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The Budyko functions under non-steady state conditions: new approach and comparison with previous formulations

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Abstract. The Budyko functions relate the evaporation ratio E/P (E is evaporation and P precipitation) to the aridity index $\Phi = E_p/P$ (E_p is potential evaporation) and are valid on long timescales under steady state conditions. A new formulation physically based (noted ML) is proposed to extend the Budyko framework under non-steady state conditions taking into account the change in soil water storage S. The ML formulation introduces an additional parameter $S^* = S/E_p$ and can be applied with all classical Budyko functions. In the standard Budyko space $(E_p/P, E/P)$, and for the particular case where the Fu-Zhang equation is used as a Budyko function, the ML formulation yields similar results to the analytical solution of Greve et al. (2016), and a simple relationship can be established between their respective parameters. Then, the ML formulation is extended to the space $[(E_p/(P + S), E/(P + S)]$ and compared to the formulations of Chen et al. (2013) and Du et al. (2016). We show that the ML and Greve et al. formulations have similar upper feasible domain but their lower feasible domain is different from those of Chen et al. (2103) and Du et al. (2016). Moreover, the domain of variation of $E_p/(P+S)$ differs: it is bounded by an upper limit $1/S^*$ in the ML formulation, while it is bounded with a lower limit in Chen et al.'s and Du et al.'s formulations.

1 Introduction

The Budyko framework has become a simple tool widely used within the hydrological community to estimate the evaporation ratio E/P at catchment scale (E is evaporation and P precipitation) as a function of the aridity index $\Phi = E_p/P$ (E_p is potential evaporation) through simple mathematical formulations $E/P = B_1(\Phi)$ and with long-term averages of the variables. Most of the formulations (Table 1) were empirically derived (e.g. Oldekop, 1911; Turc, 1954; Tixeront, 1964; Budyko, 1974; Choudhury, 1999; Zhang et al., 2001; Zhou et al., 2015), but some of them were analytically obtained from simple physical assumptions: (i) the one derived by Mezentsev (1955) and then by Yang et al. (2008), which has the same form as the one initially proposed by Turc (1954) and is noted hereafter Turc-Mezentsev; (ii) the one derived by Fu (1981) and reworked by Zhang et al. (2004), noted hereafter Fu-Zhang. As an example, Figure 1 shows the Fu-Zhang equation for different values of its shape parameter ω: when ω increases from 1 to $+\infty$, actual evaporation gets closer to the maximum

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rate. Simple approaches based on a modification of the parameter of the Budyko functions were used to take into account the temporal change of climatic characteristics and vegetation dynamics at catchment scale (Donohue et al., 2007; Yang et al., 2009; Li et al., 2013; Carmona et al., 2014).

The Budyko framework was initially derived and applied on long timescales (many years) and limited to steady-state conditions under the assumption of negligible change in soil water storage or in groundwater. Hydrological processes leading to changes in water storage are not represented and the catchment is considered as close without any anthropogenic intervention, precipitation being the only input and evaporation and runoff Q the only outputs ($P = E + Q$). Recently the Budyko framework has been downscaled to the year (Istanbulluoglu et al., 2012; Wang, 2012; Carmona et al., 2014; Du et al., 2016), the season (Gentine et al., 2012; Chen et al., 2013), and the month (Zhang et al., 2008; Du et al., 2016; Greve et al., 2016). However, the soil water storage variation ΔS cannot be considered as negligible when dealing with these finer timescales (annual, seasonal or monthly) or for unclosed basins (e.g. interbasin water transfer, withdrawing groundwater). In these cases, the basin water balance should be written: $P = E + Q + \Delta S$, and the catchment is considered in non-steady state conditions. Table 2 shows some recent formulations of the Budyko framework extended to take into account the change in water storage ΔS. Chen et al. (2013) (used in Fang et al., 2016) and Du et al. (2016) proposed empirical modifications of the Turc-Mezentsev and Fu-Zhang equations respectively, precipitation P being replaced by the available water supply defined as $(P + S)$ with $S = -\Delta S$, Du et al. (2016) including the inter-basin water transfer into ΔS. Greve et al. (2016) analytically modified the Fu-Zhang equation, keeping the standard Budyko space $(E_p/P, E/P)$ with one additional parameter, whereas Wang and Zhou (2016) proposed in the same Budyko space a formulation issued from the hydrological ABCD model (Alley, 1984), but with two additional parameters.

The extension of the Budyko framework to non-steady state conditions being a real challenge, this paper aims to propose a new formulation, characterised by a clear physical background, which is compared to the previous ones cited above. The paper is organized as follows. First, we present the new formulation under non-steady state conditions, its upper and lower limits and some particular cases. Second we compare the new formulation in the standard Budyko space $(E_p/P, E/P)$ to the analytical solution of Greve et al (2016). Finally, the new formulation is extended to the space $[E_p/(P+S), E/(P+S)]$ and compared to the formulations of Chen et al. (2013) and Du et al. (2016).

2 New formulation under non-steady state conditions

2.1 Upper and lower limits of the Budyko framework

Each catchment is characterized by the three hydrologic variables P, E and E_p which can be represented in a 2D space using dimensionless variables equal to the ratio between two of these variables and the third one. First, we present the upper and lower limits under steady state conditions, when all the water consumed by evaporation comes from the precipitation, the change in water storage ΔS being nil ($E = P - Q$). In the rest of the paper, following Andréassian et al. (2016), the space defined by $(\Phi = E_p/P, E/P)$ is called "Budyko space" and the one defined by $(\Phi^{-1} = P/E_p, E/E_p)$ "Turc space". Figure 2a

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represents the variation of maximum and minimum actual evapotranspiration, respectively E_x and E_n, as a function of precipitation P with dimensionless variables in the Turc space. The upper solid line represents the dimensionless maximum evaporation rate E_x/E_p: it follows the precipitation up to $P/E_p = 1$ (the water limit is presented in bold blue on all graphs) and then is limited by potential evaporation $E_x/E_p = 1$ (the energy limit is in bold green). The lower solid line (in bold black) represents the dimensionless minimum evaporation rate E_n/E_p which follows the x-axis: $E_n/E_p = 0$. The feasible domain between the upper and the lower limits is shown in grey. In the Budyko space we have the following relationships: i) when evaporation is maximum, for $E_p/P < 1$, $E_x/P = E_p/P$ and for $E_p/P > 1$, $E_x/P = 1$; ii) when evaporation is minimum: $E_n/P = 0$. The corresponding Budyko non-dimensional graph is shown in Figure 2b and represents the upper and lower limits of the feasible domain of $E/P = B_1(E_p/P)$.

Under non-steady state conditions, a given amount of water stored in the area at stake also participates to the evaporation process ($E = P + S - Q$). Consequently, the upper and lower limits are different. The storage amount S available and used for evaporation is supposed to be positive and lower than E_p ($0 \leqslant S \leqslant E_p$), because if $S > E_p$, E_x is systematically equal to E_p. The case where evaporation is at its maximum value is visualized as the upper limit in Figure 2c (all the available water is used for evaporation). With dimensionless variables and with $S^* = S/E_p$ we have:

$$if\ \frac{P}{E_p} < 1 - \frac{S}{E_p} \qquad then \qquad \frac{E_x}{E_p} = \frac{P}{E_p} + \frac{S}{E_p} = \Phi^{-1} + S^* \,. \tag{1}$$

$$if\ \frac{P}{E_p} > 1 - \frac{S}{E_p} \qquad then \qquad \frac{E_x}{E_p} = 1\,. \tag{2}$$

The minimal value of evapotranspiration E_n is obtained for $E = S$:

$$\frac{E_n}{E_p} = S^*\,, \tag{3}$$

The translation into the Turc-Budyko framework (Figure 2d) yields:

$$if\ \frac{E_p}{P} > \frac{E_p}{E_p - S} \qquad then \qquad \frac{E_x}{P} = 1 + \frac{S}{P} = 1 + S^*\frac{E_p}{P} = 1 + S^*\Phi \,. \tag{4}$$

$$if\ \frac{E_p}{P} < \frac{E_p}{E_p - S} \qquad then \qquad \frac{E_x}{P} = \frac{E_p}{P} = \Phi\,. \tag{5}$$

$$\frac{E_n}{P} = \frac{S}{P} = S^*\frac{E_p}{P} = S^*\Phi\,. \tag{6}$$

Eq. (4) has two limits: when $S^* = 0$, $E_x/P = 1$, and when $S \rightarrow E_p$ corresponding to $S^* = 1$, $E_x/P \rightarrow (1+\Phi)$.

The case where part of the precipitation is used for storing water ($\Delta S > 0$ and $S < 0$) is equivalent to the case where E is lower than the maximum rate E_x; the difference between precipitation and evaporation is transformed into streamflow and water storage ($P - E = Q + \Delta S$) without the possibility of differentiating them.

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2.2 General equations with restricted evaporation

We examine now the case where the evaporation rate is lower than its maximum rate. For steady state conditions *(S = 0)* it should be recall that any Budyko function B_1 defined in the Budyko space $(E_p/P, E/P)$ generates an equivalent function B_2 in the Turc space expressed as:

$$\frac{E}{E_p} = B_2(\Phi^{-1}) = \frac{B_1(\Phi)}{\Phi}, \tag{7}$$

and that any Budyko function verifies the following conditions under steady state conditions: i) $E = 0$ if $P = 0$; ii) $E \leqslant P$ if $P \leqslant E_p$ (water limit); iii) $E \leqslant E_p$ if $P \geqslant E_p$ (energy limit); iv) $E \rightarrow E_p$ if $P \rightarrow \infty$.

Under non-steady state conditions ($S > 0$; Figure 2c), Eq. (7) should be transformed to take into account the contribution of soil water storage to the evaporation process: $E = S$ for $P = 0$. We search a mathematical formulation which transforms the upper and lower limits on Figures 2a and 2b for the steady state conditions to the corresponding ones respectively on Figures 2c and 2d for the non-steady state conditions. The mathematical formulation is searched under the following form $E/E_p = \alpha\, B_2(\gamma \Phi^{-1}) + \beta$, which combines translation (β) and homothetic functions (α and γ). In Figure 2a the lower limit corresponding to $B_2(\Phi^{-1}) = 0$ transforms into S^*, which means that $\beta = S^*$; the upper limit corresponding to $B_2(\Phi^{-1}) = 1$ transforms into 1, which means that $\alpha = 1 - S^*$; and the upper limit corresponding to $B_2(\Phi^{-1}) = \Phi^{-1}$ transforms into $(\Phi^{-1} + S^*)$, which translates into $\Phi^{-1} + S^* = (1-S^*)\, \gamma \Phi^{-1} + S^*$ and means that $\gamma = 1/(1-S^*)$. Consequently Eq. (7) should be transformed into:

$$\frac{E}{E_p} = (1 - S^*) B_2\left(\frac{\Phi^{-1}}{1-S^*}\right) + S^* . \tag{8}$$

By introducing Eq. (7) into Eq. (8), we obtain the formulation in the Budyko space (Figures 2b and 2d):

$$\frac{E}{P} = (1 - S^*)\Phi B_2\left(\frac{\Phi^{-1}}{1-S^*}\right) + S^*\Phi = B_1[(1 - S^*)\Phi] + S^*\Phi \ . \tag{9}$$

In the following this new formulation is called ML formulation (ML standing for Moussa-Lhomme). The derivative of Eq. (9) is

$$\frac{d\left(\frac{E}{P}\right)}{d\Phi} = (1 - S^*)\frac{dB_1[(1-S^*)\Phi]}{d\Phi} + S^* . \tag{10}$$

Given that $\frac{dB_1[(1-S^*)\Phi]}{d\Phi} = 1$ for $\Phi = 0$ and $\frac{dB_1[(1-S^*)\Phi]}{d\Phi} = 0$ when $\Phi \rightarrow \infty$, the derivative $\frac{d\left(\frac{E}{P}\right)}{d\Phi}$ (i.e. the slope of the curve) is equal to 1 for $\Phi = 0$, and when $\Phi \rightarrow \infty$, the derivative tends to S^*.

Any Budyko equation $B_1(\Phi)$ from Table 1 can be used in Eq.(9) as detailed in Table 3. It is worth noting that both Turc-Mezentsev and Fu-Zhang functions, which are obtained from the resolution of a Pfaffian differential equation, have the following remarkable simple property: $F(1/x) = F(x)/x$. This means that the same mathematical expression is valid for B_1 and B_2: $B_1 = B_2$. Both Turc-Mezentsev and Fu-Zhang functions have similar shapes, and a simple linear relationship was established by Yang et al. (2008) between their parameters (see Table 1): $\omega = \lambda + 0.72$. The ML formulation is used

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hereafter with the Fu-Zhang function (Table 1) for comparison with the analytical solution of Greve et al. (2016) based upon the same function. Replacing B_1 in Eq. (9) by Fu-Zhang's equation gives:

$$\frac{E}{P} = 1 + \Phi - [1 + (1 - S^*)^{\omega}\Phi^{\omega}]^{\frac{1}{\omega}} . \quad (11)$$

For $\Phi = 0$ we have $E/P = 0$, and when $\Phi \to \infty$, $E/P \to \infty$. Figure 3 shows an example of the shape of the ML formulation Eq. (11) for $\omega = 1.5$ and for different values of S^* (*0, 0.25, 0.5, and 1*). Note that for $S^* = 0$ we obtain the curves of Figure 1, while for $S^* = 1$, upper and lower limits are superimposed, and the domain is restricted to the 1:1 line. We can easily verify that all functions in Table 3 have similar shapes.

3 Comparing the new formulation with other formulae from the literature

3.1 In the standard Budyko space *(E_p/P, E/P)*

Greve et al. (2016) analytically developed a Budyko type equation where the water storage is taken into account through a parameter y_0 ($0 \leqslant y_0 \leqslant 1$) introduced into the Fu-Zhang formulation (Table 2). This equation writes (Greve et al., 2016; Eq. 9)

$$\frac{E}{P} = 1 + \Phi - [1 + (1 - y_0)^{\kappa-1}\Phi^{\kappa}]^{1/\kappa} . \quad (12)$$

They used the shape parameter κ to avoid confusion with the traditional ω of Fu-Zhang equation. Despite different physical and mathematical backgrounds Eqs. (11) and (12) are exactly similar and a simple relationship between S^* and y_0 can be easily obtained. Equating Eqs. (11) and (12) with $\omega = \kappa$ yields:

$$S^* = 1 - (1 - y_0)^{\frac{\omega-1}{\omega}} . \quad (13)$$

The relationship between y_0 and S^* is independent from Φ. It is shown in Figure 4 for different values of ω. For a given value of ω, we have $S^* < y_0$. For $\omega = 1$, we have $S^* = 0$, and when $\omega \to \infty$ we have $S^* = y_0$.

The derivative of Eq. (12) gives

$$\frac{d\left(\frac{E}{P}\right)}{d\Phi} = 1 - (1 - y_0)^{\kappa-1}\Phi^{\kappa-1}[1 + (1 - y_0)^{\kappa-1}\Phi^{\kappa}]^{\frac{1-\kappa}{\kappa}} . \quad (14)$$

For $\Phi = 0$ the derivative is equal to *1*, and when $\Phi \to \infty$, the derivative tends to a value noted *m* by Greve et al. (2016; Eq. 12):

$$m = 1 - (1 - y_0)^{\frac{\kappa-1}{\kappa}} . \quad (15)$$

The value of the derivative (slope of the curve) is the same in both ML and Greve et al.'s formulations: for $\Phi = 0$ the derivative is equal to 1, and when $\Phi \to \infty$ we have $m = S^*$ (assuming $\omega = \kappa$). Greve et al. (2016; section 4) show that y_0 is the maximum value of m reached when $\omega \to \infty$

Figure 5 compares the ML formulation Eq. (11) with Greve et al.'s analytical solution Eq. (12) for $\omega = \kappa = 2$ and different values of y_0 (*0, 0.2, 0.4, 0.6, 0.8 and 1*). The corresponding values of S^* (respectively *0, 0.106, 0.225, 0.367, 0.553*

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and 1) are calculated using Eq. (13). The new ML formulation with $\omega = \kappa$ gives exactly the same curves as those obtained by Greve et al. (2016). Both formulations are identical and have the same upper and lower limits (Greve et al. (2016), however, did not mention the lower limit).

3.2 In the space *[E_p/(P+S), E/(P+S)]*

As mentioned in the introduction, some authors (Chen et al., 2013; Du et al., 2016) have tackled the non-steady conditions by modifying the Budyko reference space, replacing the precipitation P by $P + S$. These new formulations are presented hereafter and compared with the ML formulation rewritten in this new space *[E_p/(P+S), (E/(P+S)]*.

3.2.1 The formulations of Chen et al. (2013) and Du et al. (2016)

The formulations proposed by Chen et al. (2013) and Du et al. (2016) in the space *[E_p/(P+S), (E/(P+S)]* are essentially empirical. Chen et al. (2013) function (Table 2) is derived from the Turc-Mezentsev equation and written as

$$\frac{E}{P+S} = \left[1 + \left(\frac{E_p}{P+S} - \Phi_t\right)^{-\lambda}\right]^{-\frac{1}{\lambda}} . \qquad (16)$$

An additional parameter Φ_t is empirically introduced in order "to characterize the possible non-zero lower bound of the seasonal aridity index"; this parameter causes a shift of the curve $E/(P+S)$ along the horizontal axis such as for $E_p/(P+S) = \Phi_t$ we have $E/(P+S) = 0$. The derivative of Eq. (16) when $E_p/(P+S) \to \infty$ is equal to 0. Similarly, Du et al. (2016) function (Table 2) is an empirical modification of Fu-Zhang equation (Fu, 1981; Zhang et al., 2004) written as

$$\frac{E}{P+S} = 1 + \frac{E_p}{P+S} - \left[1 + \left(\frac{E_p}{P+S}\right)^{\omega} + \mu\right]^{\frac{1}{\omega}} . \qquad (17)$$

A supplementary parameter, noted here μ (> *-1*), is added to modify the lower bound of the aridity index $E_P/(P+S)$. The parameter μ plays a similar role as Φ_t in Eq. (16). For $\mu = 0$, Eq. (17) takes the original form of Fu-Zhang equation, *(P+S)* replacing P. When μ becomes positive, the lower end of the curve $E/(P+S)$ shifts to the right. The function $E/(P+S)$ in Eq. (17) is equal to zero for the particular value of $E_p/(P+S) = \Phi_d$ such as

$$(1 + \Phi_d)^{\omega} = 1 + (\Phi_d)^{\omega} + \mu . \qquad (18)$$

Figure 6 shows an example of the curves obtained with the shape parameter $\omega = 2$ for Du et al.'s equation, while for Chen et al.'s equation we take the corresponding value of the parameter $\lambda = 1.28$ (such as $\lambda = \omega - 0.72$, as proposed by Yang et al., (2008)). In both cases, the abscissa-intercept is taken at $\Phi_t = \Phi_d = 1$ (corresponding to $\mu = 2$ from Eq. (18)). Both curves tend to 1 when $E_p/(P+S) \to \infty$.

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3.2.2 The new formulation in the space *[E_p/(P+S), E/(P+S)]*

The upper limits of the ML formulation in the new space can be obtained by transforming Eqs. (4) and (5). We get respectively:

$$if\ \frac{E_p}{P+S} > 1 \qquad then \qquad \frac{E_x}{P+S} = 1 \ , \tag{19}$$

$$if\ \frac{E_p}{P+S} < 1 \qquad then \qquad \frac{E_x}{P+S} = \frac{E_p}{P+S} \ . \tag{20}$$

The lower limit is obtained from Eq. (6):

$$\frac{E_n}{P+S} = \frac{S}{P+S} = S^* \frac{E_p}{P+S} \ . \tag{21}$$

The upper limits of the ML formulation are similar to those defined by Chen et al. (2013) and Du et al. (2016), but the lower limit differs. In the new space, we put:

$$\Phi' = \frac{E_p}{P+S} = \frac{\Phi}{1+S^*\Phi} \quad \text{or} \quad \Phi = \frac{\Phi'}{1-S^*\Phi'} \ . \tag{22}$$

Consequently the relationship between *E/(P+S)*, Φ' and *E/P* is given by

$$\frac{E}{P+S} = \frac{E}{P}\frac{P}{P+S} = \frac{1}{1+S^*\Phi}\frac{E}{P} = (1 - S^*\Phi')\frac{E}{P} \ . \tag{23}$$

Inserting Eq. (9) into Eq. (23) and expressing Φ as a function of Φ' (Eq. 22) lead to the ML formulation in the new space:

$$\frac{E}{P+S} = \frac{1}{1+S^*\Phi}\left[B_1\big((1-S^*)\Phi\big) + S^*\Phi\right] = (1-S^*\Phi')B_1\left[\frac{(1-S^*)\Phi'}{1-S^*\Phi'}\right] + S^*\Phi' \ . \tag{24}$$

For $\Phi = 0$, i.e. $P \to \infty$, we have $\Phi' = 0$, $B_1 = 0$ and $E/(P+S) = 0$. When $\Phi \to \infty$ which corresponds to $P \to 0$, we have $\Phi' = 1/S^*$, $B_1 = 1$, and $E/(P+S) \to 1$. We note that the domain of variation of Φ' is limited by 0 and $1/S^*$, which is a major difference when compared to the previous formulations (Chen et al., 2013; Du et al., 2016).

Any Budyko formulation B_1 in Table 1 can be used with Eq. (24). When the Fu-Zhang equation (Table 1) is used, Eq. (24) becomes:

$$\frac{E}{P+S} = 1 + (1-S^*)\Phi' - [(1-S^*\Phi')^\omega + (1-S^*)^\omega(\Phi')^\omega]^{1/\omega} \ . \tag{25}$$

Figure 7 shows the ML formulation Eq. (25) in the space *[E_p/(P+S), E/(P+S)]* for $\omega = 1.5$ and different values of S^* *(0, 0.25, 0.5 and 1)*. For $S^* = 0$ we retrieve the original Fu-Zhang equation and when $\omega = 1$, we can easily verify that Eq. (25) is equal to the lower limit of the domain $E/(P+S) = S^*E_p/(P+S)$.

Greve et al.'s formulation can be also written in the space *[E_p/(P+S), E/(P+S)]*. Inserting Eq. (12) into Eq. (23) and expressing Φ as a function of Φ' (Eq. 22) leads to

$$\frac{E}{P+S} = 1 + (1-S^*)\Phi' - [(1-S^*\Phi')^\kappa + (1-y_0)^{\kappa-1}(\Phi')^\kappa]^{1/\kappa} \ . \tag{26}$$

It can be mathematically shown that expressing $(1 - y_0)$ in Eq. (26) as a function of S^* by inverting Eq. (13) (assuming $\omega = \kappa$) leads to the exact ML formulation of Eq. (25) because of the similarity of both formulations. Consequently, similar curves to those shown in Figure 7 are obtained with Greve et al.'s formulation.

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3.2.3 Discussion

All four formulations, ML, Greve et al. (2016), Chen et al. (2013) and Du et al. (2016), have two parameters each, one for the shape of the curve and another for its shift due to non-steady conditions: ω and S^* for the ML formulation (with the Fu-Zhang function), κ and y_0 for Greve et al. (2016), λ and Φ_t for Chen et al. (2013), and ω and μ for Du et al. (2016). If $S^* = y_0 = \Phi_t = \mu = 0$, the four formulations are identical. They also have similar upper limits. However, as demonstrated above, the ML formulation and the one of Greve et al. (2016) behave very differently from the previous formulations in the space $[E_p/(P+S), E/(P+S)]$. The major difference between the ML formulation and those of Chen et al.'s and Du et al.'s equations is the domain of variation of Φ': respectively $[0, 1/S^*]$, $[\Phi_t, \infty]$ and $[\Phi_d, \infty]$. The lower end of the curve $E/(P+S)$ corresponds respectively to $(0, 0)$, $(\Phi_t, 0)$ and $(\Phi_d, 0)$ and the upper end to $(1/S^*, 1)$for the ML formulation, and $(\infty, 1)$ for the other two.

It is worth noting that the limits of Chen et al. (2013) and Du et al. (2016) functions are not completely sound from a strict physical standpoint: for very high precipitation, when $P >> E_p$, Φ and Φ' should logically tend to zero and not to Φ_t and Φ_d; similarly, when $P \rightarrow 0$, i.e., $\Phi \rightarrow \infty$, it is physically logical that $\Phi' \rightarrow E_p/S=1/S^*$, as predicted by our Eq. (24). This tends to prove that the ML formulation, corroborated by Greve et al. (2016) formulation, is physically more correct. Additionally, at simple glimpse, we note that the ML curves could be easily adjusted to the set of experimental points shown in Chen et al. (2013; Figures 2 and 9) and in Du et al. (2016; Figures 8 and 9).

4 Conclusion

A new formulation Eq. (9), called ML formulation, was developed to extend on physical basis the Budyko functions under non-steady conditions taking into account the change in soil water storage S. The ML formulation involves any classical Budyko function $B_1(\Phi)$ valid under steady-state conditions (Table 1) and introduces an additional parameter S^* to account for the change in water storage (Table 3). In the standard Budyko space $(E_p/P, E/P)$ and for the particular case where the Fu-Zhang equation is used for $B_1(\Phi)$, the ML formulation can be compared to the analytical solution developed by Greve et al. (2016). Assuming that the shape parameters of the ML formulation and that of Greve et al. are identical ($\omega = \kappa$), a simple relationship can be established between S^* and the corresponding parameter y_0 of Greve et al., both formulations giving similar results. Moving to the space $[E_p/(P+S), (E/(P+S)]$, the ML formulation is compared to those of Chen et al. (2013) and Du et al. (2016). We show that they have different feasible domain, the ML and Greve et al. formulations having similar behaviour. The domain of variation of $E_p/(P+S)$ on the x-axis differs: it is bounded by an upper limit $1/S^*$ in the ML formulation, while it is bounded by a lower limit in Chen et al.'s and Du et al.'s formulations.

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Notation

$B_1(\Phi)$ relationship between E/P and Φ in the Budyko space $(E_p/P, E/P)$ such as $E/P = B_1(\Phi)$ [-].

$B_2(\Phi^{-1})$ relationship between E/E_p and $\Phi^{-1} = P/E_p$ in the Turc space (P/E_p, E/E_p) such as $E/E_p = B_2(P/E_p)$ [-].

E actual evaporation [LT^{-1}].

E_n lower limit of the feasible domain of evaporation [LT^{-1}].

E_p potential evaporation [LT^{-1}].

E_x upper limit of the feasible domain of evaporation [LT^{-1}].

m slope of the equation of Greve et al. (2016) when $\Phi \rightarrow \infty$ [-].

ML new formulation Eqs (8) and (9) (stands for Moussa-Lhomme)

P precipitation [LT^{-1}].

Q runoff [LT^{-1}].

S change in soil water storage $(0 \leqslant S \leqslant E_p)$ [LT^{-1}].

S^* $= S/E_p$ $(0 \leqslant S^* \leqslant 1)$ [-].

y_0 parameter in the Greve et al. (2016) equation accounting for non-steady state conditions ($0 \leqslant y_0 \leqslant 1$) [-].

κ shape parameter in the Greve et al. (2016) equation corresponding to ω in the Fu-Zhang equation [-].

λ shape parameter in the Turc-Mezentsev equation ($\lambda > 0$) [-].

μ parameter in the Du et al. (2013) equation [-].

Φ aridity index ($\Phi = E_p/P$) [-].

Φ_d aridity index threshold in the Du et al. (2016) equation corresponding to $E/(P+S) = 0$ [-].

Φ_t aridity index threshold in the Chen et al. (2013) equation [-].

Φ' $= E_p/(P + S)$ [-].

ω shape parameter of the Fu-Zhang equation ($\omega > 1$) [-].

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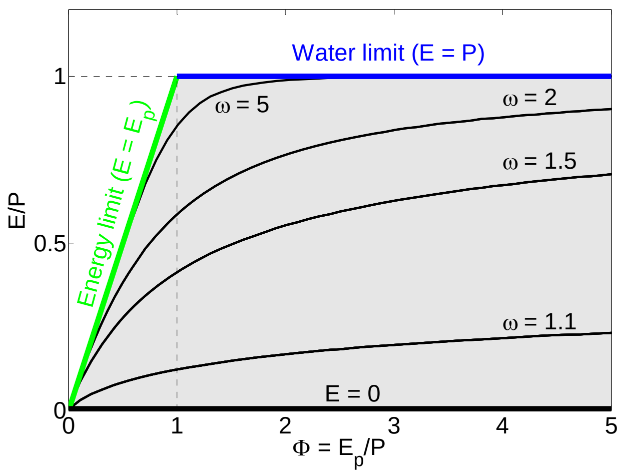

Figure 1: Representation in the Budyko space of the Fu-Zhang $B_1(\Phi)$ function (Table 1) between the ratio *E/P* and the aridity index $\Phi = E_p/P$ for four values of the parameter ω (*1.1, 1.5, 2 and 5*). The bold line indicates the upper and lower limits of the feasible domain: in blue the water limit *(E = P)*, in green the energy limit $(E = E_p)$ and in black the lower limit *(E = 0)*. The grey zone indicates the feasible domain.

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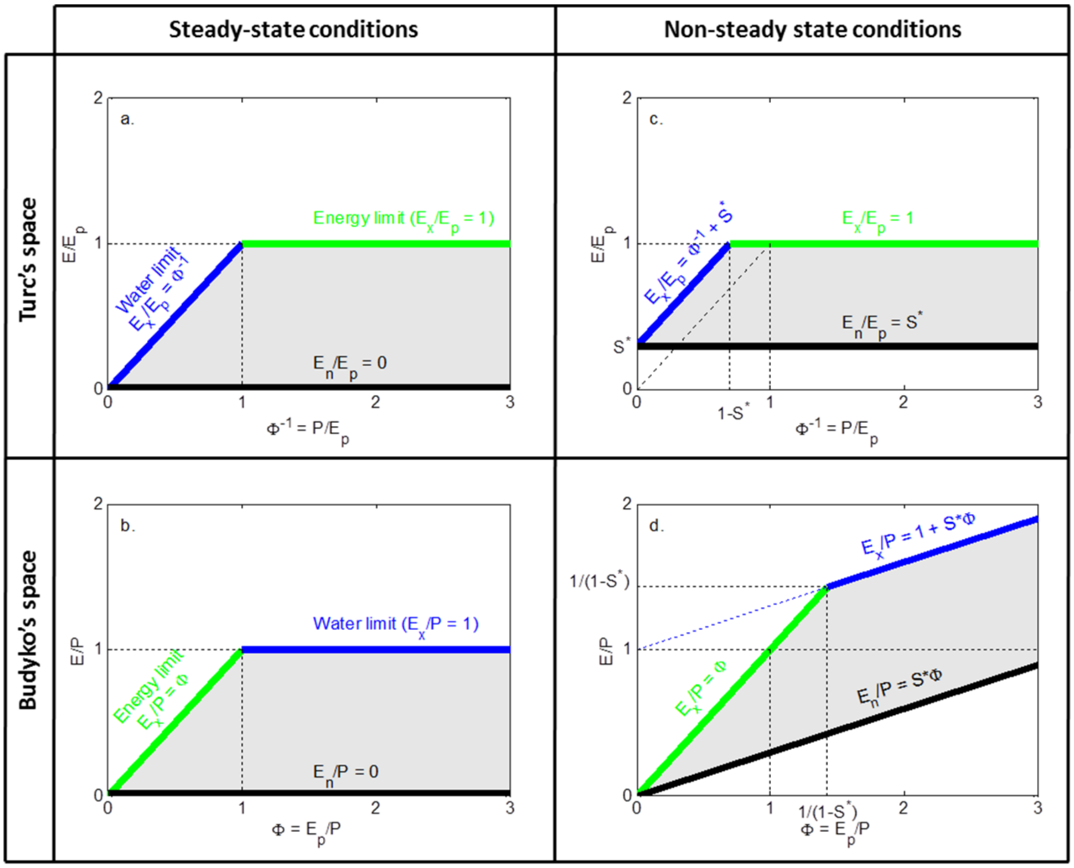

Figure 2: Upper and lower limits of the feasible domain (in grey) of evaporation in the Turc space (*P/E_p, E/E_p*) and in the Budyko space *(E_p/P, E/P)* (water limit in blue, energy limit in green and lower limit in black): (a and b) for steady state conditions; (c and d) for non-steady state conditions with a storage term *S*.

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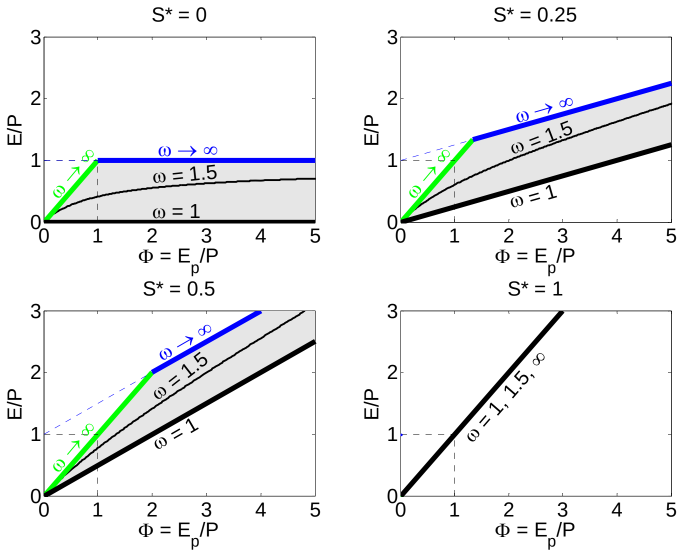

Figure 3: The ML formulation with the Fu-Zhang relationship Eq. (11) for $\omega = 1.5$ and for different values of S^* (*0, 0.25, 0.5* and *1*). The bold lines indicate the upper and lower limits of the feasible domain of evaporation shown in grey.

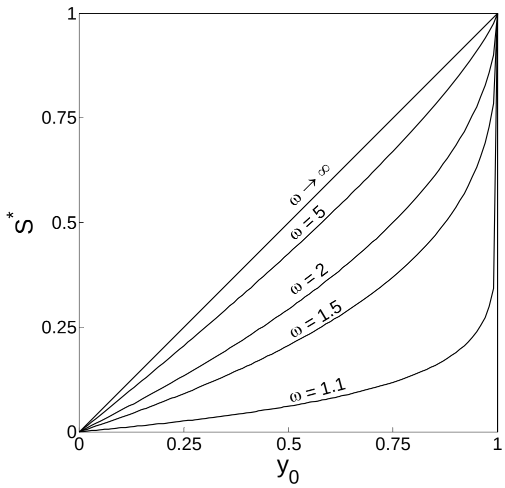

Figure 4: Relationship between the S^* of the ML formulation Eq. (11) and the parameter y_0 of the Greve et al. (2016) equation Eq. (12) for different values of ω (*1.1, 1.5, 2, 5* and *8*) and $\omega = \kappa$.

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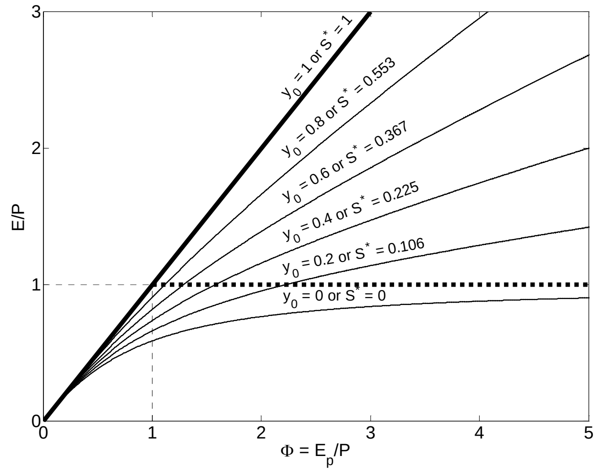
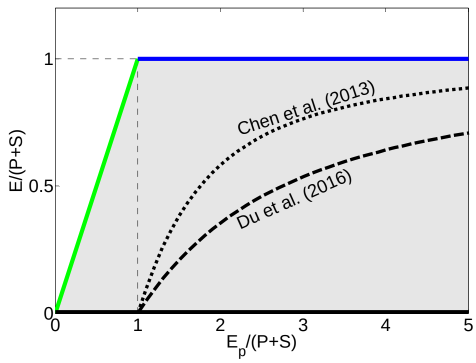


Figure 5: Example showing the similarity of the ML formulation Eq. (11) and the equation of Greve et al. (2016) Eq. (12) (with $\omega = \kappa = 2$) for different values of y_0; the corresponding values of S^* are calculated using Eq. (13).



5 **Figure 6: Formulations of Chen et al. (2013) (with $\lambda = 1.28$ and $\Phi_t = 1$) and Du et al. (2016) (with $\omega = 2$ and $\mu = 1$) in the space $[E_p/(P+S), E/(P+S)]$. The bold lines indicate the upper and lower limits of the feasible domain in grey.**

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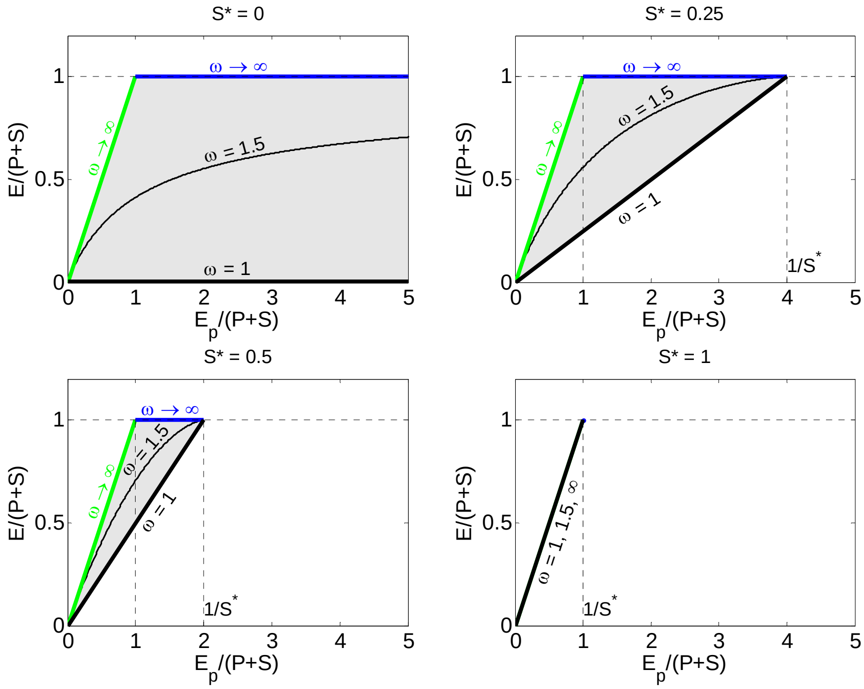

Figure 7: The ML formulation Eq. (25) with the Fu-Zhang equation in the space *[E_p/(P+S), E/(P+S)]* for *ω = 1.5* and four values of *S^**. All curves have a common upper end at *Φ' = 1/S^** corresponding to *E/(P+S) = 1*. The bold lines indicate the upper and lower limits of the feasible domain shown in grey. For *S^* = 0* the curve is similar to that in Figure 1.

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Table 1: Different expressions for the Budyko curves under steady state conditions.

Reference	Equation $E/P = B_1(\Phi)$
Budyko (1974)	$\frac{E}{P} = \left\{\Phi \tanh\left(\frac{1}{\Phi}\right)[1-\exp(-\Phi)]\right\}^{1/2}$
Turc (1954) with $\lambda = 2$, Mezentsev (1955), Yang et al. (2008)	$\frac{E}{P} = \Phi(1+\Phi^{\lambda})^{-\frac{1}{\lambda}}$
Fu (1981), Zhang et al. (2004)	$\frac{E}{P} = 1+\Phi-(1+\Phi^{\omega})^{\frac{1}{\omega}}$
Zhang et al. (2001)	$\frac{E}{P} = \frac{1+w\Phi}{1+w\Phi+\Phi^{-1}}$
Zhou et al. (2015)	$\frac{E}{P} = \Phi\left(\frac{k}{1+k\Phi^{n}}\right)^{1/n}$

Table 2: Different expressions for the Budyko curves under non-steady state conditions.

Reference	Steady state conditions $B_1(\Phi)$	Non-steady state conditions
Greve et al. (2016)	Fu-Zhang	$\frac{E}{P} = 1+\frac{E_p}{P}-\left[1+(1-y_0)^{\kappa-1}\left(\frac{E_p}{P}\right)^{\kappa}\right]^{1/\kappa}$ with κ and y_0 parameters.
Chen et al. (2013)	Turc-Mezentsev	$\frac{E}{P+S} = \left[1+\left(\frac{E_p}{P+S}-\Phi_t\right)^{-\lambda}\right]^{-\frac{1}{\lambda}}$ with λ and Φ_t parameters.
Du et al. (2016)	Fu-Zhang	$\frac{E}{P+S} = 1+\frac{E_p}{P+S}-\left[1+\left(\frac{E_p}{P+S}\right)^{\omega}+\mu\right]^{\frac{1}{\omega}}$ with ω and μ parameters.

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Table 3: The ML formulation under non-steady state conditions applied for the different Budyko curves in Table 1.

Reference Budyko curve under steady state conditions	The ML formulation under non-steady state conditions using Eq. (9)
Budyko (1974)	$\frac{E}{P}=\left\{(1-S^*)\Phi\tanh\left[\frac{1}{(1-S^*)\Phi}\right]\left[1-\exp(-\Phi+S^*\Phi)\right]\right\}^{1/2}+S^*\Phi$
Turc (1954), Mezentsev (1955), Yang et al. (2008)	$\frac{E}{P}=\left[1+(1-S^*)^{-\lambda}\Phi^{-\lambda}\right]^{-\frac{1}{\lambda}}+S^*\Phi$
Fu (1981), Zhang et al. (2004)	$\frac{E}{P}=1+\Phi-\left[1+(1-S^*)^{\omega}\Phi^{\omega}\right]^{\frac{1}{\omega}}$
Zhang et al. (2001)	$\frac{E}{P}=\frac{(1-S^*)\Phi+w(1-S^*)^2\Phi^2}{(1-S^*)\Phi+w(1-S^*)^2\Phi^2+1}+S^*\Phi$
Zhou et al. (2015)	$\frac{E}{P}=(1-S^*)\Phi\left[\frac{k}{1+k(1-S^*)^n\Phi^n}\right]^{1/n}+S^*\Phi$