Supplementary materials to “Matching the Turc-Budyko functions with the complementary evaporation relationship: consequences for the drying power of the air and the Priestley-Taylor coefficient”

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S.1 Properties of the function $E/P = B_1'(\Phi_0)$

Putting $y = E/P$ with $0 \leqslant y \leqslant 1$, $x = \Phi_0$ and $a = \alpha_0/(2\alpha_w)$, function B_1' given by Eq. (20) can be rewritten as

$$x = a\left[\left(y^{-\lambda}-1\right)^{-1/\lambda} + y\right]. \qquad \text{(S1.1)}$$

For $x = 0$, y is obviously equal to 0. When x tends to infinite the result is less evident. Eq. (S1.1) can be rewritten as:

$$\frac{x}{a} - y = \frac{1}{\left(\frac{1}{y^{\lambda}}-1\right)^{\frac{1}{\lambda}}}\,. \qquad \text{(S1.2)}$$

When x tends to infinite, given that y is limited by 1, the right-hand term of the equation should tend to infinite. This means that y should tend to 1 so that the denominator tends to zero.

The derivative of the function in Eq. (S1.1) is given by

$$\frac{dx}{dy} = a\left[1 + y^{-(\lambda+1)}\left(y^{-\lambda}-1\right)^{-\frac{1}{\lambda}-1}\right], \qquad \text{(S1.3)}$$

which can be rewritten as

$$\frac{dy}{dx} = \frac{1}{a}\left[1 + \left(1-y^{\lambda}\right)^{-(1+\lambda)/\lambda}\right]^{-1}. \qquad \text{(S1.4)}$$

Close to $x = 0$, y is close to zero and the derivative can be approximated by

$$\frac{dy}{dx} \approx \frac{2}{a}\left[1 - \left(\frac{1+\lambda}{2\lambda}\right) y^{\lambda}\right] \approx \frac{\alpha_w}{\alpha_0}\,. \qquad \text{(S1.5)}$$

If Eq. (22) is taken into account:

$$\frac{\alpha_w}{\alpha_0} = \frac{1+\left(1+x^{\lambda}\right)^{-\frac{1}{\lambda}}}{2}\,. \qquad \text{(S1.6)}$$

This means that α_w/α_0 and dy/dx tend to 1 when x tends to zero.

S.2 Properties of the function $E/E_0 = B_2'(\Phi_0^{-1})$

With $X = \Phi_0^{-1}$, $Y = E/E_0$ and the parameter a defined as above, function B_2' given in Eq. (21) can be written as

$$X^{-\lambda} = Y^{-\lambda} - \left(\frac{1}{a} - Y\right)^{-\lambda} . \tag{S2.1}$$

When X tends to zero, Y which is limited by 1 necessarily tends to zero, and when X tends to infinite Y tends to $1/2a = \alpha_w/\alpha_0$ which is equal to 1 according to Eq. (S1.6) ($x = 1/X = 0$).

The derivative of B_2' can be written as

$$\frac{dY}{dX} = \frac{X^{-\lambda-1}}{Y^{-\lambda-1} + \left(\frac{1}{a} - Y\right)^{-\lambda-1}} = \frac{1}{\left(\frac{X}{Y}\right)^{\lambda+1}\left[1 + \frac{1}{\left(\frac{1}{aY} - 1\right)^{\lambda+1}}\right]} . \tag{S2.2}$$

When X tends to zero, Y also tends to zero and the term into square brackets tends to 1 which means that

$$\frac{dY}{dX} \rightarrow \left(\frac{Y}{X}\right)^{\lambda+1} . \tag{S2.3}$$

Taking into account Eq. (S2.1), we have

$$\left(\frac{Y}{X}\right)^{\lambda+1} = \left[1 - \left(\frac{1}{aY} - 1\right)^{-\lambda}\right]^{\frac{\lambda+1}{\lambda}} . \tag{S2.4}$$

which tends to 1.

S.3 Transcendental forms of the basic equations $E/P = B_1(\Phi_p)$ and $E/E_p = B_2(\Phi_p^{-1})$

Eqs. (4) and (5) have the same following form

$$y = \left(1 + x^{-\lambda}\right)^{-1/\lambda} , \tag{S3.1}$$

with $x = \Phi_p$ (or $X = \Phi_p^{-1}$) and $y = E/P$ (or $Y = E/E_p$). Eq. (S3.1) can be also written as

$$x = \left(y^{-\lambda} - 1\right)^{-1/\lambda} . \tag{S3.2}$$

With similar notations, Eq. (23) can be written as

$$y + \left(y^{-\lambda} - 1\right)^{-1/\lambda} = x + \left(1 + x^{-\lambda}\right)^{-1/\lambda} . \tag{S3.3}$$

Eq. (S3.3) is equivalent to $y + x = x + y$, which means that S3.1 or S3.2 are solutions of Eq. (S3.3). A similar reasoning can be conducted with Eq. (24) which can be written

$$\left[1 - Y + \left(1 + X^{-\lambda}\right)^{-1/\lambda}\right]^{-\lambda} = Y^{-\lambda} - X^{-\lambda} . \tag{S3.4}$$

Taking into account Eq. (S3.1) and given that

$$Y^{-\lambda} = 1 + X^{-\lambda} . \tag{S3.5}$$

Eq. (S3.4) is equivalent to $1 = 1$, which means that S3.1 or S3.2 are solutions of Eq. (S3.4).

S.4 Calculations made with the Fu-Zhang equation

The Fu-Zhang equation is given by

$$\frac{E}{P} = 1 + \Phi_p - \left[1 + \left(\Phi_p\right)^{\omega}\right]^{\frac{1}{\omega}} \ . \qquad \text{(S4.1)}$$

First, we study the feasible domain of the drying power of the air and the correspondence with the evaporation rate. Inserting Eq. (S4.1) into Eq. (9) gives

$$\frac{E_a}{E_P} = D(\Phi_P^{-1}) = \left(1 + \frac{\Delta}{\gamma}\right)\left\{1 - \frac{1}{2\alpha_w}\left[2 + \Phi_p - (1 + \Phi_P^{\omega})^{\frac{1}{\omega}}\right]\right\}. \qquad \text{(S4.2)}$$

The limits given in Eq. (11), (12) and (13) are independent from the TB equation used, and consequently $D*$ remains unchanged

$$D^* = \frac{1}{2\alpha_w}\left(1 + \frac{\Delta}{\gamma}\right) . \qquad \text{(S4.3)}$$

Using a similar reasoning as in Eqs (14), (15), (16) and (17), we obtain

$$d^* = 2^{\frac{1}{\omega}} - 1 \ , \qquad \text{(S4.4)}$$

$$\omega = \frac{ln2}{\ln(d^* - 1)} \ , \qquad \text{(S4.5)}$$

$$\delta^* = \left(1 + \frac{\Delta}{\gamma}\right)\frac{1}{2\alpha_w}\left(2^{\frac{1}{\omega}} - 1\right) = D^* d^* \ . \qquad \text{(S4.6)}$$

Second, we link the Priestley-Taylor coefficient to the Fu-Zhang shape parameter ω. Substituting E_P in Eq. (S4.1) by its value given by Eq. (18) and putting $\Phi_0 = E_0/P$ gives

$$\frac{E}{P} = 1 + \frac{2\alpha_w}{\alpha_0}\Phi_0 - \frac{E}{P} - \left[1 + \left(\frac{2\alpha_w}{\alpha_0}\Phi_0 - \frac{E}{P}\right)^{\omega}\right]^{\frac{1}{\omega}} . \qquad \text{(S4.7)}$$

Eq. (S4.7) can be rewritten as

$$\left(1 + \frac{2\alpha_w}{\alpha_0}\Phi_0 - 2\frac{E}{P}\right)^{\omega} = 1 + \left(\frac{2\alpha_w}{\alpha_0}\Phi_0 - \frac{E}{P}\right)^{\omega} \ . \qquad \text{(S4.8)}$$

An equation similar to Eq. (21) can be obtained expressing E/E_0 as a function of $\Phi_0^{-1} = P/E_0$

$$\left[1 + \frac{2}{\Phi_0^{-1}}\left(\frac{\alpha_w}{\alpha_0} - \frac{E}{E_0}\right)\right]^{\omega} = 1 + \left(\frac{1}{\Phi_0^{-1}}\right)^{\omega}\left(\frac{2\alpha_w}{\alpha_0} - \frac{E}{E_0}\right)^{\omega} . \qquad \text{(S4.9)}$$

Eqs. (S4.8) and (S4.9) obtained from the Fu-Zhang equation correspond respectively to $E/P = B_1'(\Phi_0)$ in Eq. (20) and $E/E_0 = B_2'(\Phi_0^{-1})$ in Eq. (21) obtained with the Turc-Mezentsev equation.

Using a similar reasoning as in Eq. (22), the expression of α_0 can be inferred by matching Eq. (S4.8) and Eq. (S4.1): for a given value of the aridity index Φ, we have the same value of E/P. This leads to

$$\left[-1 + 2\left(\frac{\alpha_w}{\alpha_0} - 1\right)\Phi + 2(1 + \Phi^{\omega})^{\frac{1}{\omega}}\right]^{\omega} = 1 + \left[-1 + \left(\frac{2\alpha_w}{\alpha_0} - 1\right)\Phi + (1 + \Phi^{\omega})^{\frac{1}{\omega}}\right]^{\omega} . \qquad \text{(S4.10)}$$

Eq. (S4.10) is equivalent to Eq. (22), but with a transcendental form. It can be resolved numerically and Fig. (S1) shows the variation of the Priestley-Taylor coefficient α_0 as a function of the aridity index Φ for different values of the ω parameter. The shape of the curves is very similar to those of Fig. (5a) obtained with the parameter λ of the Turc-Mezentsev function.

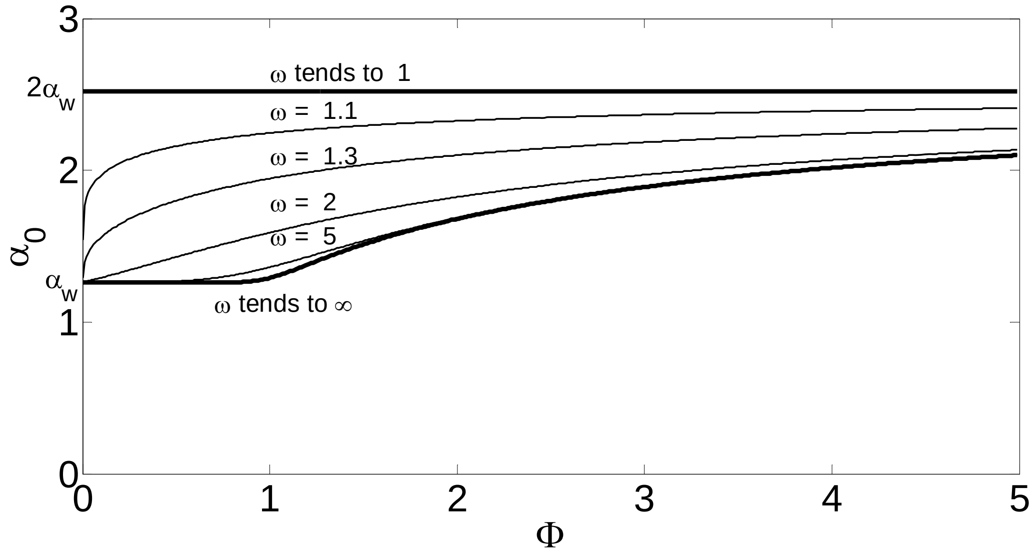

Figure S1: Variation of the Priestley-Taylor coefficient α_0 as a function of the aridity index Φ for different values of the shape parameter ω of the Fu-Zhang function. The bold lines indicate the upper and lower limits of the feasible domain.