Governing equations of transient soil water flow and soil water flux in multi–dimensional fractional anisotropic media and fractional time

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Abstract. In this study dimensionally-consistent governing equations of continuity and motion for transient soil water flow and soil water flux in fractional time and in fractional multiple space dimensions in anisotropic media are developed. Due to the anisotropy in the hydraulic conductivities of natural soils, the soil medium within which the soil water flow occurs is essentially anisotropic. Accordingly, in this study the fractional dimensions in two horizontal and one vertical directions are considered to be different, resulting in multi-fractional multi-dimensional soil space within which the flow takes place. Toward the development of the fractional governing equations, first a dimensionally-consistent continuity equation for soil water flow in multi-dimensional fractional soil space and fractional time is developed. It is shown that the fractional soil water flow continuity equation approaches the conventional integer form of the continuity equation as the fractional derivative powers approach integer values. For the motion equation of soil water flow, or the equation of water flux within the soil matrix in multi-dimensional fractional soil space and fractional time, a dimensionally consistent equation is also developed. Again, it is shown that this fractional water flux equation approaches the conventional Darcy's equation as the fractional derivative powers approach integer values. From the combination of the fractional continuity and motion equations, the governing equation of transient soil water flow in multi-dimensional fractional soil space and fractional time is obtained. It is shown that this equation approaches the conventional Richards equation as the fractional derivative powers approach integer values. Then by the introduction of the Brooks-Corey constitutive relationships for soil water into the fractional transient soil water flow equation, an explicit form of the equation is obtained in multi-dimensional fractional soil space and fractional time. The governing fractional equation is then specialized to the case of only vertical soil water flow and of only horizontal soil water flow in fractional time-space. It is shown that the developed governing equations, in their fractional time but integer space forms, show behavior consistent with the previous experimental observations concerning the diffusive behavior of soil water flow.

1. Introduction

Various laboratory (Silliman and Simpson, 1987; Levy and Berkowitz, 2003) and field studies (Peaudecerf and Sauty, 1978; Sudicky et al., 1983; Sidle et al., 1998) of transport in subsurface porous media have shown significant deviations from Fickian behavior. As one approach to the modeling of the generally non-Fickian behavior of transport, Meerschaert, Benson, Baumer, Schumer, Zhang and their co-workers (Meerschaert et al. 1999, 2002, 2006; Benson et al. 2000a,b; Baumer et al. 2005, 2007; Schumer et al. 2001, 2009; Zhang et al. 2007, 2008 and 2009) have introduced the fractional advection-dispersion equation (fADE) as a model for transport in heterogeneous subsurface media. By theoretical and numerical studies the above authors have shown that fADE has a nonlocal structure that can model well the heavy tailed non-Fickian dispersion in subsurface media, mainly by means of a fractional spatial derivative in the dispersion term of the equation. Meanwhile, they have also shown that fADE, with a fractional time derivative, can also model well the long particle waiting times in transport in both surface and subsurface environments. However, while the above-mentioned studies provided extensive treatment of the fractional differential equation modeling of transport in fractional time-space by subsurface flows, few studies have addressed the detailed modeling of the actual subsurface flows in porous media in fractional time-space.

He (1998) seems to be the first scholar who proposed a fractional form of Darcy's equation for water flux in porous media. Based on this fractional water flux equation, in his pioneering work He (1998) then proposed a fractional governing equation of flow through saturated porous media. The left-hand-side (LHS) and the right-hand-side (RHS) of He's fractional Darcy flux formulation have different units. As saturated flow equations, He's proposed governing equations address the groundwater flow instead of the unsaturated soil water flow. Since the focus of our study is soil water flow in fractional time-space, below we shall discuss the literature that specifically addresses the fractional soil water flow equations.

As early as in 1960's Gardner and his co-workers (Ferguson and Gardner, 1963; Rawlins and Gardner, 1963) questioned the classical diffusivity expression in the diffusion form of the conventional Richards equation for soil water flow being only dependent on the soil water content. Based on their experimental observations, they reported that diffusivity was also dependent explicitly on time besides being dependent on the soil water content. Following on these experimental observations, Guerrini and Swartzendruber (1992) hypothesized a new form for Richards equation for horizontal unsaturated soil water flow in semi-rigid soils. Unlike the assumption that the soil hydraulic conductivity K and soil water pressure head ψ are only dependent on the soil water content, they hypothesized that K and ψ are also dependent explicitly on time. This hypothesis led them to the formulation of the diffusivity coefficient D within the diffusion form of the Richards equation as function of not only the soil water content but also explicitly on time, that is $D = D(\theta,t) = E(\theta)\, t^m$ where E is a function of water content θ while m is a power value. The application of their theory to the field data of Rawlins and Gardner (1963) proved successful, yielding fractional values of m less than unity in t^m. In a field experimental study of horizontal water absorption into porous construction materials (fired-clay and siliceous brick), El-Abd and Milczarek (2004) arrived at a formulation of diffusivity coefficient again in the form $D(\theta,t) = E(\theta)\, t^m$. The application of this form to their experimental data produced satisfactory results.

The study by Pachepsky et al. (2003) appears to be the first to propose a fractional model of horizontal, unsaturated soil water flow in field soils. Motivated by the observations of Nielsen et al. (1962) on the jerky movements of the infiltration front in field soils, that can be explained by long recurrence time intervals in-between motions, Pachepsky et al. (2003) proposed a time-fractional model of horizontal soil water flow in field soils. While the space component of their model has integer derivatives, they proposed a fractional form for the diffusivity, and expressed the Darcy water flux formulation in diffusive form with their proposed fractional diffusivity. Pachepsky et al. (2003) showed that the cause for fractional diffusivity is the scaling of time in the Boltzmann relationship not with the power of 0.5 (which corresponds to Brownian motion) but with a power less than 0.5, an experimental observation that was already made by Guerrini and Swartzendruber (1992). Pachepsky et al. (2003) supported their claim by various previous experimental studies' results, and showed that their proposed time-fractional form of the Richards equation with fractional diffusivity can explain experimental data. Meanwhile, Gerolymatou et al. (2006) proposed a fractional integral form for the Richards equation in fractional time but in integer horizontal space for unsaturated soil water flow in one horizontal dimension. Comparing their model simulations against the field experimental data of El-Abd and Milczarek (2004), they showed that their fractional Richards equation describes the evolution of soil water content in time and space better than the corresponding integer Richards equation. Again considering horizontal unsaturated soil water flow in fractional time but integer space, Sun et al. (2013) utilized the concept of fractal ruler in time, due to Cushman et al. (2009), to define a fractional derivative in time which they used to modify the integer time derivative in the conventional Richards equation. By means of this fractional derivative definition they were able to model the anomalous Boltzmann scaling in the wetting front movement and were able to obtain good fits to water content experimental data. Sun et al. (2013) conjectured that the time-dependent diffusivity $D(\theta,t) = E(\theta)\, t^m$ (for a fractional value of m) due to Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004), in the conventional Richards equation can be expressed essentially by representing the conventional integer derivative of the soil water content with respect to time by a product of the fractional time derivative of the soil water content and a fractional power of time.

The above-cited studies on the governing equations of soil water flow only treat time with fractional dimension, while keeping space with integer dimension. Furthermore, these studies address only one spatial dimension. Accordingly, our study in the following will attempt to develop a fractional continuity equation and a fractional water flux (motion) equation for unsaturated soil water flow in both fractional time and in multi-dimensional fractional space, starting from the conventional mass conservation and Darcy's law. Due to the anisotropy in the hydraulic conductivities of natural soils, the soil medium within which the soil water flow occurs is essentially anisotropic. Accordingly, in this study the fractional dimensions in two horizontal and one vertical directions will be considered different, resulting in multi-fractional space within which the flow takes place. Toward the development of the fractional governing equations, first a dimensionally-consistent continuity equation for soil water flow in multi-fractional, multi-dimensional space and fractional time will be developed. For the motion equation of soil water flow, or the equation of water flux within the soil matrix in multi-fractional multi-dimensional space and fractional time, a dimensionally consistent equation will also be developed. From the combination of the fractional continuity and motion equations, the governing equation of transient soil water flow in multi-fractional, multi-dimensional space and fractional time will be obtained. It will be shown that this equation approaches the conventional Richards equation as the fractional derivative powers approach integer values. Then by the introduction of the Brooks-Corey constitutive relationships for soil water (Brooks and Corey, 1964) into the fractional transient soil water flow equation, an explicit form of the equation will be obtained in multi-dimensional, multi-fractional space and fractional time. The governing fractional equation is then specialized to the case of only vertical soil water flow and of only horizontal soil water flow in fractional time-space.

2. Derivation of the continuity equation for transient soil water flow in multi-dimensional fractional space and fractional time

Let $D_a^{k\beta} f(x)$ be a Caputo fractional derivative of the function f (x), defined as (Podlubny, 1999; Odibat and Shawagfeh, 2007; Usero, 2008; Li et al. 2009),

$$D_a^{k\beta} f(x) = \frac{1}{\Gamma(m-k\beta)} \int_a^x \frac{f^m(\xi)}{(x-\xi)^{k\beta+1-m}} d\xi, \qquad m\text{-}1 < \beta < m, \; m \epsilon N, \; x \geq a \; . \tag{1}$$

Specializing the integer m =1 reduces equation (1) to

$$D_a^{k\beta} f(x) = \frac{1}{\Gamma(1-k\beta)} \int_a^x \frac{f'(\xi)}{(x-\xi)^{k\beta}} d\xi, \qquad 0 < \beta < 1, \quad x \geq a \quad . \tag{2}$$

Then to β-order

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f'(\xi)}{(x-\xi)^{\beta}} d\xi \qquad 0 < \beta < 1, \quad x \geq a \quad . \tag{3}$$

One can obtain a β-order approximation to a function f (·) around "a" as

$$f(x) = f(a) + \frac{(x-a)^{\beta}}{\Gamma(\beta+1)} D_a^{\beta} f(x), \qquad 0 < \beta < 1 \qquad . \tag{4}$$

This result follows by taking the upper limit value of the Caputo derivative at "x" in the mean value representation of a function in terms of fractional Caputo derivative (Usero, 2008; Li et al., 2009; Odibat and Shawagfeh, 2007) in order to have a distinct value for the above β-order approximation of the function f around "a". Within this framework the governing equations, based on this approximation, become prognostic equations that shall be known from the outset of model simulation for the whole time-space modeling domain. The next issue is what to take for the value of "a". If one expresses equation (4) with a = x-Δx, that is,

$$f(x) = f(\mathrm{x} - \Delta\mathrm{x}) + \frac{(\Delta\mathrm{x})^{\beta}}{\Gamma(\beta+1)} D_{\mathrm{x}-\Delta\mathrm{x}}^{\beta} f(x), \tag{5}$$

then the evaluation of the Caputo fractional derivative for f(x) = x will result in an expression that will contain a binomial expansion with a fractional power, which has infinite number of terms. As will be discussed in a later section, in order to render the developed fractional governing equations to become purely differential equations, it is necessary to establish an analytical relationship between Δx and $(\Delta x)^{\beta}$ that will be universally applicable throughout the modeling domain. This is possible when one takes the lower limit in the above Caputo derivative in equation (5) as zero (0) (that is, Δx = x) for f(x) = x. Then under such a construct, it will be possible to develop purely differential forms (with only fractional differential operators and no finite difference operators) for the governing equations of soil water flow, as will be shown in the following.

Within the above framework one can express the net mass outflow rate from the control volume in Figure 1 as

$$\left[\rho q_{x_1}(x_1\,, x_2, x_3; t) - \rho q_{x_1}(x_1 - \Delta x_1\,, x_2, x_3; t)\right]\Delta x_2 \Delta x_3 +$$
$$\left[\rho q_{x_2}(x_1\,, x_2, x_3; t) - \rho q_{x_2}(x_1\,, x_2 - \Delta x_2, x_3; t)\right]\Delta x_1 \Delta x_3 +$$
$$\left[\rho q_{x_3}(x_1\,, x_2, x_3; t) - \rho q_{x_3}(x_1\,, x_2, x_3 - \Delta x_3; t)\right]\Delta x_1 \Delta x_2 \qquad (6)$$

Then by introducing equation (5) into equation (6) with Δx = x, and expressing the resulting Caputo derivative $D_0^\beta f(x)$ (taking Δx = x renders the lower limit in the Caputo derivative of equation (5) to be 0) by $\frac{\partial^\beta f(x)}{(\partial x)^\beta}$ for convenience, the net mass flux from the soil control volume in Figure 1 may be expressed to β-order in fractional space as,

$$= \frac{(\Delta x_1)^{\beta_1}}{\Gamma(\beta_1+1)}\left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\left(\rho q_{x_1}(x_1\,, x_2, x_3; t)\right)\Delta x_2 \Delta x_3 + \frac{(\Delta x_2)^{\beta_2}}{\Gamma(\beta_2+1)}\left(\frac{\partial}{\partial x_2}\right)^{\beta_2}\left(\rho q_{x_2}(x_1\,, x_2, x_3; t)\right)\Delta x_1 \Delta x_3$$

$$+ \frac{(\Delta x_3)^{\beta_3}}{\Gamma(\beta_3+1)}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(\rho q_{x_3}(x_1\,, x_2, x_3; t)\right)\Delta x_1 \Delta x_2 \qquad (7)$$

where different fractional powers are considered in the three Cartesian directions in space due to the general anisotropy in the soil permeabilities and in the resulting flows in the soil media. It also follows from equation (5) with $f(x_i) = x_i$ that to β-order one obtains the approximation,

$$\Delta x_i = \frac{(\Delta x_i)^{\beta_i}}{\Gamma(\beta_i+1)}\frac{\partial^{\beta_i} x_i}{(\partial x_i)^{\beta_i}} \qquad i=1,2,3 \qquad (8)$$

With respect to the Caputo derivative $D_0^\beta x$;

$$\frac{\partial^{\beta_i} x_i}{(\partial x_i)^{\beta_i}} = \frac{x_i^{\,1-\beta_i}}{\Gamma(2-\beta_i)} \quad , \qquad i=1,2,3 \qquad (9)$$

Hence, combining equations (8) and (9) yields,

$$(\Delta x_i)^{\beta_i} = \frac{\Gamma(\beta_i+1)\Gamma(2-\beta_i)}{x_i^{\,1-\beta_i}}\,(\Delta x_i), \qquad i=1,2,3 \qquad (10)$$

with respect to β_i-order fractional space in the i-th direction, i=1,2,3.

Introducing equation (10) into equation (7) yields for the net mass outflow rate

$$= \frac{\Gamma(2-\beta_1)}{x_1^{\,1-\beta_1}}\left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\left(\rho q_{x_1}(\bar{x}; t)\right)\Delta x_1 \Delta x_2 \Delta x_3 + \frac{\Gamma(2-\beta_2)}{x_2^{\,1-\beta_2}}\left(\frac{\partial}{\partial x_2}\right)^{\beta_2}\left(\rho q_{x_2}(\bar{x}; t)\right)\Delta x_1 \Delta x_2 \Delta x_3$$

$$+\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(\rho q_{x_3}(\bar{x};t)\right)\Delta x_1\Delta x_2\Delta x_3\ , \qquad \bar{x}\ = (x_1\ , x_2, x_3) \tag{11}$$

to β-order, reflecting multi-fractional scaling in the anisotropic soil medium.

Denoting the volumetric water content by $\theta(\bar{x},t)$, the water volume V_w within the control volume in Figure 1 may be expressed as

$$V_w = \theta\ \Delta x_1\Delta x_2\Delta x_3 \qquad . \tag{12}$$

Hence, the time rate of change of mass within the control volume in Figure 1 is

$$\frac{\rho(\bar{x},t)\theta(\bar{x},t)-\rho(\bar{x},t-\Delta t)\theta(\bar{x},t-\Delta t)}{\Delta t}\ \Delta x_1\Delta x_2\Delta x_3 \qquad . \tag{13}$$

Introducing equation (5) with fractional power β replaced by α, x replaced by t and with Δt = t, into equation (13), and expressing the resulting Caputo derivative operator with its lower limit as 0, by $\frac{\partial^\alpha}{(\partial t)^\alpha}$ for convenience, yields the time rate of change of mass within the control volume with respect to α-fractional time increments:

$$\frac{(\Delta t)^\alpha}{\Delta t\ \Gamma(\alpha+1)}\left(\frac{\partial}{\partial t}\right)^\alpha \rho(\bar{x},t)\theta(\bar{x},t) \qquad . \tag{14}$$

to α-order. With respect to the Caputo derivative $D_0^\alpha t = \frac{\partial^\alpha t}{(\partial t)^\alpha}$:

$$\frac{\partial^\alpha t}{(\partial t)^\alpha} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tag{15}$$

which when combined with equation (5) (with x replaced by t and β replaced by α) yields the approximation,

$$(\Delta t)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(2-\alpha)}{t^{1-\alpha}}\ (\Delta t) \quad . \tag{16}$$

to α-order. Introducing equation (16) into equation (14) yields for the time rate of change of mass within the control volume in Figure 1 with respect to α-order fractional time increments:

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\ \frac{\partial^\alpha \rho(\bar{x},t)\theta(\bar{x},t)}{(\partial t)^\alpha}\ \Delta x_1\Delta x_2\Delta x_3 \qquad . \tag{17}$$

Since the time rate of change of mass within the control volume of Figure 1 is inversely related to the net flux through the control volume, equations (11) and (17) can be combined to yield

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\ \frac{\partial^\alpha \rho(\bar{x},t)\theta(\bar{x},t)}{(\partial t)^\alpha} =$$

$$-\left[\frac{\Gamma(2-\beta_1)}{x_1{}^{1-\beta_1}}\left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\left(\rho q_{x_1}(\bar{x};t)\right)+\frac{\Gamma(2-\beta_2)}{x_2{}^{1-\beta_2}}\left(\frac{\partial}{\partial x_2}\right)^{\beta_2}\left(\rho q_{x_2}(\bar{x};t)\right)+\right.$$

$$\left.\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(\rho q_{x_3}(\bar{x};t)\right)\right] \qquad ,$$

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\ \frac{\partial^\alpha \rho(\bar{x},t)\theta(\bar{x},t)}{(\partial t)^\alpha} = -\sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(\rho(\bar{x};t)q_{x_i}(\bar{x};t)\right) \tag{18}$$

as the fractional continuity equation of transient soil water flow in multi-fractional space of a generally anisotropic soil medium in fractional time.

If one further assumes an incompressible soil medium with constant density, then the fractional soil water flow continuity equation (18) simplifies further to

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^\alpha \theta(\bar{x},t)}{(\partial t)^\alpha} = -\sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(q_{x_i}(\bar{x};t)\right)\ ,\ 0< \alpha,\ \beta_1,\ \beta_2,\ \beta_3 <1\ ;\ \bar{x} = (x_1\ , x_2, x_3). \tag{19}$$

In the following, Equation (19) will be used as the fractional continuity equation for soil water flow for further study.

Performing a dimensional analysis of Equation (19), one obtains

$$\frac{1}{T^{1-\alpha}} \cdot \frac{1}{T^{\alpha}} = \frac{1}{L^{1-\beta_i}} \frac{1}{L^{\beta_i}} \frac{L}{T} = \frac{1}{T} \tag{20}$$

where L denotes length and T denotes time. Hence, Equation (20) shows the dimensional consistency of the left hand and right hand sides of the continuity Equation (19) for transient soil water flow in multi-fractional space and fractional time.

Podlubny (1999) has shown that for $n-1 < \alpha, \beta_i < n$ where n is any positive integer, as α and $\beta_i \rightarrow n$, the Caputo fractional derivative of a function f(y) to order α or β_i (i = 1, 2, 3) becomes the conventional n-th derivative of the function f(y). Therefore, specializing Podlubny's (1999) result to n = 1, for α and $\beta_i \rightarrow 1$ (i = 1, 2, 3), the continuity equation (19) reduces to

$$\frac{\partial \theta(\bar{x},t)}{\partial t} = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(q_{x_i}(\bar{x};t)\right) \tag{21}$$

which is the conventional continuity equation for soil water flow.

3. An equation for soil water flux (specific discharge) in fractional time-space

The experiments of Darcy (1856) showed that the specific discharge q_i is directly proportional to the change in hydraulic head, $\Delta h = h(x_i) - h(x_i - \Delta x)$, i=1,2,3, and is inversely proportional to the spatial displacement in any direction i, $\Delta x_i = x_i - (x_i - \Delta x_i)$, i= 1,2,3 (Freeze and Cherry, 1979). Hence, one can express the Darcy law in integer time-space as

$$q_{x_i} \Delta x_i = -K_i \Delta h_i \quad , i = 1,2,3 . \tag{22}$$

where $K_i = K_i(\bar{x})$ denotes the hydraulic conductivity in the i-th spatial direction (i=1,2,3), and the negative sign on the right-hand-side (RHS) of equation (22) is due to soil water flow being in the direction of decreasing hydraulic head.

In equation (22), using the β-order approximation to a function around x-Δx in equation (5) to β_i-order (i= 1,2,3) yields (with $D_0^{\beta_i} h = \frac{\partial^{\beta_i} h}{(\partial x_i)^{\beta_i}}$) :

$$\Delta h_i = \frac{(\Delta x_i)^{\beta_i}}{\Gamma(\beta_i + 1)} \frac{\partial^{\beta_i} h}{(\partial x_i)^{\beta_i}} \quad , \qquad i = 1, 2, 3 \tag{23}$$

where the lower limit in the integral of the Caputo derivative is again taken at zero. Combining equations (10) and (23) with equation (22) yields,

$$q_i \left[\frac{{x_i}^{1-\beta_i}}{\Gamma(2-\beta_i)}\right] = -K_i \left[\frac{\partial^{\beta_i} h}{(\partial x_i)^{\beta_i}}\right] \text{ , i=1,2,3 .} \tag{24}$$

Expressing equation (24) for the specific discharge q_i , one obtains

$$q_i(\bar{x},t) = -K_i(\bar{x}) \frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}} \frac{\partial^{\beta_i} h}{(\partial x_i)^{\beta_i}} \qquad i = 1,2,3 \tag{25}$$

as the equation of water flux through anisotropic soil media in multi-fractional multi-dimensional space.

Performing a dimensional analysis on equation (25), one obtains:

$$[q_i(\bar{x},t)] = {}^{L}/_{T} \qquad \text{and} \qquad \left[K_i(\bar{x})\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\frac{\partial^{\beta_i}h}{(\partial x_i)^{\beta_i}}\right] = \frac{L}{T}\frac{L}{L^{1-\beta_i}L^{\beta_i}} = \frac{L}{T} \tag{26}$$

which establishes the dimensional consistency of equation (25) as the fractional equation for soil water flux. Furthermore, it is well-known that for unsaturated soil water flow, the hydraulic conductivity is function of the volumetric soil water content θ and of spatial location (Freeze and Cherry, 1979). In fact, K_i may be expressed in terms of the saturated hydraulic conductivity K_s and the relative hydraulic conductivity $K_r(\theta)$ as

$$K_i(\overline{x},\theta) = K_{s,i}(\overline{x})\mathrm{K_r}(\theta) \quad . \tag{27}$$

Hence, the equation of soil water flux (specific discharge) in multi-dimensional, multi-fractional anisotropic soil space may be expressed as

$$q_i(\bar{x},t) = -K_i(\overline{x},\theta)\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\frac{\partial^{\beta_i}h(\overline{x},t)}{(\partial x_i)^{\beta_i}} \quad , \quad \text{i}= 1,2,3\ . \tag{28}$$

Equation (28) is dimensionally consistent in that both the LHS and RHS of the equation have the unit L/T.

As noted above, Podlubny (1999) has shown that for n-1< $\beta_i < n$ (i = 1, 2, 3) where n is any positive integer, as $\beta_i \rightarrow$ n, the Caputo fractional derivative of a function f(y) to order β_i (i = 1, 2, 3) becomes the conventional n-th derivative of the function f(y). Therefore, specializing Podlubny's (1999) result to n = 1, for $\beta_i \rightarrow 1$ (i = 1, 2, 3), the fractional soil water flux equation (28) becomes

$$q_i(\bar{x},t) = -K_i(\overline{x},\theta)\frac{\partial h(\overline{x},t)}{\partial x_i} \quad , \quad \text{i}= 1,2,3\ . \tag{29}$$

which is the conventional Darcy's equation for soil water flux. As such the derived fractional soil water flux Equation (28) is consistent with the conventional Darcy's equation for the integer power case.

4. Governing equation of transient soil water flow in multi-dimensional fractional soil space and fractional time

Combining the fractional continuity equation (19) with the fractional soil water flux equation (28) yields,

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_i(\overline{x},\theta)\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\frac{\partial^{\beta_i}h(\overline{x},t)}{(\partial x_i)^{\beta_i}}\right) \text{ for } 0<\alpha, \beta_1, \beta_2, \beta_3 <1\ ;$$
$$\bar{x} = (x_1, x_2, x_3). \tag{30}$$

Since $K_i(\overline{x},\theta) = K_{s,i}(\overline{x})\mathrm{K_r}(\theta)$, one obtains

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)\frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}}\frac{\partial^{\beta_i}h(\overline{x},t)}{(\partial x_i)^{\beta_i}}\right) \text{ for } 0<\alpha,\beta_1, \beta_2, \beta_3 <1;$$
$$\bar{x} = (x_1, x_2, x_3) \tag{31}$$

as the governing equation of transient soil water flow in anisotropic multi-dimensional fractional soil media and fractional time.

Meanwhile, the soil hydraulic head h is related to the elevation head x_3 and soil capillary pressure head ψ by

$$h = \psi(\theta) + x_3 \tag{32}$$

Substituting equation (32) into equation (31) results in

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\ \frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\ \frac{\partial^{\beta_i}}{(\partial x_i)^{\beta_i}}(\psi(\theta)+x_3)\right) \quad . \tag{33}$$

With respect to the Caputo derivative:

$$\frac{\partial^{\beta_3}x_3}{(\partial x_3)^{\beta_3}} = \frac{x_3{}^{1-\beta_3}}{\Gamma(2-\beta_3)} \quad . \tag{34}$$

Opening equation (33) further and introducing equation (34) yields

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\ \frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3}\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)\frac{\Gamma(2-\beta_i)}{x_i{}^{1-\beta_i}}\ \frac{\partial^{\beta_i}\psi(\theta)}{(\partial x_i)^{\beta_i}}\right)$$
$$+\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\right);\ 0<\alpha,\beta_1,\beta_2,\beta_3<1\ ;\ \bar{x}=(x_1\ ,x_2,x_3) \tag{35}$$

as the governing equation of transient soil water flow in anisotropic multi-dimensional fractional media and fractional time. This governing equation may also be written as

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3}\frac{1}{\Gamma(2-\alpha)}\frac{(\Gamma(2-\beta_i))^2}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)\frac{t^{1-\alpha}}{x_i{}^{1-\beta_i}}\ \frac{\partial^{\beta_i}\psi(\theta)}{(\partial x_i)^{\beta_i}}\right)$$
$$+\frac{1}{\Gamma(2-\alpha)}\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(t^{1-\alpha}K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\right);\ 0<\alpha,\beta_1,\beta_2,\beta_3<1\ ;\ \bar{x}=(x_1\ ,x_2,x_3) \tag{36}$$

As noted above, Podlubny (1999) has shown that for n-1< $\alpha, \beta_i < n$ (i=1,2,3) where n is any positive integer, as α and $\beta_i \rightarrow$ n, the Caputo fractional derivative of a function f(y) to order α or β_i (i = 1, 2, 3) becomes the conventional n-th derivative of the function f(y). Therefore, specializing Podlubny's (1999) result to n = 1, for α and $\beta_i \rightarrow 1$ (i = 1, 2, 3), the fractional governing equation (33) of soil water flow becomes

$$\frac{\partial\theta(\bar{x},t)}{\partial t} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)\ \frac{\partial}{\partial x_i}(\psi(\theta)+x_3)\right) \tag{37}$$

which is the conventional Richards equation for transient soil water flow.

With respect to dimensional consistency, one may note that both sides of the fractional governing equation (33) or equation (35) for transient soil water flow have the unit 1/T. Meanwhile, both sides of equation (36) have the unit 1/T$^{\alpha}$. Hence, these fractional equations are dimensionally consistent.

5. Fractional governing equation of transient soil water flow in the vertical direction

In the case of vertical transient unsaturated flow for the infiltration process in soils in fractional time-space, Equation (35) reduces further to

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\ \frac{\partial^{\beta_3}\psi(\theta)}{(\partial x_3)^{\beta_3}}\right)+$$
$$+\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\right)\qquad ;\ 0<\alpha,\ \beta_3<1\ ;\ \bar{x}=(x_1\ ,x_2,x_3) \tag{38}$$

as the governing equation. This governing equation for vertical transient soil water flow in fractional time-space can also be expressed as;

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\frac{\Gamma(2-\beta_3)}{\Gamma(2-\alpha)}\frac{t^{1-\alpha}}{x_3{}^{1-\beta_3}}\ \frac{\partial^{\beta_3}\psi(\theta)}{(\partial x_3)^{\beta_3}}\right)+$$

$$+\frac{1}{\Gamma(2-\alpha)}\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(t^{1-\alpha}K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\right)\quad ;\ 0<\alpha,\ \beta_3<1\ ;\ \bar{x}=(x_1\,,x_2,x_3). \tag{39}$$

As in the integer case of Richards equation (37), equations (35), (36), (38) and (39) have both the hydraulic conductivity K and the capillary pressure head ψ as functions of the soil volumetric water content θ. As such, characteristic soil water relationships, such as those given by Brooks and Corey (1964), may be utilized to obtain an explicit form of the governing equation of transient, unsaturated soil water flow in fractional time-space, as explained in the following.

6. Soil water content-based explicit form of the governing equation of transient soil water flow in fractional time-space

One can utilize the Brooks-Corey (1964) formula for the soil characteristic relationship between the capillary soil water pressure head ψ and the soil water content θ as follows:

$$\psi(\theta)=\ \psi_b\theta_e{}^{1/\lambda}(\theta-\theta_r)^{-1/\lambda} \tag{40}$$

where ψ_b is the air entry pressure head (bubbling pressure), $\theta_e = (\theta_s - \theta_r)$ is the effective porosity, θ_s is the saturation volumetric soil water content, θ_r is the residual water content, and λ is the pore size distribution index. Therefore, the β_i-order Caputo fractional derivative of the capillary pressure head ψ with respect to x_i in the interval $(0,x_i)$ may be expressed in terms of the Brooks-Corey relationship (40) as (Podlubny, 1999; Odibat and Shawagfeh, 2007)

$$\frac{\partial^{\beta_i}\psi(\theta)}{(\partial x_i)^{\beta_i}}=\ \frac{\psi_b\theta_e{}^{1/\lambda}}{\Gamma(1-\beta_i)}\int_0^{x_i}\left(\frac{\partial}{\partial\xi_i}(\theta-\theta_r)^{-1/\lambda}\right)(x_i-\xi_i)^{-\beta_i}\,d\xi_i=\ \psi_b\theta_e{}^{1/\lambda}\frac{\partial^{\beta_i}(\theta-\theta_r)^{-1/\lambda}}{(\partial x_i)^{\beta_i}} \tag{41}$$

Under the Brooks-Corey (1964) relationship between the hydraulic conductivity and the volumetric soil water content, the relative hydraulic conductivity $K_r(\theta)$ is expressed as

$$K_r(\theta)=\theta_e{}^{-3-2/\lambda}(\theta-\theta_r)^{3+2/\lambda} \tag{42}$$

and using expression (42) within $K_i(\overline{x},\theta)=\ K_{s,i}(\overline{x})\mathrm{K_r}(\theta)$, the β_i-order fractional Caputo derivative of $K_i(\overline{x},\theta)$ with respect to x_i in the interval $(0,x_i)$ may be expressed as

$$\frac{\partial^{\beta_i}K_{s,i}(\overline{x})\mathrm{K_r}(\theta)}{(\partial x_i)^{\beta_i}}=\ \theta_e{}^{-3-2/\lambda}\frac{\partial^{\beta_i}\left(K_{s,i}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right)}{(\partial x_i)^{\beta_i}}\qquad ,\ i=1,2,3 \tag{43}$$

Substituting equations (41) and (43) into equation (35) results in an explicit form of the governing equation of transient soil water flow in anisotropic multi-dimensional fractional soil space and fractional time in terms of the volumetric water content θ as

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^\alpha\theta(\bar{x},t)}{(\partial t)^\alpha}=\sum_{i=1}^3\psi_b\theta_e{}^{-3-1/\lambda}\frac{(\Gamma(2-\beta_i))^2}{x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})\frac{(\theta-\theta_\mathrm{r})^{3+2/\lambda}}{x_i{}^{1-\beta_i}}\frac{\partial^{\beta_i}(\theta-\theta_\mathrm{r})^{-1/\lambda}}{(\partial x_i)^{\beta_i}}\right)$$

$$+\theta_e{}^{-3-2/\lambda}\frac{\Gamma(2-\beta_3)}{x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right)\qquad ;\ 0<\alpha,\beta_1,\ \beta_2,\ \beta_3<1 \tag{44}$$

in terms of the Brooks-Corey soil water characteristics relationships. This governing equation can also be expressed as

$$\frac{\partial^\alpha\theta(\bar{x},t)}{(\partial t)^\alpha}=\sum_{i=1}^3\psi_b\theta_e{}^{-3-1/\lambda}\frac{(\Gamma(2-\beta_i))^2}{\Gamma(2-\alpha)x_i{}^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left(K_{s,i}(\overline{x})(\theta-\theta_\mathrm{r})^{3+2/\lambda}\frac{t^{1-\alpha}}{x_i{}^{1-\beta_i}}\frac{\partial^{\beta_i}(\theta-\theta_\mathrm{r})^{-1/\lambda}}{(\partial x_i)^{\beta_i}}\right)$$

$$+\theta_e{}^{-3-2/\lambda}\frac{\Gamma(2-\beta_3)}{\Gamma(2-\alpha)x_3{}^{1-\beta_3}}\left(\frac{\partial}{\partial x_3}\right)^{\beta_3}\left(t^{1-\alpha}K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right)\quad ;\ 0<\alpha,\beta_1,\ \beta_2,\ \beta_3<1. \tag{45}$$

Upon dimensional analysis of equation (44) one can see that it is dimensionally consistent since both of its sides have the unit of 1/T where T denotes time. Meanwhile, equation (45) is also dimensionally consistent with both sides of the equation having the unit $1/T^{\alpha}$.

Specializing equation (45) to only the vertical direction, the governing equation of transient soil water flow in the vertical direction in fractional space-time may be expressed as,

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \psi_b \theta_e^{-3-1/\lambda} \frac{\left(\Gamma(2-\beta_3)\right)^2}{\Gamma(2-\alpha)x_3^{1-\beta_3}} \left(\frac{\partial}{\partial x_3}\right)^{\beta_3} \left(K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda} \frac{t^{1-\alpha}}{x_3^{1-\beta_3}} \frac{\partial^{\beta_3}(\theta-\theta_r)^{-1/\lambda}}{(\partial x_3)^{\beta_3}} \right) +$$

$$+ \theta_e^{-3-2/\lambda} \frac{\Gamma(2-\beta_3)}{\Gamma(2-\alpha)x_3^{1-\beta_3}} \left(\frac{\partial}{\partial x_3}\right)^{\beta_3} \left(t^{1-\alpha} K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right) \quad ; 0< \alpha, \beta_3 <1 \ . \qquad (46)$$

Upon dimensional analysis of equation (46) one can find that both sides of this equation have the unit of $1/T^{\alpha}$ where T denotes time. Hence, the fractional equation of vertical transient soil water flow, in its explicit form, is dimensionally consistent.

Finally, specializing equation (45) to only the horizontal directions, the governing equation of transient soil water flow in the horizontal directions in fractional space-time may be expressed as,

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{2} \psi_b \theta_e^{-3-1/\lambda} \frac{(\Gamma(2-\beta_i))^2}{\Gamma(2-\alpha)x_i^{1-\beta_i}} \left(\frac{\partial}{\partial x_i}\right)^{\beta_i} \left(K_{s,i}(\overline{x})(\theta-\theta_r)^{3+2/\lambda} \frac{t^{1-\alpha}}{x_i^{1-\beta_i}} \frac{\partial^{\beta_i}(\theta-\theta_r)^{-1/\lambda}}{(\partial x_i)^{\beta_i}} \right)$$

$$; 0< \alpha, \beta_1, \beta_2 <1 \ . \qquad (47)$$

Upon dimensional analysis of equation (47) one can find that both sides of this equation have the unit of $1/T^{\alpha}$ where T denotes time. Hence, the fractional equation of horizontal transient soil water flow, in its explicit form, is dimensionally consistent.

7. Physical framework for the developed time-space fractional governing equations of soil water flow

In parallel to the conventional governing equations of soil water flow processes (Freeze and Cherry, 1979; Bear, 1979), the corresponding governing equations of the soil water flow processes in fractional time-space must have certain properties: i) The fractional governing equations must be purely differential equations, containing only differential operators, and no difference operators. ii) They must be prognostic equations. That is, they are solved from the initial conditions and boundary conditions in order to describe the evolution of the flow field in time and space. As such, from the outset the form of the governing equation must be known in its entirety. Once its physical parameters (such as the saturated hydraulic conductivity, etc.) are estimated, the governing equation is fixed throughout the simulation time and the simulation space for the simulation of the soil water flow in question. iii) These equations must be dimensionally consistent. iv) The fractional governing equations of soil water flow with fractional powers must converge to the corresponding conventional governing equations with integer powers as the fractional powers approach the corresponding integer powers.

However, a distinct difference of the fractional governing equations of soil water flow from the corresponding conventional equations is that they are based on fractional derivatives which are nonlocal. Being nonlocal, the fractional governing equations of soil water flow have the potential to account for the effect of the initial conditions on the soil water flow for long times, and for the effect of the upstream boundary conditions on the flow for long distances from the upstream boundary. The physical meaning of the fractional governing equation may be explained most easily in the case of vertical soil water flow. In the context of upstream-to-downstream vertical soil water flow from the soil surface downward, in the integer form of the soil water flow mass conservation equation (the conventional equation) the time rate of change of mass within a control volume grid (x-Δx, x) is determined by the mass flux coming from the upstream neighbour grid (x-2Δx, x-Δx) into (x-Δx, x), and the mass flux that is moving from the control volume grid (x-Δx, x) to the downstream neighbour grid (x, x+Δx). This framework holds also for the soil water flow in the two horizontal directions. As such, the mass evolution in the case of the integer governing equation of soil water flow is local (at the scale of the specific computational grid), due to interaction only among neighbouring computational grids. On the other hand, in the case of the fractional governing equation of mass of vertical upstream-to-downstream soil water flow from the soil surface downward, we deal with the Caputo fractional derivative

$$\frac{\partial^{\beta} f}{(\partial x_3)^{\beta}} = D_0^{\beta} f(x_3) \tag{48}$$

defined by,

$$D_0^{\beta} f(x_3) = \frac{1}{\Gamma(1-\beta)} \int_0^{x_3} \frac{f^{`}(\xi)}{(x_3-\xi)^{\beta}} d\xi \qquad 0 < \beta < 1, \quad x_3 \geq 0. \tag{49}$$

As such, each local integer derivative $f^{`}(\xi)$ at each depth ξ in the interval (0, x_3) contributes to the Caputo fractional derivative of the interval (0, x_3) with weight $(x_3 - \xi)^{-\beta}$. Within this framework, for example, in the case of one-dimensional downward vertical soil water flow in fractional time-space, the effect of the upstream boundary condition at depth zero is still accounted for at any depth x_3 below the soil surface by means of the fractional spatial derivatives that appear in the corresponding governing equation (Equation (39) or Equation (46) above). It also follows from equation (49) that this memory effect is modulated by the value of the fractional power β . This is also the case in the time dimension where the effect of the initial condition at time zero is accounted for at any time t after the initial time. Also, the effects of the local derivatives at each time s ($0 \leq s \leq t$) on the Caputo derivative of the interval (0,t) are accounted for with the weights $(t-s)^{-\alpha}$. Hence, the fractional governing equations of soil water flow are nonlocal, and, as such, can quantify the influence of the initial and boundary conditions on the flow process more effectively than the corresponding conventional governing equations that are local.

Referring to equation (4) above, it is necessary to take the upper limit value of the Caputo derivative at "x" in the mean value representation of a function in terms of the fractional Caputo derivative (Usero, 2008; Li et al., 2009; Odibat and Shawagfeh, 2007) in order to have the governing equations, based on this approximation, become prognostic equations that shall be known from the outset of model simulation for the whole time-space modeling domain. Then referring to equation (5) above, in order to have the governing equations to have purely differential forms (with only the differential operators (and no difference operators) existing in these equations), it is necessary to establish an analytical relationship between Δx and $(\Delta x)^{\beta}$. This is possible by taking the origin of the Caputo derivative in equation (5) at zero (the upstream boundary location in space or initial time location in time). Otherwise, when one evaluates the Caputo derivative of the function x at the integral limits (x-Δx, x), one ends up with a fractional binomial expansion that has infinite number of terms, which prevents an analytical relationship between Δx and $(\Delta x)^{\beta}$. This is also the case for the time dimension. The Caputo derivative of the function t in the time dimension must again be evaluated at the lower limit of the integral set at the initial time zero in order to obtain purely differential operators for the evolution in time for the governing equations. It is also important to note that under these approximations, the resulting governing equations are all dimensionally-consistent, and all the resulting fractional governing equations converge to their corresponding conventional counterparts with integer powers as their fractional powers approach unity.

8. Discussion and conclusion

The governing equations that were developed in this study are for the fractional time dimension and for multi-dimensional fractional space that represents the fractal spatial structure of a soil field. If one were to simplify the developed theory above to only fractional time but integer-space soil fields, then the developed equations would simplify substantially. The governing equation (36) of transient soil water flow in anisotropic multi-dimensional fractional soil media in fractional time would simplify to (with $\beta_i = 1$, i =1,2,3):

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3} \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x_i}\left(K_{s,i}(\overline{x})\mathrm{K_r}(\theta)t^{1-\alpha}\,\frac{\partial\psi(\theta)}{\partial x_i}\right) + \frac{1}{\Gamma(2-\alpha)}\frac{\partial}{\partial x_3}\left(t^{1-\alpha}K_{s,3}(\overline{x})\mathrm{K_r}(\theta)\right) \quad ; 0< \alpha <1;\ \bar{x} = (x_1\,, x_2, x_3) \qquad (50)$$

for the governing equation of transient soil water flow in integer multi-dimensional soil media and in fractional time. In terms of the Brooks-Corey soil characteristic relationships, the explicit governing equation of transient soil water flow in integer multi-dimensional soil space and in fractional time is obtained from the simplification of equation (45) as (with $\beta_i = 1$, i =1,2,3):

$$\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3} -{}^{1}/_{\lambda}\,\psi_b\theta_e^{\,-3-{}^{1}/_{\lambda}} \frac{1}{\Gamma(2-\alpha)}\,\frac{\partial}{\partial x_i}\left(t^{1-\alpha}K_{s,i}(\overline{x})(\theta-\theta_\mathrm{r})^{2+{}^{1}/_{\lambda}}\,\frac{\partial\theta}{\partial x_i}\right) + \theta_e^{\,-3-2/\lambda}\frac{1}{\Gamma(2-\alpha)}\frac{\partial}{\partial x_3}\left(t^{1-\alpha}K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right) \ ; 0< \alpha <1 ;\ \bar{x} = (x_1\,, x_2, x_3)\ . \qquad (51)$$

As mentioned before, Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004), in their explanation of the anomalous behavior of the diffusivity coefficient in their experiments, have proposed that the diffusivity coefficient in the diffusion-based formulation of the Richards equation of soil water flow must depend not only on the water content but also on time. Hence, they formulated this diffusivity coefficient D as $D = D(\theta,t) = E(\theta)\, t^m$ where E is a function of water content θ while m is a power value. This formulation proved to be successful in modeling various experimental data on horizontal soil water flow. If one were to formulate the diffusivity $D_i(\theta,t)$ in the explicit governing equation (51) of transient soil water flow in fractional time and in anisotropic multi-dimensional integer soil space as

$$\mathrm{D_i}(\theta,\mathrm{t}) = K_{s,i}(\overline{x})(\theta-\theta_\mathrm{r})^{2+{}^{1}/_{\lambda}}t^{1-\alpha} \quad , \quad \mathrm{i} = 1,2,3, \qquad (52)$$

this diffusivity coefficient is in the same form as the diffusivity coefficient $D(\theta,t) = E(\theta)\, t^m$ that was formulated by Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004) based on experimental observations. As such, within the framework of Brooks-Corey soil water relationships, the explicit governing equations that were developed in this study for the transient soil water flow in multi-dimensional fractional soil media and fractional time, when simplified to integer soil space, are consistent with the experimental observations of Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004) when their power value m = 1-α .

Sun et al. (2013) conjectured that the time-dependent diffusivity D(θ,t) = E(θ) t^m (for a fractional value of m) due to Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004), in the conventional Richards equation can be expressed essentially by representing the conventional integer derivative of the soil water content with respect to time by a product of the fractional time derivative of the soil water content and a fractional power of time (Sun et al. 2013, Eqn. (12)), that is, $\frac{\partial\theta(\bar{x},t)}{\partial t} = \frac{C}{t^{1-\alpha}}\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}}$ where C denotes a constant. In order to examine the conjecture of Sun et al. (2013), one can re-write the explicit governing equation (51) for soil water flow in integer space but fractional time in equivalent form as

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}\theta(\bar{x},t)}{(\partial t)^{\alpha}} = \sum_{i=1}^{3} - {}^{1}/_{\lambda}\,\psi_b\theta_e^{-3-{}^{1}/_{\lambda}}\frac{\partial}{\partial x_i}\left(K_{s,i}(\overline{x})(\theta-\theta_r)^{2+{}^{1}/_{\lambda}}\frac{\partial\theta}{\partial x_i}\right)$$

$$+\,\theta_e^{-3-2/\lambda}\frac{\partial}{\partial x_3}\left(K_{s,3}(\overline{x})(\theta-\theta_r)^{3+2/\lambda}\right) \;;0<\alpha<1\;;\; \bar{x}=(x_1\,,x_2,x_3)\;. \qquad (53)$$

Equation (53) shows that the fractional soil water flow equation (51) which accounts for the time-dependent diffusivity expression of Guerrini and Swartzendruber (1992) and El Abd and Milczarek (2004), does have an equivalent form where the integer time derivative of the soil water content in the conventional Richards equation is replaced by a product of the fractional time derivative of the soil water content and a fractional power of time, thereby supporting Sun et al.'s (2013) conjecture, although in this study the fractional derivative is defined in the Caputo sense while in Sun et al. (2013) the fractional derivative is defined with respect to a fractal ruler in time.

In conclusion, in this study first a dimensionally-consistent continuity equation for soil water flow in multi-fractional, multi-dimensional space and fractional time was developed. For the motion equation of soil water flow, or the equation of water flux within the soil matrix in multi-fractional multi-dimensional space and fractional time, a dimensionally consistent equation was also developed. From the combination of the fractional continuity and motion equations, the governing equation of transient soil water flow in multi-fractional, multi-dimensional space and fractional time was then obtained. It is shown that this equation approaches the conventional Richards equation as the fractional derivative powers approach integer values. Then by the introduction of the Brooks-Corey constitutive relationships for soil water (Brooks and Corey, 1964) into the fractional transient soil water flow equation, an explicit form of the equation was obtained in multi-dimensional, multi-fractional space and fractional time. Finally, the governing fractional equation was specialized to the cases of vertical soil water flow and horizontal soil water flow in fractional time-space. It is shown that the developed governing equations, in their fractional time but integer space forms, show behavior consistent with the previous experimental observations concerning the diffusive behavior of soil water flow.

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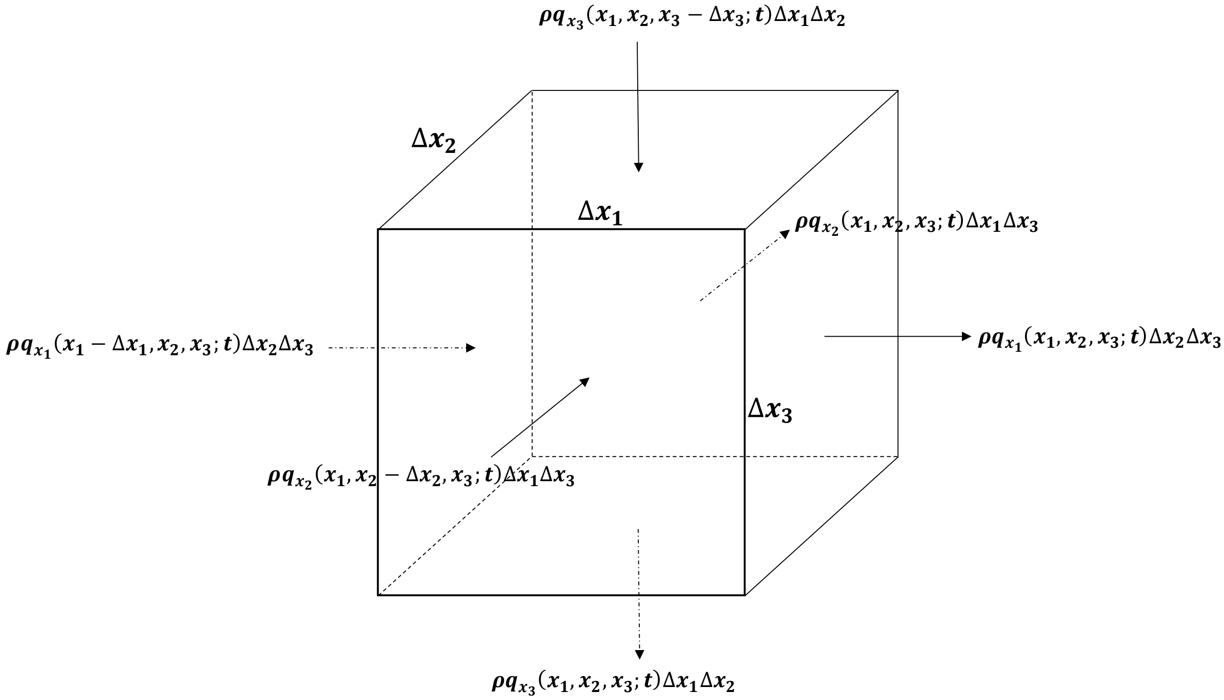

Figure 1. The control volume for the three-dimensional soil water flow.