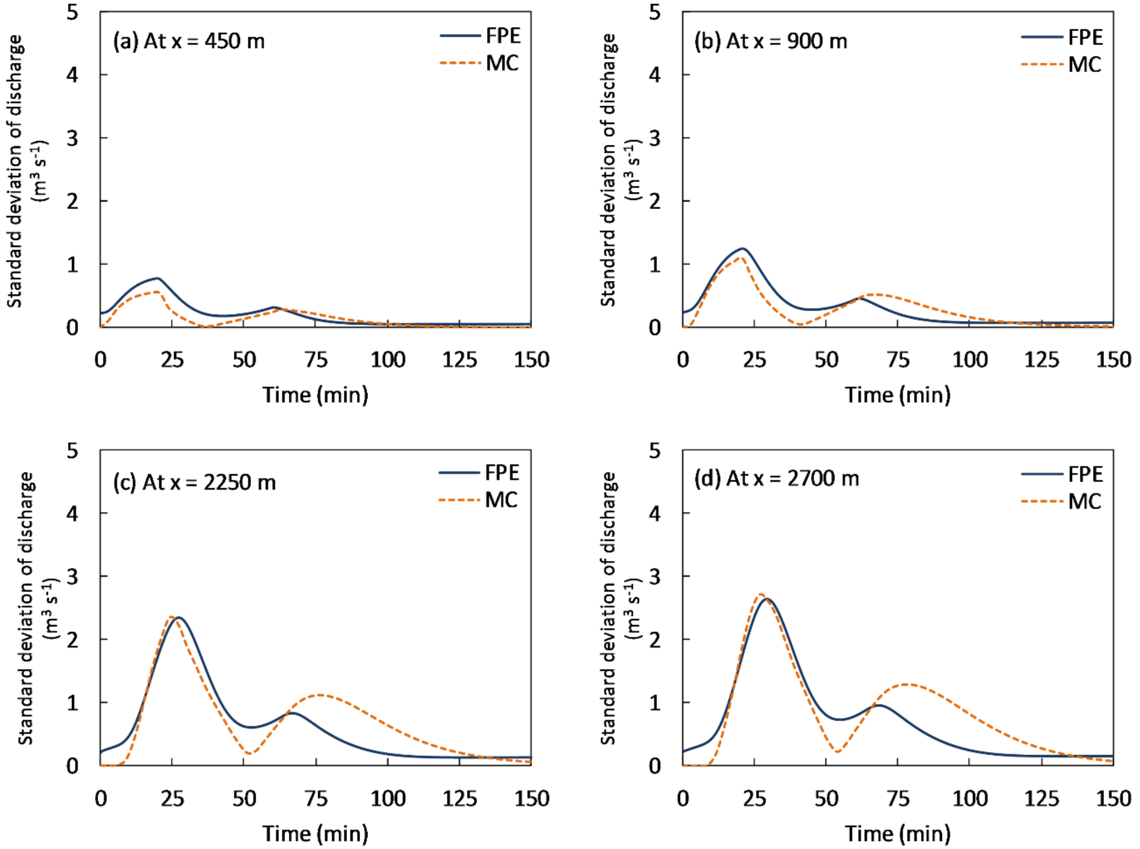

Figure 10: Comparison of the standard deviation of the flow discharge obtained by the FPE methodology and the MC approach as a function of time, at different channel locations.

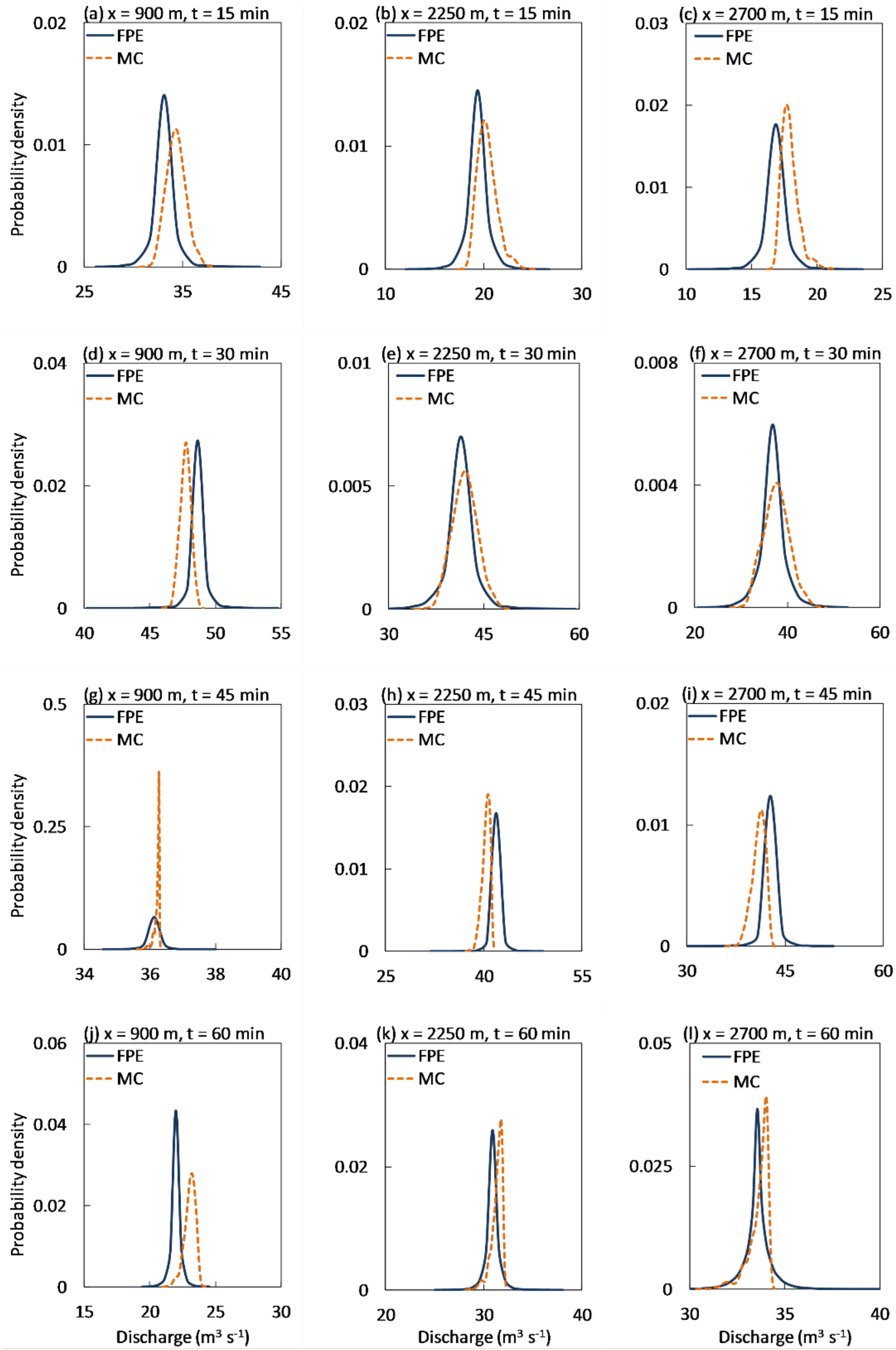

Figure 11: Comparison of the probability density functions of the flow discharge obtained by the FPE methodology and the MC approach, plotted at different times and channel locations.