Short summary of the hydraulic equations used in the manuscript under discussion

The phase lag equation (Eq. (6) in the manuscript under discussion), the scaling equation (Eq. (7)), the damping equation (Eq.(8)) and the celerity equation (Eq.(9)). Have been developed in different stages by Savenije (1992, 1998, 2001) and by Savenije and Veling (2005). They are summarised in Savenije (2005). These equations can be written in dimensionless form and subsequently they can be solved, as demonstrated by Savenije et al. (2008). In dimensionless form these equations are presented in the manuscript as Eqs. (17), (18) and (19), representing respectively: the damping equation, the celerity equation and the phase lag equation. The scaling equation then follows as:

$$\mu = \frac{\sin(\varepsilon)}{\lambda} = \frac{1}{\sqrt{\delta^2 + \lambda^2}}$$

The solution of this system is given in the Table below. It shows the dimensionless variables and the four equations in the left column. The other two columns present classical solutions by Battjes (well known in The Netherlands, but not in the formal literature) and the commonly known solution by Harleman and Ippen (1966). What we can see is that the equations for the velocity number, tidal propagation and phase lag are exactly the same. So in that sense the methods don't differ. The difference lies in two aspects. The first is the assumption of exponential damping, and the other is the linearization of the damping term in the momentum balance equation. By the way, these two aspects are one and the same thing, since the fully linearised momentum balance equation leads to exponential damping.

The assumption of exponential damping is shown in the top of the right column. This assumption is made by most classical methods, but it is seldom seen as an assumption. It is assumed to be correct. However, it is an implicit assumption that is not used in the left column. Related to the linearization of the friction term, most methods linearize by the Lorentz linearization. Savenije (1998), however used an envelope method that does not require the linearization of the friction term and results in a different damping equation, which is not exponential. In dimensionless form this damping equation is presented in the third row of the first column and it can be seen to be different from the classical equations which contain the Lorentz factor.

There is an additional benefit of this new damping equation (that does not assume exponential damping and does not use a linearized friction term), which is that it allows fully explicit solution of the set of equations. The second row in the first column presents the explicit solution for the velocity number, after which all equations have an explicit solution. In the manuscript under discussion this solution is also presented. A full description of this solution is given in Savenije et al. (2008).

In the framework of this discussion it is not useful to repeat the full mathematical handling of this solution, but as the editor indicated, some more background about what is really different in the method used is needed. That's why here below, we summarize the derivation of the damping equation which is the main and essential difference with other methods used in the classical literature. It uses a Lagrangean transform and subsequently determines the tidal damping by subtraction of the envelopes of maximum high and low water. The subtraction of these extreme water levels yields the differential equation for the tidal range, which is the damping equation. We start with the Lagrangean form of the water balance equation (based on Savenije, 1992) and subsequently we present the envelope method presented in Savenije (1998). The entire method is fully worked out in the book "Salinity and Tides in Alluvial Estuaries" by

Savenije (2005). The equation numbers below are related to this publication, which is presently being revised for a fully revised and updated edition.

Savenije & Toffolon	Battjes	Harleman & Ippen
$\zeta = \frac{\eta}{\overline{h}}$ $\mu = \frac{\upsilon}{U_0} = \frac{1}{r_S}\frac{\upsilon}{\zeta c_0} = \frac{1}{r_S}\frac{\upsilon \overline{h}}{\eta c_0}$ $\lambda = \frac{L_0}{L} = \frac{c_0}{c}$ $\chi = r_S f \frac{c_0}{\omega \overline{h}} \zeta$ $\delta = \frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{c_0}{\omega}$	$\mu\chi = f\frac{\upsilon}{\omega \overline{h}}$ $\tan 2\alpha = \frac{8}{3\pi}\mu\chi$ $\lambda = \frac{1}{\sqrt{1-\tan^2\alpha}}$ $\delta = -\frac{\tan\alpha}{\sqrt{1-\tan^2\alpha}}$	$\eta = \eta_0 \exp\left(\frac{\delta\omega}{c_0}\right)\cos\left(\omega t - \frac{\omega\lambda}{c_0}x\right)$ $v = \upsilon_0 \exp\left(\frac{\delta\omega}{c_0}x\right)\cos\left(\omega t - \frac{\omega\lambda}{c_0}x + \alpha\right)$ $\upsilon_0 = \frac{\eta_0 c_0}{h}\frac{1}{\sqrt{\delta^2+\lambda^2}}$
velocity number $\mu = \frac{1}{\sqrt{\delta^2+\lambda^2}}$ $\mu = \sqrt{\frac{m_0^2-6}{3m_0\chi}}$ $m_0 = 3\left(\chi + \sqrt{\chi^2 + \frac{8}{27}}\right)^{1/3}$	required input (given): $\upsilon, \overline{h}$ then follow: $\mu\chi, \alpha, \lambda, \delta$	conversion: $\tan\varepsilon = 1/\tan\alpha$ $\delta = -\frac{1}{\sqrt{\tan^2\varepsilon - 1}}$ $\tan\alpha = -\frac{\delta}{\lambda} = -\frac{\delta}{\sqrt{1+\delta^2}}$
damping number $\delta = -\mu^2\frac{\chi}{2}$	damping number $\delta = -\frac{8}{3\pi}\frac{\mu}{\lambda}\frac{\chi}{2}$	damping number $\delta = -\frac{8}{3\pi}\frac{\mu}{\lambda}\frac{\chi}{2} = -\frac{8}{3\pi}\frac{\mu^2}{\sin\varepsilon}\frac{\chi}{2}$
propagation number $\lambda^2 = 1 + \left(\frac{\chi\mu^2}{2}\right)^2 = 1+\delta^2$	propagation number $\lambda^2 = 1+\delta^2$	propagation number $\lambda^2 = 1+\delta^2$
phase lag $\tan\varepsilon = -\frac{\lambda}{\delta} = -\frac{\sqrt{1+\delta^2}}{\delta}$	phase lag $\tan\varepsilon = -\frac{\lambda}{\delta} = -\frac{\sqrt{1+\delta^2}}{\delta}$	phase lag $\tan\varepsilon = -\frac{\lambda}{\delta} = -\frac{\sqrt{1+\delta^2}}{\delta}$

Lagrangean form of the water balance equation

The tidal flow velocity U can be expressed both in Eulerian and Lagrangean terms. The connection between the two is given by Eqs. (2.76) and (2.75):

$$U = \upsilon \sin(\xi) \tag{2.76}$$

$$\xi = \omega t - \frac{\omega(x - x_0 - S)}{c} \tag{2.75}$$

where $U=U(x,t)$ is the Eulerian velocity of flow at a certain location at a certain time and c is the tidal wave celerity. If we move with the water particle, then $x = x_0 + S$, $\xi = \omega t$ and $U = V$, where V is the Lagrangean velocity of the moving particle. Using (2.75) and (2.76), the Eulerian continuity equation for one-dimensional flow, can be transformed to the Lagrangean reference frame.

The water balance equation (2.27) can be combined with the exponential variation of the cross-section to yield equation (2.77)

$$r_S \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} + \frac{hU}{B} \frac{\partial B}{\partial x} = 0 \tag{2.27}$$

$$B = B_0 \exp\left(- \frac{x}{b} \right) \tag{2.39}$$

$$r_S \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} - \frac{hU}{b} = 0 \tag{2.77}$$

Partial differentiation of (2.76) with respect to x for a moving particle where $x=x_0+S$, and combination of the result with (2.72) and (2.75), yields a Lagrangean expression for the partial derivative of U with respect to x:

$$\frac{\partial U}{\partial x} = -\frac{1}{c} \frac{\mathrm{d}V}{\mathrm{d}t} + \delta_U V \tag{2.78}$$

Moreover, the variation of the water depth with time for a moving water particle is defined by:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x} \tag{2.79}$$

Substitution of (2.78) and (2.79) into (2.77) yields the continuity equation for a moving volume of water ($U=V$) in a Lagrangean reference frame. We can further assume that: 1) r_S is close to unity; and 2) the Froude number is small. As a result, the second term of (2.79) is much smaller than the first. Therefore the introduction of the storage width ratio in the second term of the water balance equation creates only a third order error. Hence:

$$r_S \frac{\mathrm{d}h}{\mathrm{d}t} = +\frac{h}{c} \frac{\mathrm{d}V}{\mathrm{d}t} + hV \frac{(1 - \delta_U b)}{b} \tag{2.80}$$

We can rearrange (2.80) as follows:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = r_S \frac{c}{h} \frac{\mathrm{d}h}{\mathrm{d}t} - \frac{cV}{b} + \frac{cV}{\upsilon} \frac{\mathrm{d}\upsilon}{\mathrm{d}x} \tag{3.1}$$

We shall now make use of (2.94), stating that the damping of the velocity amplitude is almost equal to the damping of the tidal range (including an error term Δ). This error term is zero when the phase lag ε and the damping/amplification δ_U are constant (implying exponential damping or no damping). This assumption is valid in long estuaries that gradually transform into a river. In short and closed estuaries this assumption may not be correct, but we'll see further on in Section 3.4 that in short estuaries we may use a simple linearised equation that performs well under those conditions. Making use of this assumption, (3.1) becomes:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = r_S \frac{c}{h} \frac{\mathrm{d}h}{\mathrm{d}t} - \frac{cV}{b} + cV\left(\frac{1}{H} \frac{\mathrm{d}H}{\mathrm{d}x} - \Delta \right) \tag{3.2}$$

Lagrangean form of the momentum balance equation

Next we combine (3.2) with the Lagrangean momentum balance equation. Written in a Lagrangean reference frame it reads:

$$\frac{\mathrm{d}V}{\mathrm{d}t}+g\frac{\partial h}{\partial x}+g(I_b-I_r)+g\frac{V|V|}{C^2h}=0 \tag{3.3}$$

Combination of (3.2) and (3.3), and making use of the Lagrangean relationship $V=\mathrm{d}v/\mathrm{d}t$, yields:

$$r_S\frac{cV}{gh}\frac{\mathrm{d}h}{\mathrm{d}x}-\frac{cV}{g}\left(\frac{1}{b}-\frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}x}+\Delta\right)+\frac{\partial h}{\partial x}+I_b-I_r+\frac{V|V|}{C^2h}=0 \tag{3.4}$$

To derive an explicit relation for the tidal damping, we shall condition this differential equation for the occurrence of HW and LW. We shall then obtain two differential equations describing the envelopes of the water levels at HW and LW. At HW and LW the special condition applies that $\partial h/\partial t=0$, and hence:

$$\left.\frac{\mathrm{d}h}{\mathrm{d}x}\right|_{HW,LW}=\left.\frac{\partial h}{\partial x}\right|_{HW,LW} \tag{3.5}$$

Using this relation we can write (3.4) completely in Lagrangean derivatives for the conditions of HW and LW. Moreover, since the tidal range H is the difference between h_{HW} and h_{LW}, the Lagrangean gradient of the tidal range is defined by:

$$\frac{\mathrm{d}h_{HW}}{\mathrm{d}x}-\frac{\mathrm{d}h_{LW}}{\mathrm{d}x}=\frac{\mathrm{d}H}{\mathrm{d}x} \tag{3.6}$$

And similarly because the sum of the two depth is twice the average depth (for a symmetrical wave), which we may assume to be correct if the tidal amplitude to depth ratio is small:

$$\frac{\mathrm{d}h_{HW}}{\mathrm{d}x}+\frac{\mathrm{d}h_{LW}}{\mathrm{d}x}\approx 2\frac{\mathrm{d}\bar{h}}{\mathrm{d}x}=2\,I \tag{3.7}$$

where I is the residual water level slope. Finally the following conditions apply for HW and LW if the tidal amplitude to depth ratio is not too large:

$$h_{HW}\approx\bar{h}+\eta \tag{3.8}$$

$$h_{LW}\approx\bar{h}-\eta \tag{3.9}$$

where $\eta=H/2$. Moreover, if the velocity has a sinus shape:

$$V_{HW}=\upsilon\sin\varepsilon \tag{3.10}$$

$$V_{LW}=-\upsilon\sin\varepsilon \tag{3.11}$$

The envelopes for HW and LW lead to the damping equation

Combination of (3.4), (3.5), (3.8) and (3.10) yields for the condition of HW:

$$\frac{r_S c\upsilon\sin\varepsilon}{g(\bar{h}+\eta)}\frac{\mathrm{d}h_{HW}}{\mathrm{d}x}-\frac{c\upsilon\sin\varepsilon}{g}\left(\frac{1}{b}-\frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}+\Delta\right)+\frac{\mathrm{d}h_{HW}}{\mathrm{d}x}+\frac{(\upsilon\sin\varepsilon)^2}{C^2(\bar{h}+\eta)}=-I_b+I_r \tag{3.12}$$

This is the differential equation that describes the upper envelope of all water levels in the estuary, because no water level can rise above the point of HW. Similarly for the condition of LW we find the envelope for LW, which is the lower boundary of all the water levels in the estuary:

$$\frac{-r_S c\upsilon \sin\varepsilon}{g(\bar{h}-\eta)}\frac{\mathrm{d}h_{LW}}{\mathrm{d}x}+\frac{c\upsilon\sin\varepsilon}{g}\left(\frac{1}{b}-\frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}+\Delta\right)+\frac{\mathrm{d}h_{LW}}{\mathrm{d}x}-\frac{(\upsilon\sin\varepsilon)^2}{C^2(\bar{h}-\eta)}=-I_b+I_r \tag{3.13}$$

Subtraction of these two envelopes yields:

$$\frac{r_S c\upsilon\sin\varepsilon}{2\bar{h}}\left(\frac{\mathrm{d}h_{HW}}{\mathrm{d}x}\frac{\bar{h}}{(\bar{h}+\eta)}+\frac{\mathrm{d}h_{LW}}{\mathrm{d}x}\frac{\bar{h}}{(\bar{h}-\eta)}\right)-\frac{c\upsilon\sin\varepsilon}{\bar{h}}\left(\frac{\bar{h}}{b}-\frac{\bar{h}}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}+\Delta\right)+g\frac{\mathrm{d}\eta}{\mathrm{d}x}+f'\frac{(\upsilon\sin\varepsilon)^2}{\bar{h}}=0 \tag{3.14}$$

with:

$$f'=\frac{g}{C^2}\left(1-\left(\frac{\eta}{\bar{h}}\right)^2\right)^{-1}=f\left(1-\left(\frac{\eta}{\bar{h}}\right)^2\right)^{-1} \tag{3.15}$$

where f' is the adjusted friction factor taking account of the friction being larger at LW than at HW. One could also determine this friction factor on the basis of Strickler's formula. It would then read:

$$f'=\frac{g}{K^2\bar{h}^{1/3}}\left(1-\left(\frac{1.33\eta}{\bar{h}}\right)^2\right)^{-1} \tag{3.15a}$$

The coefficient 1.33 in this equation follows from a Taylor series expansion of $(h+\eta)^{1.33}\approx h^{1.33}(1+1.33\ \eta/h)$, if $\eta<h$. Due to the factor 1.33, this equation only makes sense as long as $\eta/h<0.7$ and may only be applied for smaller amplitude to depth ratios. We can see that if the tidal amplitude to depth ratio is small $f'\approx f=g/C^2$.

The part between brackets in the first term of (3.14) can be replaced by the residual water level slope I defined in (3.7), provided $\eta/h<1$. Elaboration yields:

$$\frac{\bar{h}}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}\left(1+\frac{g\eta}{c\upsilon\sin\varepsilon}\right)=\frac{\bar{h}}{b}-f'\frac{\upsilon\sin\varepsilon}{c}-r_S I+\bar{h}\Delta \tag{3.16}$$

Now let us consider the order of magnitude of the terms containing the residual slope and $\bar{h}\Delta$ and compare them to h/b. First the residual slope. Over the distance L of the tidal influence the bottom slope is negligible, hence $I<<h/L$. Moreover in all estuaries b is several times smaller than L. As a result, I is at least an order of magnitude smaller than h/b during periods of low river discharge. Therefore, the residual slope term can be disregarded in most practical cases. However, as the residual slope gains prominence as we move further up the estuary, where river discharge becomes more dominant and where there is more bottom slope, it causes additional damping and the tide gradually dies out. Further research into the relative importance of I versus h/b may be needed (particularly in the upstream part of an estuary), which is complicated by the fact that I is difficult to observe accurately. Regarding the term $\bar{h}\Delta$. It is zero in a near ideal estuary where: a) there is no bottom slope, b) the tide is modestly damped/amplified or δ_U is constant, and c) the phase lag is constant. In long coastal plain estuaries this is generally an acceptable assumption. If there is amplification or damping in a coastal plain estuary, then this is generally modest. In that case the term $\bar{h}\Delta$ is non-zero, but since the gradient of the tidal velocity amplitude small compared to the convergence length ($\delta_U b<0.1$), $\bar{h}\Delta$ is still much smaller than h/b. In short (amplified) estuaries, there may be a bottom gradient, a gradient in the phase lag (gradually moving towards a standing wave) and a gradient in the tidal velocity amplitude (gradually reducing to zero). So in short estuaries the $\bar{h}\Delta$ term may become prominent and may need to be accounted for. In coastal plain estuaries, however, particularly in the downstream part, this term may be disregarded.

Hence, the analytical solution of the St. Venant's equations yields the **damping equation**:

$$\frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x}\left(1+\frac{g\eta}{c\upsilon\sin\varepsilon}\right)=\frac{1}{b}-f'\frac{\upsilon\sin\varepsilon}{\bar{h}c} \qquad (3.17)$$

This is a differential equation describing the damping of the tidal amplitude as a function of the estuary shape, the friction and the residual slope. The subtraction of the two envelopes for HW and LW resulted in a differential equation that describes the tidal range. It will prove to be a very useful equation to elaborate further. In the following text, for convenience sake, we shall drop the over-bar for the average depth and *h* will stand for the tidal average depth.

It is interesting to check the origin of the terms in this equation. The first term on the right hand side obviously comes from the convergence term in the continuity equation. The second term stems from the friction term in the momentum balance equation. On the left hand side, it is less obvious. The 1 stems from the last term in (3.1) and is the term that determines the effect of tidal damping on the mass balance equation; the second term between brackets stems from the depth gradient in the momentum balance equation (clearly an important term). Scaling (see below) shows that this term is indeed larger than 1.

This equation is a general version of Green's law, a rule of thumb often quoted. Green (1837) assumed that the amount of energy in a progressive tidal wave ($E=0.5\rho g\eta^2 BcT$) would remain constant under frictionless flow as it travels up a converging estuary. If we use the classical equation for wave propagation ($c^2=gh$), this leads to the tidal range being inversely proportional to the square root of the width and the 0.25th power of the depth. In an ideal estuary with constant depth, it implies that $\delta_H=1/(2b)$. We shall see further on in equation (3.64) that zero friction ($f'=0$) indeed leads to Green's law. So Green's law is a special case of (3.17), and (3.17) is a general version of Green's law.

From (3.17), it can be seen that in an ideal estuary where there is no tidal damping or amplification:

$$\frac{1}{b}=f'\frac{\upsilon\sin\varepsilon}{\bar{h}c}=\frac{R'}{c} \qquad (3.18)$$

This is a similar result as in (2.57), which was the condition for an ideal estuary to occur. The resistance term R'/c is also presented in Table 2.2. We can indeed verify the earlier remark that there is tidal amplification if $1/b>R'/c$, see Figure 3.1. We can also verify that since $b\,f$, h and υ are constant along the estuary, the wave celerity and $\sin(\varepsilon)$ are proportional. Since $\sin(\varepsilon)$ indicates the type of tidal wave (it equals zero for a standing wave and 1 for a progressive wave) it is called the Wave-type number N_E (Savenije, 1998). In alluvial estuaries, where the tidal wave is of a mixed character, the Wave-type number is between 0 and 1. Since it has been observed that the phase lag is constant along an estuary, the wave celerity also is, which can indeed be observed in estuaries, at least for considerable stretches where convergence and depth are constant.

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