Hydrol. Earth Syst. Sci. Discuss., 8, 9469–9480, 2011
www.hydrol-earth-syst-sci-discuss.net/8/9469/2011/
doi:10.5194/hessd-8-9469-2011

Hydrology and Earth System Sciences Discussions

Technical Note: Analytical sensitivity analysis of a two parameter recursive digital baseflow separation filter

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Received: 19 September 2011 – Accepted: 6 October 2011 – Published: 25 October 2011

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Published by Copernicus Publications on behalf of the European Geosciences Union.

Abstract

A sensitivity analysis for a well established baseflow separation technique, a two parameter recursive digital filter, is presented. The propagation of errors or uncertainties of the two filter parameters into the calculated baseflow index is analytically ascertained. Representative sensitivity indices (defined as the ratio between the relative error of the baseflow index and the relative error of the respective parameter) are derived by application of the resulting equations to a great number of catchments. It is found that in the mean the parameter *a*, the recession constant, has a stronger influence on the calculated baseflow index than the second filter parameter BFI_{max}. This is favourable in that *a* can be determined by a recession analysis and therefore should be less uncertain. Whether this finding also applies for a specific catchment can easily be checked by means of the derived equations.

1 Introduction

The aim of baseflow separation is to distinguish two streamflow components: baseflow (groundwater discharging into the stream) and quick flow (surface runoff and interflow). In the past, many separation methods have been proposed, amongst them the two parameter recursive digital filter of Eckhardt (2005), which has since been applied in numerous studies, sometimes under the name of "Eckhardt filter". The equation of this low-pass filter is

$$b_k = \frac{(1 - BFI_{max})\, a\, b_{k-1} + (1 - a)\, BFI_{max}\, y_k}{1 - a\, BFI_{max}} \quad (1)$$

subject to $b_k \leq y_k$, where b is the baseflow, y is the streamflow, and k is the time step number. Furthermore, the filter has two parameters: the recession constant a and the maximum value BFI_{max} of the baseflow index (the long-term ratio of baseflow to total streamflow) that can be modelled by the algorithm.

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A key question is how errors and uncertainties in these two parameters affect the results of the separation. A first attempt to answer this question was the empirical sensitivity analysis in Eckhardt (2005). An empirical sensitivity analysis consists of three steps, which are repeated several times:

1. Input of a model (consisting of one or more equations) is varied.
2. The model is run.
3. The model output is analysed.

One can also speak of an experimental sensitivity analysis. However, an empirical sensitivity analysis is only a makeshift if an analytical sensitivity analysis, that is an analytical calculation of the error propagation through the model, is not feasible. In case of Eq. (1), such a calculation of the error propagation is possible and will be presented in the following.

2 Analytical sensitivity analysis

The analytical sensitivity analysis of the filter Eq. (1) requires the calculation of the partial derivatives of b_k with respect to a and $\mathrm{BFI}_{\max}$:

$$\frac{\partial b_k}{\partial a} = \frac{(1 - \mathrm{BFI}_{\max})\left(b_{k-1} - \mathrm{BFI}_{\max}\, y_k\right)}{(1 - a\,\mathrm{BFI}_{\max})^2} \tag{2}$$

$$\frac{\partial b_k}{\partial \mathrm{BFI}_{\max}} = \frac{(a - 1)\left(a\, b_{k-1} - y_k\right)}{(1 - a\,\mathrm{BFI}_{\max})^2} \tag{3}$$

(see Appendix A).

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In the following, the considered model output is the baseflow index

$$\mathrm{BFI} = \frac{\sum_{k=1}^{n} b_k}{\sum_{k=1}^{n} y_k} = \frac{b}{y} \tag{4}$$

where b denotes the sum of baseflow and y the sum of streamflow over the whole period of the available streamflow measurements.

The error propagation into the model output BFI is described by the partial derivatives of *BFI* with respect to a and $\mathrm{BFI}_{\max}$:

$$\frac{\partial \mathrm{BFI}}{\partial a} = \frac{1}{y} \frac{1 - \mathrm{BFI}_{\max}}{(1 - a\,\mathrm{BFI}_{\max})^2} (b - b_n - \mathrm{BFI}_{\max}\, y) \tag{5}$$

$$\frac{\partial \mathrm{BFI}}{\partial \mathrm{BFI}_{\max}} = \frac{1}{y} \frac{a - 1}{(1 - a\,\mathrm{BFI}_{\max})^2} [a\,(b - b_n) - y] \tag{6}$$

(see Appendix A).

In order to get representative BFI values, the filtered hydrographs should be long. In this case the term $- b_n$ in the Eqs. (3) and (4) can be neclected:

$$\frac{\partial \mathrm{BFI}}{\partial a} = \frac{1}{y} \frac{1 - \mathrm{BFI}_{\max}}{(1 - a\,\mathrm{BFI}_{\max})^2} (b - \mathrm{BFI}_{\max}\, y) \tag{7}$$

$$\frac{\partial \mathrm{BFI}}{\partial \mathrm{BFI}_{\max}} = \frac{1}{y} \frac{a - 1}{(1 - a\,\mathrm{BFI}_{\max})^2} (a\,b - y). \tag{8}$$

Now, the question how an error Δa in the filter parameter a propagates into the calculated baseflow index BFI can be answered. Small errors in a cause an error in BFI of

$$\Delta_a \mathrm{BFI} = \frac{\partial \mathrm{BFI}}{\partial a} \Delta a = \frac{1}{y} \frac{1 - \mathrm{BFI}_{\max}}{(1 - a\,\mathrm{BFI}_{\max})^2} (b - \mathrm{BFI}_{\max}\, y)\, \Delta a. \tag{9}$$

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Correspondingly, small errors $\Delta \mathrm{BFI}_{\max}$ in the filter parameter $\mathrm{BFI}_{\max}$ cause an error in BFI of

$$\Delta_{\mathrm{BFI}_{\max}} \mathrm{BFI} = \frac{\partial \mathrm{BFI}}{\partial \mathrm{BFI}_{\max}} \Delta \mathrm{BFI}_{\max} = \frac{1}{y} \frac{a-1}{(1 - a\, \mathrm{BFI}_{\max})^2} (a\, b - y)\, \Delta \mathrm{BFI}_{\max}. \tag{10}$$

As a measure for the sensitivity of the baseflow index BFI with respect to the parameters a and $\mathrm{BFI}_{\max}$ a dimensionless sensitivity index S is calculated as the ratio between the relative error of BFI and the relative error of the respective parameter. The sensitivity index for the parameter a is

$$S(\mathrm{BFI}|a) = \frac{\Delta_a \mathrm{BFI}}{\mathrm{BFI}} \Big/ \frac{\Delta a}{a} = \frac{(1 - \mathrm{BFI}_{\max})\,(\mathrm{BFI} - \mathrm{BFI}_{\max})}{(1 - a\, \mathrm{BFI}_{\max})^2} \frac{a}{\mathrm{BFI}} \tag{11}$$

(see Appendix B). In this notation, S stands for "sensitivity index", the first symbol in the parentheses (here BFI) indicates the output that is assessed, and the second symbol (here a) the uncertain input. Sometimes this dimensionless index is also called "elasticity index".

The sensitivity index for the parameter $\mathrm{BFI}_{\max}$ is

$$S\left(\mathrm{BFI}|\mathrm{BFI}_{\max}\right) = \frac{\Delta_{\mathrm{BFI}_{\max}} \mathrm{BFI}}{\mathrm{BFI}} \Big/ \frac{\Delta \mathrm{BFI}_{\max}}{\mathrm{BFI}_{\max}} = \frac{(a-1)\,(a\, \mathrm{BFI} - 1)}{(1 - a\, \mathrm{BFI}_{\max})^2} \frac{\mathrm{BFI}_{\max}}{\mathrm{BFI}} \tag{12}$$

(see Appendix B).

For specific values of a, $\mathrm{BFI}_{\max}$, and BFI, the sensitivity indices $S(\mathrm{BFI}|a)$ and $S(\mathrm{BFI}|\mathrm{BFI}_{\max})$ can now be calculated and compared.

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3 Application

3.1 Data and results

An empirical sensitivity analysis requires several runs of the filter over the hydrograph of a specific stream, each one with different values of the two filter parameters. Subsequently, the resulting time series of baseflow have to be analysed to ascertain how the baseflow index varies and which values the sensitivity indices have. Alternatively, only one filter run and calculation of the baseflow index is sufficient, if Eqs. (11) and (12) are used for calculating the sensitivity indices.

This method has been applied to the 65 catchments whose baseflow indices BFI were calculated by Eckhardt (2008). The results are summarised in Table 1. Two catchment types are distinguished: catchments with a perennial stream and porous aquifer, and catchments with an ephemeral stream and porous aquifer. Eckhardt (2005) suggested to attribute a $\mathrm{BFI}_{\max}$ value of 0.80 to the former and of 0.50 to the latter. The recession constant a of each catchment was determined by a recession analysis as described by Eckhardt (2008), the respective sensitivity indices were calculated with Eqs. (11) and (12).

3.2 Discussion

The analytical sensitivity analysis shows that the recession constant a influences the calculated baseflow index BFI more than the filter parameter $\mathrm{BFI}_{\max}$. In the case of the catchments with perennial stream and porous aquifer, for example, the value $S(\mathrm{BFI}|a) = -0.77$ signifies that a relative error of X percent in a causes a relative error of -0.77 times X percent in BFI. A relative error of X percent in $\mathrm{BFI}_{\max}$ causes only a relative error of 0.26 times X percent in BFI. This is good news because the recession constant a can be determined by a recession analysis whereas an optimal $\mathrm{BFI}_{\max}$ value cannot be derived from the streamflow measurements alone. Therefore, the value of $\mathrm{BFI}_{\max}$ will be more uncertain than the value of a.

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At first glance, the finding that the parameter a has a stronger influence on the calculated baseflow index than the parameter BFI_{max} seems to contradict the result of the empirical sensitivity analysis of Eckhardt (2005). In the latter, hydrographs of two catchments were used:

- For a catchment with a perennial stream and porous aquifer, and assuming values of $a = 0.925$ and $BFI_{max} = 0.75$, the baseflow index was found to be $BFI = 0.72$ and sensitivity indices $S(\bar{b}|a) = -0.55$ and $S(\bar{b}|BFI_{max}) = 0.96$ were calculated ($\bar{b}$: mean value of the baseflow).

- For a catchment with a perennial stream and hard rock aquifer, and assuming values of $a = 0.925$ and $BFI_{max} = 0.25$, the baseflow index was found to be $BFI = 0.25$ and the sensitivity indices were $S(\bar{b}|a) = 0.00$ and $S(\bar{b}|BFI_{max}) = 0.98$.

Therefore, the conclusion was that BFI_{max} is the more critical parameter.

Indeed, this is confirmed if we insert only the two afore-mentioned sets of values into Eqs. (11) and (12). With $a = 0.925$, $BFI_{max} = 0.75$, and $BFI = 0.72$ one gets $S(BFI|a) = -0.10$ and $S(BFI|BFI_{max}) = 0.28$. With $a = 0.925$, $BFI_{max} = 0.25$, and $BFI = 0.25$ one gets $S(BFI|a) = 0.00$ and $S(BFI|BFI_{max}) = 0.10$. This, however, is obviously a non-representative result.

We have here a further argument for the analytical sensitivity analysis: Because it requires less effort than an empirical sensitivity analysis once the Eqs. (11) and (12) are derived, more catchments can be included and hence a more reliable conclusion can be drawn.

4 Conclusions

The finding that BFI_{max} is the less critical parameter in the filter Eq. (1) is favourable in that BFI_{max} cannot be derived from the streamflow measurements and therefore is more uncertain than the other filter parameter, the recession constant a. Optimal

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BFI_{max} values have to be found by calibration. Gonzales et al. (2009), for example, have calibrated the filter (Eq. 1) by means of a tracer-based separation using dissolved silica and found an optimal BFI_{max} value of 0.92 for a Dutch catchment. Eckhardt (2005) suggested $BFI_{max} = 0.80$ for such a catchment with a perennial stream and porous aquifer. Thus there may be an uncertainty of about 0.15 or 19 %, respectively, in the filter parameter BFI_{max}. The sensitivity index $S(BFI|BFI_{max}) = 0.26$ indicates that such an error leads to a mean error in the calculated baseflow index BFI of only $0.26 \times 19\,\% = 5\,\%$. For catchments with ephemeral stream and porous aquifer, the uncertainty is smaller yet.

Of course, these values only characterise mean conditions. The baseflow index of a specific catchment can show another sensitivity to uncertainties in the filter parameters. However, this can easily be checked by means of the Eqs. (11) and (12), which herewith provide an important additional information to this baseflow separation technique.

Appendix A

Calculation of the partial derivatives

$$\begin{aligned}
\frac{\partial b_k}{\partial a} &= \frac{\partial}{\partial a}\,\frac{(1-\mathrm{BFI}_{max})\,a\,b_{k-1} + (1-a)\,\mathrm{BFI}_{max}\,y_k}{1-a\,\mathrm{BFI}_{max}} \\
&= (1-\mathrm{BFI}_{max})\,b_{k-1}\,\frac{\partial}{\partial a}\,\frac{a}{1-a\,\mathrm{BFI}_{max}} + \mathrm{BFI}_{max}\,y_k\,\frac{\partial}{\partial a}\,\frac{1-a}{1-a\,\mathrm{BFI}_{max}} \\
&= (1-\mathrm{BFI}_{max})\,b_{k-1}\,\frac{1}{(1-a\,\mathrm{BFI}_{max})^2} + \mathrm{BFI}_{max}\,y_k\,\frac{\mathrm{BFI}_{max}-1}{(1-a\,\mathrm{BFI}_{max})^2} \\
&= \frac{(1-\mathrm{BFI}_{max})\left(b_{k-1}-\mathrm{BFI}_{max}\,y_k\right)}{(1-a\,\mathrm{BFI}_{max})^2} \qquad \text{(A1)}
\end{aligned}$$

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$$
\begin{aligned}
\frac{\partial b_k}{\partial \mathrm{BFI_{max}}} &= a\, b_{k-1} \frac{\partial}{\partial \mathrm{BFI_{max}}} \frac{1 - \mathrm{BFI_{max}}}{1 - a\,\mathrm{BFI_{max}}} + (1-a)\, y_k \frac{\partial}{\partial \mathrm{BFI_{max}}} \frac{\mathrm{BFI_{max}}}{1 - a\,\mathrm{BFI_{max}}} \\
&= a\, b_{k-1} \frac{a-1}{(1 - a\,\mathrm{BFI_{max}})^2} + (1-a)\, y_k \frac{1}{(1 - a\,\mathrm{BFI_{max}})^2} \\
&= \frac{(a-1)\,(a\, b_{k-1} - y_k)}{(1 - a\,\mathrm{BFI_{max}})^2}
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
\frac{\partial \mathrm{BFI}}{\partial a} &= \frac{\partial}{\partial a} \frac{b}{y} = \frac{1}{y} \sum_{k=1}^{n} \frac{\partial b_k}{\partial a} \\
&= \frac{1}{y} \sum_{k=1}^{n} \frac{(1 - \mathrm{BFI_{max}})\,(b_{k-1} - \mathrm{BFI_{max}}\, y_k)}{(1 - a\,\mathrm{BFI_{max}})^2} \text{ (see Eq. A1)} \\
&= \frac{1}{y} \frac{1 - \mathrm{BFI_{max}}}{(1 - a\,\mathrm{BFI_{max}})^2} \sum_{k=1}^{n} (b_{k-1} - \mathrm{BFI_{max}}\, y_k) \\
&= \frac{1}{y} \frac{1 - \mathrm{BFI_{max}}}{(1 - a\,\mathrm{BFI_{max}})^2} (b - b_n - \mathrm{BFI_{max}}\, y)
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
\frac{\partial \mathrm{BFI}}{\partial \mathrm{BFI_{max}}} &= \frac{1}{y} \sum_{k=1}^{n} \frac{\partial b_k}{\partial \mathrm{BFI_{max}}} = \frac{1}{y} \sum_{k=1}^{n} \frac{(a-1)\,(a\, b_{k-1} - y_k)}{(1 - a\,\mathrm{BFI_{max}})^2} \text{ (see Eq. A2)} \\
&= \frac{1}{y} \frac{a-1}{(1 - a\,\mathrm{BFI_{max}})^2} \sum_{k=1}^{n} (a\, b_{k-1} - y_k) \\
&= \frac{1}{y} \frac{a-1}{(1 - a\,\mathrm{BFI_{max}})^2} [a\,(b - b_n) - y]
\end{aligned} \tag{A4}
$$

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Appendix B

Calculation of the sensitivity indices

$$S(\mathrm{BFI}|a) = \frac{\Delta_a \, \mathrm{BFI}}{\mathrm{BFI}} / \frac{\Delta a}{a} = \frac{(1 - \mathrm{BFI}_{max})\,(b - \mathrm{BFI}_{max}\, y)}{y\,(1 - a\,\mathrm{BFI}_{max})^2}\,\Delta a\,\frac{a}{\mathrm{BFI}\,\Delta a}$$

(see Eq. 9). With $b = \mathrm{BFI}\, y$ (Eq. 4) one can also write

$$S(\mathrm{BFI}|a) = \frac{(1 - \mathrm{BFI}_{max})\,(\mathrm{BFI}\, y - \mathrm{BFI}_{max}\, y)}{y\,(1 - a\,\mathrm{BFI}_{max})^2}\,\frac{a}{\mathrm{BFI}}$$

$$= \frac{(1 - \mathrm{BFI}_{max})\,(\mathrm{BFI} - \mathrm{BFI}_{max})}{(1 - a\,\mathrm{BFI}_{max})^2}\,\frac{a}{\mathrm{BFI}} \qquad \text{(B1)}$$

$$S\left(\mathrm{BFI}|\mathrm{BFI}_{max}\right) = \frac{\Delta_{\mathrm{BFI}_{max}}\,\mathrm{BFI}}{\mathrm{BFI}} / \frac{\Delta\,\mathrm{BFI}_{max}}{\mathrm{BFI}_{max}}$$

$$= \frac{(a - 1)\,(a\,b - y)}{y\,(1 - a\,\mathrm{BFI}_{max})^2}\,\Delta\,\mathrm{BFI}_{max}\,\frac{\mathrm{BFI}_{max}}{\mathrm{BFI}\,\Delta\,\mathrm{BFI}_{max}} \quad \text{(see Eq. 10)}$$

$$= \frac{(a - 1)\,(a\,\mathrm{BFI} - 1)}{(1 - a\,\mathrm{BFI}_{max})^2}\,\frac{\mathrm{BFI}_{max}}{\mathrm{BFI}} \qquad \text{(B2)}$$

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Table 1. Results of the analysis of 65 North American catchments (mean values and their standard deviation).

catchment type	number	a	BFI_{max}	BFI	$S(BFI\|a)$	$S(BFI\|BFI_{max})$
perennial stream, porous aquifer	60	0.970 ±0.001	0.80	0.67 ±0.01	−0.77 ±0.07	0.26 ±0.01
ephemeral stream, porous aquifer	5	0.961 ±0.004	0.50	0.42 ±0.04	−0.42 ±0.22	0.11 ±0.03

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