Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences
Discussions
Open Access

Numerical Solution and Application of Time-Space Fractional Governing Equations of One-Dimensional Unsteady Open Channel Flow Process

Ali Ercan[1] and M. Levent Kavvas[2]

[1] Research Fellow, J. Amorocho Hydraulics Laboratory, Dept. of Civil and Environmental Engineering, University of California, Davis, CA, 95616, USA
[2]Professor, J. Amorocho Hydraulics Laboratory, Dept. of Civil and Environmental Engineering, University of California, Davis, CA, 95616, USA

Abstract

Although fractional integration and differentiation have found many applications in various fields of science, such as physics, finance, bioengineering, continuum mechanics and hydrology, their engineering applications, especially in the field of fluid flow processes, are rather limited. In this study, a finite difference numerical approach is proposed to solve the time-space fractional governing equations of one-dimensional unsteady/non-uniform open channel flow process. By numerical simulations, results of the proposed fractional governing equations of the open channel flow process were compared with those of the standard Saint Venant equations. Numerical simulations showed that flow discharge and water depth can exhibit heavier tails in downstream locations as space and time fractional derivative powers decrease from 1. The fractional governing equations under consideration are generalizations of the well-known Saint Venant equations, which are written in the integer differentiation framework. The new governing equations in the fractional order differentiation framework have the capability of modeling nonlocal flow processes both in time and in space by taking the global correlations into consideration. Furthermore, the generalized flow process may shed light into understanding the theory of the anomalous

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

transport processes and observed heavy tailed distributions of particle displacements in transport processes.

Introduction

The origin of integration and differentiation of arbitrary (non-integer) order dates back to a letter of Leibniz, in which derivatives of 0.5 order was described (Oldham and Spainer, 1974). Especially after the first international conference (Ross, 1975), the fractional calculus found many applications in various fields of science, including physics, finance, bioengineering, continuum mechanics and hydrology (Caputo, 1967; Bouchaud and Georges, 1990; Carpinteri and Mainardi, 1997; Podlubny, 1999; Hilfer, 2000; Barkai et al., 2000; Benson et al., 2000; Scalas et al., 2000; Baeumer et al., 2001; Meerschaert et al., 2002; Raberto et al., 2002; Zaslavsky, 2002; Magin, 2006; Tarasov, 2010; Benson et al., 2013; Kavvas et al., 2015; Kim et al., 2015; and many others). As Tarasov (2010) stated, the fractional equations are utilized to describe the fractal distributions of mass, charge and probability. However, their engineering applications, especially in the field of fluid dynamics, are limited (Kulish and Lage, 2002; Tarasov, 2010; Kavvas and Ercan, 2015, 2016; Ercan and Kavvas, 2015a).

The long-range dependence or the long memory occurs when the so called Hurst coefficient (Hurst, 1951) is between 0.5 and 1 showing that the process is outside the Brownian domain of finite memory processes. Slowly decaying autocorrelations and unbounded spectral density near zero frequency are the characteristics of the long memory signals (Beran 1994). The long-range dependency or the long memory was reported for several geophysical processes, for example, for river flows and velocity fluctuations by Nordin et al. (1972), Montanari et al. (1997), Vogel et al. (1998), and Szolgayová et al. (2013); porosity and hydraulic conductivity in sub-surface hydrology by Molz and Boman (1993); climate variability by Bloomfield (1992), Stephenson et al. (2000), Koutsoyiannis (2003), Franzke (2013), and Franzke et al. (2015); and sea levels by Barbosa et al. (2006), and Ercan et al. (2013). Hurst phenomena or the long-range dependence is also closely related to self-similarity of the geophysical processes (Pipiras and Taqqu, 2003; Beran, 1994; Ercan et al., 2014; Ercan and Kavvas, 2015b, 2015c).

In order to be able to model both the long-memory behavior as well as the Brownian finite-memory behavior of a flow process at various scales, Kavvas and Ercan (2016) stated that a model needs to be more flexible than a purely long-memory model (such as fractional Gaussian noise or Autoregressive fractionally integrated moving average model), or a finite-memory model (such as the standard integer governing equations of the river flow processes). Within this context, time-space fractional governing equations of unsteady/non-uniform open channel flow process were developed within Caputo fractional derivative framework by Kavvas and Ercan (2016). The advantage of the fractional derivatives in Caputo framework is that the traditional initial and boundary conditions, which are physically interpretable, can be utilized (Podlubny, 1999).

In this study, a numerical algorithm is proposed to solve the time-space fractional governing equations of one-dimensional unsteady/non-uniform open channel flow process and to investigate its modeling capabilities different than the standard Saint Venant equations. A first-order approximation of the Caputo's fractional time derivative (Murio, 2008) and a second-order accurate Caputo's fractional space derivative (Odibat, 2009) are coupled in the proposed numerical algorithm. When orders (or powers) of the time and space fractional derivatives become one, the proposed governing equations of unsteady/non-uniform open channel flow process in fractional differentiation framework reduce to Saint Venant equations in integer order differentiation framework (Kavvas and Ercan, 2016), and the proposed numerical algorithm reduces to the explicit finite difference scheme reported in Viessman, et al. (1977). As such, utilizing the proposed numerical solution, the capabilities of the proposed fractional governing equations of the unsteady/nonuniform open channel flow process were investigated, comparing the results of the fractional governing equations with those of the standard Saint Venant equations.

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences
Open Access
Discussions

The fractional governing equations of one-dimensional unsteady/non-uniform open channel flow process based on the Caputo fractional derivatives, as derived in Kavvas and Ercan (2016), are presented in the next section.

Governing Equations

The time-space fractional mass conservation equation and the time-space fractional motion equation, based on the Caputo fractional derivative framework, for one-dimensional unsteady/non-uniform open channel flow in straight prismatic channels which is taking place from the upstream to downstream direction, can be written as (Kavvas and Ercan, 2016),

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}A}{\partial t^{\alpha}}+\frac{\Gamma(2-\beta)}{x^{1-\beta}}\frac{\partial^{\beta}AV}{\partial x^{\beta}}=q_L(x,t)\ldots \quad (1)$$

$$S_f=-\frac{\Gamma(2-\beta)}{x^{1-\beta}}\frac{\partial^{\beta}z}{\partial x^{\beta}}-\frac{\Gamma(2-\beta)}{x^{1-\beta}}\frac{\partial^{\beta}y}{\partial x^{\beta}}-\frac{V}{g}\frac{\Gamma(2-\beta)}{x^{1-\beta}}\frac{\partial^{\beta}V}{\partial x^{\beta}}-\frac{1}{g}\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}V}{\partial t^{\alpha}}\ldots \quad (2)$$

where $0<\alpha,\ \beta\leq 1$ are the time and space fractional derivative powers, $\Gamma(\cdot)$ is the gamma function, Q is the flow rate, A is the flow cross-sectional area, x is distance in the flow direction, t is time, y is water depth, g is the acceleration due to gravity, z is the channel bed elevation. The advantage of the utilized Caputo fractional derivative approach is that the traditional initial and boundary conditions can be incorporated (Podlubny, 1999).

In the fractional continuity equation and fractional motion equation, there are three unknown flow variables, while there are only two equations. Accordingly, in order to close this system a resistance equation is needed. Such a resistance equation is the Manning's equation, which can be expressed as

$$S_f=\frac{n^2V|V|}{R^{4/3}}\ldots \quad (3)$$

The fractional mass conservation equation (1), the fractional motion equation (2) and the fractional resistance equation (3) form the complete set of governing equations of upstream-to-downstream one dimensional unsteady open channel flow in fractional time-

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

space. When α=β=1, governing equations of one dimensional fractional open channel flow reduce to classical Saint Venant equations.

A comprehensive explanation of the concepts of fractional differentiation and integration could be found in Oldham and Spanier (1974), Miller and Ross (1993), Samko et al. (1993), Podlubny (1999), Podlubny (2002), and Tarasov (2010).

Numerical Solution

Although a large number of physical problems are formulated by the fractional differential equations, the state of art is far less developed for their solution by the numerical approaches, and the analytical solutions are available for very simple linear problems (Podlubny, 1999). Therefore, as pointed out by Li and Zeng (2012), it is important to develop efficient and reliable numerical solutions for the problems governed by the fractional differential equations. Within this context, a finite difference numerical method, first order accurate in time and second order accurate in space, is introduced in this article to solve time-space fractional governing equations of the one-dimensional unsteady/non-uniform open channel flow process.

Dividing the time interval [0, T] into N subintervals of equal width k = T/N by using the nodes $t_n = nk$, n = 0, 1, 2, … , N, the first-order approximation of Caputo's fractional time derivative can be written as (Murio, 2008)

$$D_t^{\alpha} f_i^n = \sigma_{\alpha,k} \sum_{j=1}^{n} w_j^{(\alpha)} (f_i^{n-j+1} - f_i^{n-j}) \ldots \quad (4)$$

where $f_i^n = f(x_i, t_n)$, $\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(1-\alpha)} \frac{1}{k^{\alpha}}$ and $w_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$

Then, the fractional continuity equation given by Eqn. (1) can be discretized as

$$\sigma_{\alpha,k} \sum_{j=1}^{n+1} w_j^{(\alpha)} (A_i^{n-j+2} - A_i^{n-j+1}) + \frac{(t_i^n)^{1-\alpha}}{(x_i^n)^{1-\beta}} \frac{\Gamma(2-\beta)}{\Gamma(2-\alpha)} \frac{\partial^{\beta} A_i^n V_i^n}{\partial x^{\beta}} = \frac{(t_i^n)^{1-\alpha}}{\Gamma(2-\alpha)} q_{Li}^{\ n} \ldots (5)$$

where $A_i^n = A(x_i, t_n)$. Rearranging the known variables on the right hand side and the unknown flow area on the left hand side results in

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

$$A_i^{n+1} = A_i^n - \sum_{j=2}^{n+1} w_j^{(\alpha)} (A_i^{n-j+2} - A_i^{n-j+1}) - \frac{(t_i^n)^{1-\alpha}}{(x_i^n)^{1-\beta}} \frac{\Gamma(2-\beta)}{\Gamma(2-\alpha)\sigma_{\alpha,k}} \frac{\partial^\beta A_i^n V_i^n}{\partial x^\beta} + \frac{(t_i^n)^{1-\alpha}}{\Gamma(2-\alpha)\sigma_{\alpha,k}} q_{Li}^{\,n} \quad \ldots(6)$$

Fractional space derivative term $\frac{\partial^\beta}{\partial x^\beta}$ in Eqn. (6) can be obtained from the second order accurate approximation of the Caputo fractional derivatives, as proposed by Odibat (2009). Dividing the space interval [0, a] into M subintervals $[x_j, x_{j+1}]$ of equal width h = a/M using the nodes x_j = jh, for j = 0, 1, 2,…, M-1, the Caputo fractional derivative $D_x^\alpha f(x)$ at a>0 can be approximated by (Odibat, 2009)

$$(D_x^\alpha f(x))(a) \approx \frac{h^{1-\alpha}}{\Gamma(3-\alpha)} \Big\{ \left[(M-1)^{2-\alpha} - (M+\alpha-2)M^{1-\alpha} \right] f'(0) + f'(a)$$

$$+ \sum_{j=1}^{M-1} \left[(M-j+1)^{2-\alpha} - 2(M-j)^{2-\alpha} + (M-j-1)^{2-\alpha} \right] f'(x_j) \Big\} \ldots(7)$$

where $0 < \alpha \le 1$ is the arbitrary order of the derivative and $f'(x_j)$ is the first order derivative at x_j , which can be estimated by the central-difference formula as

$$f'(x_j) = \frac{f(x_j + h) - f(x_j - h)}{2h} + O(h^2) \ldots \quad (8)$$

The first order derivative of the function f can be estimated as $f'(0) = \frac{f(h) - f(0)}{h}$ at the upstream boundary and as $f'(a) = \frac{f(a) - f(a-h)}{h}$ at the downstream boundary when the further upstream and further downstream function values are not available, which is usually the case in open channel flow problems.

Analogous to the explicit finite difference scheme for the equation of motion in terms of integer order derivatives in Viessman, et al. (1977), and utilizing the first-order approximation of Caputo’s fractional time derivative (Murio, 2008), the fractional equation of motion given by Eqn. (2) can be discretized as

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences Discussions
Open Access

$$(S_f)_i^{n+1} = -\frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta z_i^n}{\partial x^\beta} - \frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta y_i^n}{\partial x^\beta} - \frac{V_i^n}{g}\frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta V_i^n}{\partial x^\beta}$$

$$-\frac{\Gamma(2-\alpha)\sigma_{\alpha,k}}{g(t_i^n)^{1-\alpha}}\sum_{j=1}^{n+1} w_j^{(\alpha)}(V_i^{n-j+2} - V_i^{n-j+1}) \ldots (9)$$

When the flow is from upstream to downstream $V|V|$ in Eqn. (3) becomes V^2 and Eqn. (9) can be written as

$$\frac{n^2(V_i^{n+1})^2}{(R_i^{n+1})^{4/3}} = -\frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta z_i^n}{\partial x^\beta} - \frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta y_i^n}{\partial x^\beta} - \frac{V_i^n}{g}\frac{\Gamma(2-\beta)}{(x_i^n)^{1-\beta}}\frac{\partial^\beta V_i^n}{\partial x^\beta}$$

$$-\frac{\Gamma(2-\alpha)\sigma_{\alpha,k}}{g(t_i^n)^{1-\alpha}}\sum_{j=1}^{n+1} w_j^{(\alpha)}(V_i^{n-j+2} - V_i^{n-j+1}) \ldots (10)$$

Space-fractional derivative terms $\frac{\partial^\beta z_i^n}{\partial x^\beta}$, $\frac{\partial^\beta y_i^n}{\partial x^\beta}$, and $\frac{\partial^\beta V_i^n}{\partial x^\beta}$ can be estimated from Eqn. (7).

If bed elevation does not change through time, then the initial estimate of $\frac{\partial^\beta z_i^n}{\partial x^\beta}$ will be the same for all time steps. On the other hand, if bed elevation changes through time, such as in the case of sediment transport processes, then this term should be calculated for each time step as the bed elevation changes through time due to erosion and deposition processes.

For a known geometry, hydraulic radius R_i^{n+1} can be calculated from the known flow cross-sectional area A_i^{n+1}. For example, water depth can be calculated as $y_i^{n+1} = A_i^{n+1}/B$ for a rectangular channel of width B and the hydraulic radius can be calculated as $R_i^{n+1} = A_i^{n+1}/(B + 2y_i^{n+1})$. Eqn. (10) is a quadratic equation in the form of $c_1(V_i^{n+1})^2 + c_2V_i^{n+1} + c_3 = 0$, from which V_i^{n+1} can be estimated.

Numerical Example

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences
Open Access
Discussions

For a numerical example problem, a 3218.7-meter long, 6.1-meter wide rectangular channel with a bottom slope of 0.0015 and an estimated Manning's roughness of 0.02 is considered. At the upstream boundary, 1.83-meter depth initial uniform flow is subjected to an increase to 56.63 m^3/s in a period of 20 minutes. Then this flow decreases uniformly to the initial flow depth during a subsequent period of 40 minutes duration. During the rest of the simulation, the initial flow conditions are applied. In this example problem, the upstream flow is routed utilizing the time-space fractional governing equations of the one-dimensional unsteady/non-uniform open channel flow (Eqns 1-3, Kavvas and Ercan, 2016), by applying the proposed numerical solution approach given above.

A classical solution of this problem by the Saint Venant equations in integer order differentiation framework was provided by Viessmann et al. (1977). In addition to the above proposed numerical solution for the time-space fractional governing equations, this problem was also simulated by the Viessmann et al. (1977)'s explicit numerical scheme for the Saint Venant equations, with the difference that the hydraulic radius was estimated by the ratio of the flow cross-sectional area to the wetted perimeter. The solutions by the Saint Venant equations were depicted as Viessmann et al. (1977) in Figures 1 and 2.

Non-dimensional flow discharges (Q/Q_0) and non-dimensional water depths (y/y_0) through time at the downstream boundary are depicted in Figure 1 when space and time fractional powers $\beta = \alpha = 1$, 0.9, 0.8, 0.7. Here, Q_0 and y_0 are initial flow and water depths. As shown in Figures 1a and 1b, when time and space derivative powers are equal to one (α=β=1), non-dimensional water depths and flows at the downstream location, simulated by the proposed numerical scheme for the fractional governing equations of the one-dimensional

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

unsteady/non-uniform open channel flow, coincide well with those simulated by the Viessmann et al. (1977)'s explicit numerical scheme for standard Saint Venant equations. In order to visualize the tails better, non-dimensional flow discharges (Q/Q_0) and non-dimensional water depths (y/y_0), which were depicted in Figures 1a and 1b, are zoomed over the [0.9-1.1] range and presented in Figures 2a and 2b, respectively. As presented in Figures 2a and 2b, non-dimensional flow discharges and non-dimensional water depths exhibit heavier tails as space and time fractional derivative powers decrease from 1.

Magnitudes and occurrence times of the peak non-dimensional flows (Q/Q_0) at the downstream boundary for various values of the fractional space derivative power β and of the fractional time derivative power α are tabulated in Table 1. Both the magnitude and the occurrence time of the peak flow decrease as the time and space fractional derivative powers decrease. The magnitude of the peak flow decreases from 2.05 to 1.82 and the occurrence time of the peak flow decreases from 0.59 to 0.50 hr.

Non-dimensional flow discharges through longitudinal length x at various simulation times (t=20, 30, 40, 60, 75, 90, 120, 180 minutes) are depicted in Figure 3 when space and time fractional derivative powers $\beta=\alpha=1$, 0.9, 0.8, 0.7. Figure 3 clearly shows that non-dimensional flow discharges reached the equilibrium value of 1 from upstream boundary to downstream boundary (from x=0 to x=3.2187 km) after the simulation time of 120 minutes when space and time fractional derivative powers are 1 ($\beta=\alpha=1$). However, when the simulation time is greater than 120 minutes, the non-dimensional flow discharges deviate from the value of 1 toward the downstream boundary as space and time fractional

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

derivative powers decrease from 1. In other words, the influence of the flood hydrograph that enters to the channel reach from the upstream boundary, is observed for a longer duration in the downstream locations when the space and time fractional derivative powers are smaller than 1. This finding also explains the heavier tails at the downstream boundary when the fractional derivative powers are less than 1.

Concluding remarks

In this study, a numerical algorithm, first order accurate in time and second order accurate in space, was proposed to solve time-space fractional governing equations of one-dimensional unsteady/non-uniform open channel flow process (Kavvas and Ercan, 2016) and to investigate its modeling capabilities that are different from the standard Saint Venant equations. The fractional governing equations under consideration are generalizations of the well-known Saint Venant equations written in the integer differentiation framework. When orders (powers) of the fractional time and fractional space derivatives become one, the proposed fractional governing equations of open channel flow process (Kavvas and Ercan, 2016) reduce to standard Saint Venant equations and the proposed numerical algorithm reduces to the explicit finite difference scheme reported in Viessman, et al. (1977). The new governing equations in the fractional order differentiation framework have the capability of modeling nonlocal flow processes both in time and in space by taking the global relations into consideration (see Eqn. 4 and Eqn. 7).

The following conclusions can be drawn from the numerical investigation:

1. When the time and space fractional derivative powers become one, the proposed numerical scheme for the time-space fractional governing equations of one-dimensional unsteady/non-uniform open channel flow process give similar results with the explicit finite difference scheme for standard Saint Venant equations (Viessman, et al., 1977).
2. Non-dimensional flow discharges and non-dimensional water depths exhibit heavier tails at the downstream boundary as space and time fractional derivative powers decrease from 1.
3. Both the magnitude and the occurrence time of the peak flow decrease as the powers of the time and space fractional derivatives decrease from 1.
4. The influence of the flood hydrograph that enters to the channel reach from the upstream boundary, is observed for a longer duration in the downstream locations when the space and time fractional derivative powers are smaller than 1. This finding also explains the heavier tails at the downstream boundary when the fractional derivative powers are less than 1.

References

Baeumer B, Benson DA, Meerschaert MM, Wheatcraft SW. 2001. Subordinated advection–dispersion equation for contaminant transport. Water Resources Research 37(6): 1543-1550.

Barbosa SM, Fernandes MJ, Silva ME. 2006. Long-range dependence in North Atlantic sea level. Physica A 371(2): 725–731.

Barkai E, Metzler R, Klafter J. 2000. From continuous time random walks to the fractional Fokker–Planck equation. Physical Review E 61: 132-138.

Benson DA, Wheatcraft SW, Meerschaert MM. 2000. Application of a fractional advection-dispersion equation. Water Resources Research 36(6): 1403-1412.

Benson DA, Meerschaert MM, Revielle J. 2013. Fractional calculus in hydrologic modeling: A numerical perspective. Advances in Water Resources 51: 479-497.

Beran J. 1994. Statistics for Long-Memory Processes. Chapman and Hall, New York.

Bloomfield P. 1992. Trends in global temperature. Climatic Change 21(1): 1-16.

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Bouchaud JP, Georges A. 1990. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Phys.Rep.195: 127-293.

Caputo M. 1967. Linear models of dissipation whose Q is almost frequency independent, part II. Geophysical Journal International, 13(5), 529–539.

Carpinteri A, Mainardi F. 1997. Fractals and Fractional Calculus in Continuum Mechanics. CISM International Centre for Mechanical Sciences Series, vol. 378. Springer-Verlag New York, Inc., New York.

Ercan A, Kavvas ML. 2015a. Fractional governing equations of diffusion wave and kinematic wave open channel flow in fractional time-space: II. Numerical Simulations. Journal of Hydrologic Engineering, 20(9), 10.1061/(ASCE)HE.1943-5584.0001081, 04014097.

Ercan A, Kavvas ML. 2015b. Scaling and Self-similarity in Two-Dimensional Hydrodynamics. Chaos 25(7). DOI:10.1063/1.4913852.

Ercan A, Kavvas ML. 2015c. Self-similarity in incompressible Navier-Stokes equations. Chaos, 25, 123126. http://dx.doi.org/10.1063/1.4938762.

Ercan A, Kavvas ML, Haltas I. 2014. Scaling and self-similarity in one-dimensional unsteady open channel flow. Hydrological Processes 28(5), 2721-2737.

Ercan A, Kavvas ML, Abbassov RK. 2013. Long-range Dependence and Sea Level Forecasting. SpringerBriefs in Statistics. Springer International Publishing, Cham. Book ISBN: 978-3-319-01505-7.

Franzke, C. 2013. Persistent regimes and extreme events of the North Atlantic atmospheric circulation. Philos. Trans. R. Soc. A, 371, 20110471.

Franzke, CLE, Osprey SM, Davini P, Watkins NW. 2015. A Dynamical Systems Explanation of the Hurst Effect and Atmospheric Low-Frequency Variability. Sci. Rep. 5, 9068; DOI:10.1038/srep09068.

Hilfer R. 2000. Applications of Fractional Calculus in Physics. World Scientific, River Edge, NJ, USA.

Hurst HE. 1951. Long-term storage capacity of reservoirs. Transactions of the American Society of Civil Engineers 116:77–779.

Kavvas ML, Ercan A. 2015. Fractional governing equations of diffusion wave and kinematic wave open channel flow in fractional time-space: I. Development of the equations. Journal of Hydrologic Engineering 20(9), 10.1061/(ASCE)HE.1943-5584.0001136, 04014096.

Kavvas ML, Ercan A. 2016. Time-space fractional governing equations of unsteady open channel flow. Journal of Hydrologic Engineering (Accepted for publication).

Kavvas ML, Kim S, Ercan A. 2015. Fractional Ensemble Average Governing Equations of Transport by Time-Space Non-Stationary Stochastic Fractional Advective Velocity and Fractional Dispersion. Journal of Hydrologic Engineering 20(2), 10.1061/(ASCE)HE.1943-5584.0000992, 04014039.

Kim S, Kavvas ML, Ercan A. 2015. Fractional Ensemble Average Governing Equations of Transport by Time-Space Non-Stationary Stochastic Fractional Advective Velocity and Fractional Dispersion: Numerical Investigation. Journal of Hydrologic Engineering 20(2), 10.1061/(ASCE)HE.1943-5584.0000994, 04014040.

Koutsoyiannis D. 2003. Climate change, the Hurst phenomenon, hydrological statistics. Hydrologic Sciences Journal 48, 3–24.

Kulish VV, Lage JL. 2002. Application of fractional calculus to fluid mechanics. Journal of Fluids Engineering 124(3), 803–806.

Langlands TAM, Henry BI. 2005. The accuracy and stability of an implicit solution method for the fractional diffusion equation. J. Comput. Phys. 205: 719–736.

Li CP, Zeng FH. 2012. Finite difference methods for fractional differential equations. Int. J. Bifurcation Chaos 22(4), 1230014 DOI:0.1142/S0218127412300145.

Magin RL. 2006. Fractional Calculus in Bioengineering. Begell House Publishers, Connecticut.

Meerschaert MM, Benson DA, Scheffler HP, Baeumer B. 2002. Stochastic solution of space-time fractional diffusion equations. Phys. Rev. E 65: 1103-1106.

Miller K, Ross B. 1993. An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York.

Molz FJ, and Boman GK. 1993. A fractal-based stochastic interpolation scheme in subsurface hydrology. Water Resources Research 29(11): 3769–3774.

Montanari A, Rosso R, and Taqqu MS. 1997. Fractionally differenced ARIMA models applied to hydrologic time series: Identification, estimation and simulation, Water Resour Res 33, 1035 – 1044.

Murio DA. 2008. Implicit finite difference approximation for time fractional diffusion equations. Computers and Mathematics with Applications 56: 1138–1145.

Nordin CF, McQuivey RS, Mejia JM. 1972. Hurst phenomenon in turbulence. Water Resources Research 8, 1480.

Odibat ZM. 2009. Computational algorithms for computing the fractional derivatives of functions. Mathematics and Computers in Simulation 79: 2013–2020.

Oldham KB, Spanier J. 1974. The fractional calculus. Academic Press, New York.

Pipiras V, Taqqu MS. 2003. Fractional calculus and its connections to fractional Brownian motion. in Theory and Applications of Long-Range Dependence. Doukhan P, Oppenheim G, and Taqqu MS, Eds., pp. 165–201, Birkhauser, Boston, Mass, USA.

Podlubny I. 1999. Fractional Differential Equations. Academic Press, New York.

Podlubny I. 2002. Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation. Fractional Calculus and Applied Analysis 5(4): 367–386.

Raberto M, Scalas E, Mainardi F. 2002. Waiting-times and returns in high-frequency financial data: an empirical study. Physica A 314: 749–755.

Ross B. (editor). 1975. Proceedings of the International Conference on Fractional Calculus and Its Applications. University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York.

Samko S, Kilbas A, Marichev O. 1993. Fractional Integrals and derivatives: Theory and Applications. Gordon and Breach, London.

Scalas E, Gorenflo R, Mainardi F. 2000. Fractional calculus and continuous-time finance. Physica A 284: 376–384.

Stephenson DB, Pavan V, Bojariu R. 2000. Is the North Atlantic Oscillation a random walk? Int. J. Clim. 20(1): 1-18.

Szolgayová, E., Laaha, G., Blöschl, G., and Bucher, C. 2013. Factors influencing long range dependence in streamflow of European rivers, Hydrological Processes 28: 1573–1586, doi:10.1002/hyp.9694.

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Tarasov VE. (2010). Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer-Verlag, New York.

Viessman W, Knapp JW, Lewis GL, Harbaugh TE. 1977. Introduction to Hydrology, Second Edition, Harper & Row Publishers Inc.

Vogel RM, Tsai Y, Limbrunner JF. 1998. The regional persistence and variability of annual streamflow in the United States. Water Resources Research 34(12): 3445-3459.

Zaslavsky GM. 2002. Chaos, fractional kinetics, and anomalous transport. Phys. Rep. 371: 461–580.

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

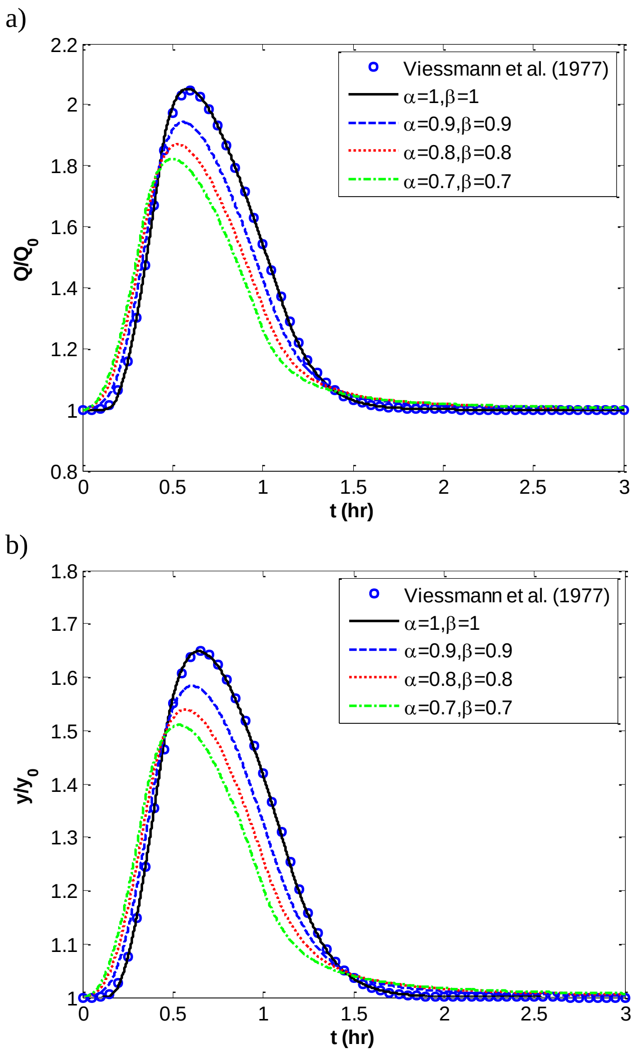
Hydrology and Earth System Sciences
Open Access
Discussions

Table 1. Magnitudes and occurrence times of the peak non-dimensional flows (Q/Q_0) at the downstream boundary for various fractional space derivative power β and fractional time derivative power α

α	β	peak Q/Q_0	time of peak Q/Q_0 (hr)
$\alpha = \beta$			
1	1	2.05	0.59
0.9	0.9	1.94	0.56
0.8	0.8	1.87	0.53
0.7	0.7	1.82	0.50

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences
Discussions
Open Access



420 **Figure 1. a)** Non-dimensional flow discharges, **b)** non-dimensional water depths through
421 time at the downstream boundary when space and time fractional derivative powers
422 $\beta = \alpha = 1$, 0.9, 0.8, 0.7

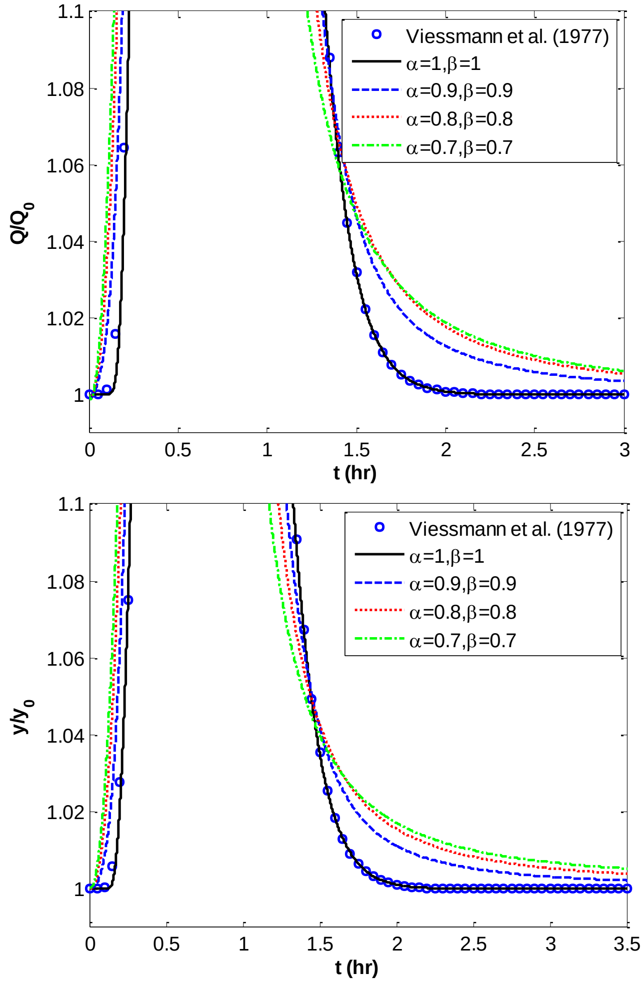


Figure 2. a) Non-dimensional flow discharges (zoomed to 0.9-1.1 range), **b)** non-dimensional water depths (zoomed over 0.9-1.1 range) through time at the downstream boundary when space and time fractional derivative powers are $\beta = \alpha = 1$, 0.9, 0.8, 0.7

Hydrol. Earth Syst. Sci. Discuss., doi:10.5194/hess-2016-364, 2016
Manuscript under review for journal Hydrol. Earth Syst. Sci.
Published: 28 July 2016

Hydrology and Earth System Sciences
Open Access
Discussions

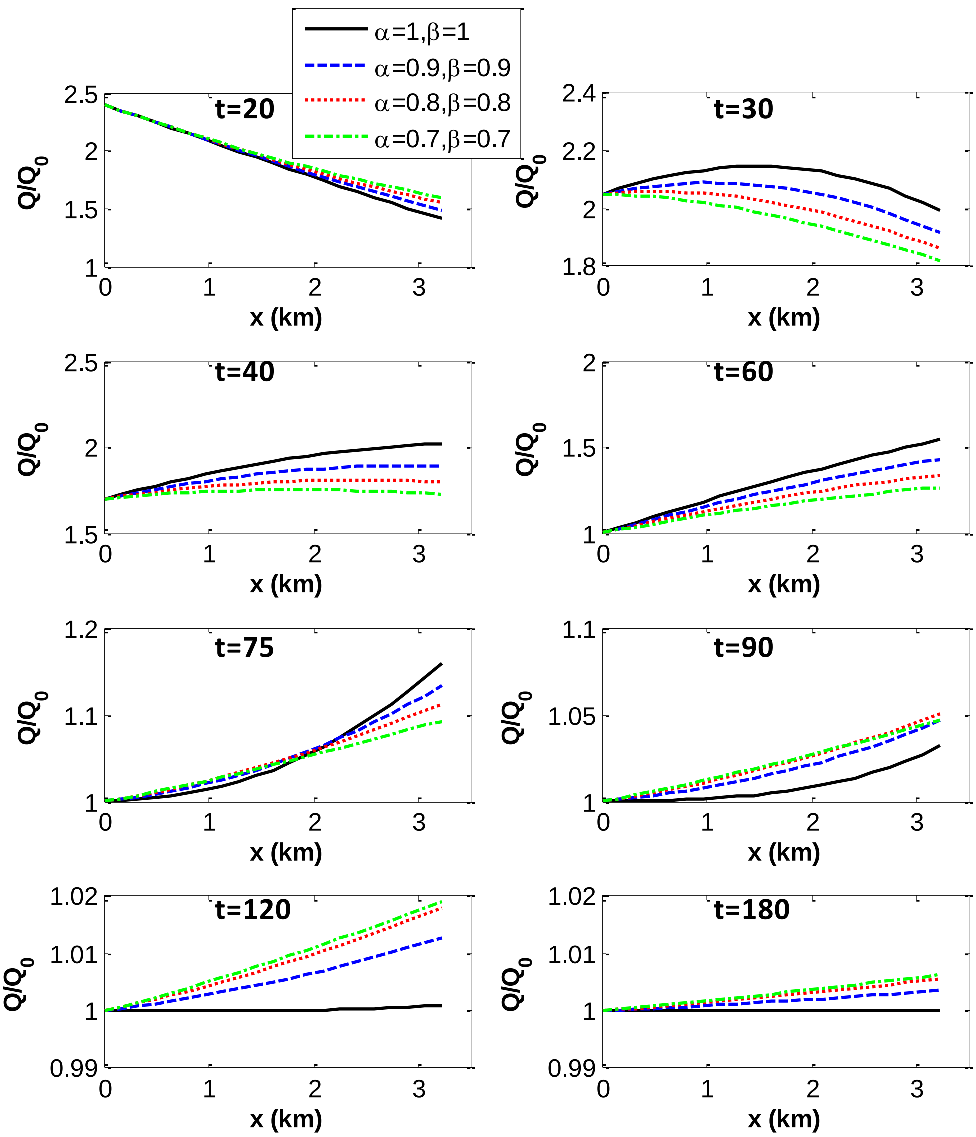


Figure 3. Non-dimensional flow discharges through longitudinal length x at simulation times t=20, 30, 40, 60, 75, 90, 120, 180 minutes when space and time fractional derivative powers are $\beta = \alpha = 1$, 0.9, 0.8, 0.7