Referee Evaluation/Discussion of the manuscript *Extended power-law scaling of air permeabilities measured on a block of tuff*, by Siena, Guadagnini, Riva and Neuman

By

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General comments: The authors work with the data set considered previously by Tidwell and Wilson (1999), and look at its scaling properties in relation to self-similarity (SS) as well as extended self-similarity (ESS) and its generalized version (GESS). More specifically, the data are the increments of the underlying process Y at lag t,

$$X(t) = \Delta Y(t) = Y(s+t) - Y(s), \tag{1}$$

and the scaling properties relate to the structure function of order q, defined as the expectation of the qth absolute moment of the increment,

$$S_q(t) = E(|X(t)|^q) = < |X(t)|^q >. \tag{2}$$

This study is mostly empirical, as the scaling properties in the data are explored through standard graphical utilities used earlier by these and other authors in similar applications. In this process one works with sample structure functions defined as sample means,

$$S_{q,N}(t) \;=\; \frac{1}{N(t)} \sum_{n=1}^{N(t)} |\Delta Y_n(t)|^q, \tag{3}$$

where *N(t)* is the number of measured increments at lag *t*. Additionally, the authors present a newly developed model called truncated fractional Brownian motion (TFBM) that is shown in the appendix to posses certain scaling features observed in the data. The general findings are that the data do not seem to follow SS, although its ESS is quite apparent.

A lot of discussion is presented in the present manuscript, as well as in most of the other references on ESS provided by the authors, about the fact that ESS may be quite apparent while SS itself is not clearly seen in the data. Here, this point is illustrated by Figures 4 and 6 - where the curves in Figure 4 are hardly straight lines, even over limited ranges of lags t, while those in Figure 6 seem to follow the straight line pattern rather well. There is a simple explanation for this, which does not seem to have been discussed by the present or previous authors. Roughly speaking, the SS property (formula (2) in the text) always implies ESS (formula (3) in the text), but not the other way around. The ESS is more general, and can hold without SS. A natural question that arises in this connection, is what the ESS is actually equivalent to? In other words, is there a scaling property that can be stated similarly to (2), which is equivalent to (3)? Such a question is important, because without an answer one does not understand clearly what satisfying the ESS criteria imply concerning the underlying process. More elaboration is presented below.

Specific comments: Let us start with the property of SS. This is often discussed in a broader context of self-similar processes. Recall that a process Y is said to be self-similar with index H (in short, H-SS) if for any $c > 0$ the finite-dimensional distributions of $Y(ct)$ are the same as those of the process $c^H Y(t)$. This implies a power-law form of the structure function,

$$S_q(t) = E|Y(t)|^q = E|t^H Y(1)|^q = E|Y(1)|^q t^{Hq}, \tag{4}$$

so that

$$S_q(t) \propto t^{Hq} \tag{5}$$

with the constant of proportionality being $C = C(q) = E|Y(1)|^q$. The emphasis here is how a given structure function of order q scales with the lag t. If the empirical structure function is plotted against t on a log-log scale, the graph should be a straight line with a slope of Hq.

The above SS property of the structure function is a special case of the following, more general, SS property, usually connected with the definition of multi-fractals.

$$S_q(t) \propto t^{\xi(q)}, \tag{6}$$

where $\xi(q)$ may be a non-linear function of q. It is easy to see, that if such an SS property of the structure function holds, then so does the ESS discussed in the present manuscript and previous papers. Indeed, if the above holds for any positive q, then for any positive m and n we have

$$S_m(t) = C(m)t^{\xi(m)} \qquad \text{and} \qquad S_n(t) = C(n)t^{\xi(n)}. \tag{7}$$

Upon solving the first equation for t and substituting into the second equation, we come to the conclusion

$$S_n(t) \propto S_m(t)^{\xi(n)/\xi(m)}. \tag{8}$$

Consequently, SS of the structure function implies ESS of the structure function. [As an aside, let us note that the quantity $\xi(n)$ that appears in the scaling relation (8) is defined only up to a scale factor; that is, if (8) holds with some function $\xi(n)$ then it also does with the function $c\xi(n)$ for any constant c. Thus, the meaning of this function, as well as its estimation, is a bit ambiguous.]

Now, let us suppose that the structure function scales as follows:

$$S_q(t) \propto [u(t)]^{\xi(q)}, \tag{9}$$

where $u(t)$ is some, possibly non-linear function of t. It is easy to see, that this property of the structure function also leads to the ESS property (8). Indeed, proceeding as above, for any positive m and n we have

$$S_m(t) = C(m)[u(t)]^{\xi(m)} \quad \text{and} \quad S_n(t) = C(n)[u(t)]^{\xi(n)}, \tag{10}$$

and solving the first equation for $u(t)$ and substituting into the second equation, leads again to relation (8). This shows that EES as given by (8) does not necessarily imply SS given by (6), so that for some data, including those studied in the manuscript under review, one can observed (8) while (6) will not show up on the log-log plot of the structure function versus lag t. [Again we note that neither $\xi(n)$ nor $u(t)$ in (9) are uniquely defined, since for any $a > 0$, the relation (9) holds for $\tilde{u}(t) = [u(t)]^a$ and $\tilde{\xi}(q) = \xi(q)/a$ if and only if it holds for $u(t)$ and $\xi(q)$. This again makes the meaning of these functions, as well as their estimation, a bit ambiguous.]

Now, coming back to the question of what the ESS (8) represents, we come to the conclusion that it is actually equivalent to (9). To see this, all we need to do is set $m = 1$ in (8), leading to

$$S_n(t) \propto [(S_1(t))^{1/\xi(1)}]^{\xi(n)}, \tag{11}$$

or, equivalently, to

$$S_n(t) \propto [S_1(t)]^{\xi(n)/\xi(1)}. \tag{12}$$

In either case, we can identify these with (9), where the scaling function $u(t)$ has to do with the first absolute moment of the process, $S_1(t) = E|Y(t)|$. Again, we have the uniqueness issue. Perhaps the most logical strategy is to follow (12), and associate the function $u(t)$ of (9) directly with $S_1(t) = E|Y(t)|$. In this case, the exponent of (9) will become $\xi(n)/\xi(1)$, so that the value of the exponent is 1 when $n = 1$. In conclusion, we can say that the ESS relation (8) is equivalent to (9), where in (9) the exponent is 1 when the input n is one, and $u(t) = E|Y(t)|$. This relation can then be taken as an alternative definition of ESS, where now the exponent function $\xi(n)$ is more meaningful. [It appears that Benzi et al. (1996, Eq. 5) hinted at these ideas, but did not elaborate.]

According to the above, the ESS relation (8) should not be considered in parallel with the SS as given by (6), particularly in connection with a further assumption that the exponent functions $\xi(n)$ of (6) and (8) are the same. Moreover, since the exponent function $\xi(n)$ of (8) is not uniquely defined, there is no way to estimate it from data. However, the authors of the present manuscript do not seem clear on this point. In their analysis of 1-face data in sections 3.1-3.2, first, in section 3.1, they are trying to fit the exponent function by plotting (sample) structure functions against the lag on a log-log scale, as if the SS relation (6) was OK. This leads to Figures 4 and 5. Then, in Section 3.2, they proceed with plotting two structure functions of orders q and q-1 against each other across different lag sizes on log-log scales, to estimate the exponent ratios $\xi(q)/\xi(q-1)$ from the slopes of the resulting straight lines, shown in Figure 6. This procedure is justified by the ESS property (8) – but as we just explained – is incompatible with the standard SS property (6) assumed to hold in Section 3.1. The authors then use an estimate of $\xi(1)$ obtained is Section 3.1 in conjunction of the ratios of this functions estimated in Section 3.2, to finally get the actual values of this function, $\xi(q)$ for a range of inputs q, which is subsequently plotted in Figure 8. Again, this procedure makes little sense, since, as explained above, this function is only defined up to a scale factor.

In connection with this analysis the authors also observe on page 7813 that the ratios $\xi(q)/\xi(q-1)$ decrease asymptotically with q towards 1 as q increases, connecting this with the pattern observed in Figure 5. In our opinion this is simply a reflection of the asymptotic behavior of the function $\xi(q)$ at infinity. For example, if this is a power function, $\xi(q) = q^{\eta}$, that ratio monotonically decreases and converges to 1:

$$\frac{\xi(q)}{\xi(q-1)} = \frac{q^{\eta}}{(q-1)^{\eta}} = (1+\frac{1}{q-1})^{\eta} \rightarrow 1 \quad \text{as} \quad q \rightarrow \infty. \tag{13}$$

In light what we stated above in connection with the equivalence of (8) and (9) the estimation should involve both functions $\xi(q)$ as well as $u(t)$, assuming that the former has the value of 1 when q = 1, and the latter is the first absolute moment of the process at lag t. Here, one might proceed as the authors in Section 3.2 to estimate the ratios of the first function, which, together with the assumption $\xi(1) = 1$, should produce all other values. On the other hand, the function $u(t)$ may be estimated from data by plotting the empirical structure function of order 1 against a range of lags.

A logical question arises as to what kind of stochastic models have this ESS structure function. One way to obtain them is to allow a time deformation in connection with a self-similar process. Namely, consider the process $Y(v(t))$, where $v(t)$ is a non-negative and deterministic time deformation, while $Y(t)$ is a H-SS process. Then, since the distribution of $Y(v(t))$ is the same as that of $[v(t)]^{H} Y(1)$, it follows that the structure function of $Y(v(t))$ is

$$S_q(t) = [v(t)]^{qH} E[|Y(1)|^{q}] \propto [v(t)]^{qH}. \tag{14}$$

We thus have a structure function of the form (9) with $\mathrm{u(t)} = [v(t)]^{H} E|Y(1)|$ and $\xi(q) = q$. The problem with this approach is that the so defined process may not have stationary increments, even though $Y(t)$ has this property.

The authors mention in their paper a stochastic process, called truncated fractional Brownian motion (TFBM), as an example of the ESS property. The structure function of this process, as presented in (A2) of Appendix A, is indeed of the form (9), so that this model seems to have the ESS property. This structure function appears consistent with the assumption that this process has stationary increments – but this is not clearly stated in the paper.

It should be mentioned in connection with this paper, that the notion of extended self-similarity has also been used in the reference Kaplan and Kao (1994), where it was defined differently than here and elsewhere in the turbulence physics literature (Benzi et al., 1993ab, 1996). Namely, this definition was used in connection with a Gaussian process, and states that a zero-mean Gaussian process $Y(t)$ with stationary increments is ESS if the variance of the lag-t increments is of the form

$$Var[Y(t)] = E[Y(t)^2] = f(t)\sigma^2, \tag{15}$$

where σ^2 is the variance of the lag-1 increment $Y(1)$. It follows that the covariance function of such a process is

$$E[Y(t)Y(s)] = E\left\{\frac{1}{2}\left([Y(t)]^2 + [Y(s)]^2 - \left(Y(t) - Y(s)\right)^2\right)\right\} = \frac{\sigma^2}{2}[f(t) + f(s) - f(t-s)]. \tag{16}$$

Citing the reference Benzi et al. (1993b), the authors of that paper state that "the term 'extended self-similarity' has been used in the area of turbulence physics to describe the hyperbolic relation between two different moments of the velocity increments for a given increment length" and that their use of extended self-similarity, defined by (15) above, "described a totally different situation". Clearly, in view of the above facts related to the concept of ESS as defined by Benzi et al., one can not agree with this statement. Indeed, since the marginal distributions of this process are Gaussian, in view of (15), it follows that Y(t) has the same distribution as $\sqrt{f(t)}\sigma Z$, where Z is standard normal. Consequently, the structure function of $Y(t)$ is of the form

$$E[|Y(t)|^q] = \sigma^q[f(t)]^{q/2}E[Z^q] \propto [f(t)]^{q/2}, \tag{17}$$

showing that this in fact is the same concept of ESS as that discussed in the paper under review, restricted to the case of Gaussian processes.

In relation to the existence of processes displaying the ESS property and stationary increments, let us note that perhaps a large class of such processes could be defined via subordination. First, observe that whenever $Y(t)$ is an H-SS process with stationary increments and $G(t)$ is a positive, non-decreasing process independent of $Y(t)$, whose increments are also stationary, then the process $Y(t)$ subordinated to $G(t)$ has stationary increments as well. Indeed, if $W(t)$=$Y(G(t))$, then

$$W(t+s) - W(s) =_d Y\big(G(t+s)\big) - Y\big(G(s)\big) =_d Y\big(G(t+s) - G(s)\big) =_d Y\big(G(t)\big) =_d W(t). \quad (18)$$

Moreover, by the properties of self-similarity, the structure function of $W(t)$ can be related to that of $G(t)$,

$$E[|W(t)|^q] = E[|Y(G(t)|^q] = E[|G(t)|^{qH}|Y(1)|^q] = E[|Y(1)|^q]E[|G(t)|^{qH}]. \quad (19)$$

Consequently, if the subordinator $G(t)$ has the property of ESS, then so does the process $W(t)$. In particular, if G(t) has the SS property (6), then a similar property is shared by $W(t)$. Let us also point out that even though $G(t)$ may not be quite SS or ESS, nevertheless the sample structure functions of $W(t)$ may still exhibit self-similarity over certain ranges of the lag. One such example is the fractional Laplace motion (FLM), a process with stationary but dependent increments defined via this scheme where $Y(t)$ is a fractional Brownian motion while $G(t)$ is an independent gamma process (the latter has independent and stationary increments, and $G(t)$ has a gamma distribution with shape parameter t). This process has been used before for modeling permeabilities as well as other data (see references below) and its structure functions exhibit the SS property (6) over the lag ranges relevant to the data studied (see, e.g., Ganti et al., 2009). Moreover, the exponent $\xi(q)$ scales non-linearly with q. As expected, plots of two FLM structure functions of different orders against each other are linear across wide ranges of parameters and lags.

Technical Corrections: The idea of this model is quite interesting and the applications (both present and previous), are important, but we are having problems with its mathematical underpinning, as described in the above comments. The reader would benefit from a better description of the TFBM model presented in Appendix A. We do not mean all the technical details of this construction as presented in the papers of Neuman. The authors need to clarify the finite-dimensional distributions of this process. If the process is Gaussian with stationary increments, this needs to be clearly stated, along with its covariance function. Regarding this point, the term "Gaussian autocorrelation function" used in connection with the TFBM model (see line 3 following equation (A7) on p. 7822) can be misleading – and we think should be avoided – along with the term "exponential autocorrelation function". These terms are closely connected with the construction of the TFBM that is not presented here, and their use can only lead to confusion.

The notation λ_m in equation (A4) is a bit confusing – we would suggest just writing λ instead.

The ratio of the structure functions of order q+1 and q presented in (A9) does not seem to be correct – which likely propagates into (A10). The problem may be error in equation (A2) on which these two are based. This needs to be carefully checked and corrected.

On page 7893, 6th line of section 3.2, the word "powers" should be replaced by "slopes".

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