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# On accuracy of upper quantiles estimation

I. Markiewicz, W. G. Strupczewski, and K. Kochanek

Institute of Geophysics Polish Academy of Sciences, Warszawa, Poland

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Correspondence to: I. Markiewicz (iwonamar@igf.edu.pl)

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## Abstract

Flood frequency analysis (FFA) entails estimation of the upper tail of a probability density function (PDF) of annual peak flows obtained from either the annual maximum series or partial duration series. In hydrological practice the properties of various estimation methods of upper quantiles are identified with the case of known population distribution function. In reality the assumed hypothetical model differs from the true one and one can not assess the magnitude of error caused by model misspecification in respect to any estimated statistics. The opinion about the accuracy of the methods of upper quantiles estimation formed from the case of known population distribution function is upheld. The above-mentioned issue is the subject of the paper. The accuracy of large quantile assessments obtained from the four estimation methods are compared for two-parameter log-normal and log-Gumbel distributions and their three-parameter counterparts, i.e., three-parameter log-normal and GEV distributions. The cases of true and false hypothetical model are considered. The accuracy of flood quantile estimates depend on the sample size, on the distribution type, both true and hypothetical, and strongly depend on the estimation method. In particular, the maximum likelihood method loses its advantageous properties in case of model misspecification.

## 1 Introduction

Flood frequency analysis provides an information about the probable size of flood flows. Obtained in this way the estimates of the quantiles of maximum flows have many practical applications. It is an information required for designing hydraulic structures, in determining the limits of flood zones with varying degrees of flood risk, in estimating the risk of exploitation of floodplains, as well as for the valuation of the contributions of many branches of the insurance market. Flood Forecasting provides support for the governing bodies of water resources in decision-making processes and plays a very important role in reducing flood risk.

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Flood frequency analysis boils down to the estimation of the upper tail, i.e. the upper quantiles of the probability density function of the annual (or partial duration) maximum flows, and the distribution function assumed is the statistical hypotheses. The problem of flood frequency modeling refers to the choice of the probability distribution describing the annual peak flows along with the method of estimation parameters, and thus quantiles of this distribution. This issue is called the distribution and estimation (D/E) procedure. The accuracy of quantile estimate is measured by the mean square error (MSE) and the bias (B). In a classical hydrological approach the properties of the estimation methods are analyzed under the assumption that the hypothetical distribution adopted is true. In the literature there are several papers concerning the analysis of the accuracy of the estimates of large quantiles for the selected probability distribution, e.g., Landwehr et al. (1980), Kuczera (1982), Hoshi et al. (1984). The properties of the estimation method observed for some distribution are often automatically generalized to other distributions. In the literature three estimation methods have been usually compared, including the method of conventional moments (MOM), the method of linear moments (LMM) and the maximum likelihood method (MLM). In this paper, another method is proposed for the comparative analysis, it is the method built on the mean deviation (MDM). Due to the analytical intractability of the mean deviation in statistics, this method was not yet widely applied in the FFA. However, using the simulation techniques can cope with this inconvenience. The application of the MDM to the estimation of the flood quantiles has been proposed in Markiewicz et al. (2006) and Markiewicz and Strupczewski (2009).

As the objective function for the selection of the probability density function to the data should be the best fit of the distribution to empirical data, primarily in the range of the upper quantiles, making allowance for low quality of largest sample data. Moreover, no simple statistical model can reproduce the data set in its entire range of variability. This would require the use of too many parameters that can not be estimated reliably and efficiently from a data series which usually is of relatively small size. The probability of correct identification of density function on the basis of short hydrological samples

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is very low, even in the ideal case, when a set of alternative distributions contains the true density function (e.g., Mitosek et al., 2006). Therefore, the traditional approach based on the knowledge of the theoretical distribution is not acceptable. In papers Strupczewski et al. (2002a,b) and Weglarczyk et al. (2002), the asymptotic bias of

5 quantile in the case of assuming the wrong distribution has been derived for various estimation methods and for selected pairs of probability functions. If the hypothetical distribution is true, for a given estimation method, the bias of quantile estimate results from a finite random sample on the basis of which we assess the value of a quantile, but when the hypothetical distribution differs from the true, the total bias of quantile estimator also includes the error resulting from the model.

10 The aim of the study is to show that the theoretical properties of various estimation methods vary significantly when the choice of a hypothetical distribution is incorrect, as in the realities of hydrology is very likely. The paper is organized as follows. After providing some introduction to the topic, the four estimation methods and the probability distributions analyzed in the paper are presented in Sects. 2 and 3, respectively. The 15 next section provides studies on accuracy of upper quantile estimates for these two- and three-parameter distributions under the assumption of true hypothetical distribution. The similar discussion for the case of false hypothetical distribution is presented in Sect. 5. The paper is concluded in the final section.

## 20 2 Estimation methods

Several systems of summary statistics describing the properties of a random sample have been developed. Basing on different principles they provide, in particular, the measures of location, dispersion, skewness and kurtosis, which consecutively serve for identifying and fitting PDFs. It is convenient to use dimensionless versions of the 25 summary statistics sets in the form of summary statistics ratios. They measure the shape of a distribution independently of its scale of measurement. Among the systems of summary statistics, the most popular are the system of the conventional moments

and that of the linear moments ( $L$ -moments). The  $L$ -moments create an attractive system because their estimators, in contrast to the classical moments estimators, are not biased and the sampling  $L$ -moment ratios have very small biases for moderate and large samples, e.g. Hosking and Wallis (1997). For both the system of conventional and linear moments, the measure of location is expressed by the mean ( $\mu \equiv \lambda_1$ ) and the measures of dispersion and skewness are presented in Table 1.

For the estimation of statistical characteristics, the method of moments (MOM), e.g., Kendall and Stuart (1969), and the method of linear moments (LMM), e.g., Hosking and Wallis (1997), have been alternatively used. The MDM is an innovative method based on applying the mean deviation  $\delta_\mu$  about the mean value ( $\mu$ ) as a measure of dispersion (see Table 1), with the mean as a measure of location and  $\delta_S$  as a measure of skewness (Markiewicz et al., 2006; Markiewicz and Strupczewski, 2009). The complement to the estimation methods based on distribution characteristics is the MLM (e.g., Kendall and Stuart, 1973), which is based on the main probability mass. The MLM is sometimes regarded as the most appropriate method because it allows to obtain the asymptotically most efficient estimators. However, the MLM is related with relatively large accounting difficulties and the maximum likelihood estimators do not always exist.

### 3 Probability distributions

The true probability distribution, which reflects the time series of extreme flows for a given gauging station is not known. The study on a distribution form which would describe the observed data series is the subject of many papers, such as Jenkinson (1969) or NERC (1975). The hydrological report of the World Meteorological Organization from 1989 (Cunnane, 1989) shows that the most commonly used and recommended were Gumbel and log-normal distributions. Nowadays, the researchers of hydrological extreme events recommend the use of the heavy-tailed distributions for modeling the annual maximum flows (e.g., FEH, 1999; Rao and Hamed, 2000; Katz

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et al., 2002). However, as yet, the certificate of a heavy tail of hydrological variables are not sufficiently convincing (e.g., Rowinski et al., 2002; Weglarczyk et al., 2002). The heavy-tailed distributions have conventional moments only in a certain range of shape parameter values and the range decreases with growing moment order. Since the hydrological samples of peak flows are usually of a relatively small size, in order to estimate many parameters reliably and efficiently, both two- and three-parameter distributions are used in FFA, while the lower bound parameter ( $\varepsilon$ ) serves as the third one (e.g., Rao and Hamed, 2000). In the paper, to assess the accuracy of the estimates of high quantiles, two two-parameter distributions have been selected, i.e. log-normal 2 (LN2) and log-Gumbel (LG) and their three-parameter counterparts, LN3 and GEV. Density functions of distributions are shown in Table 2. Both two- and three-parameter log-normal distributions represent the classical (albeit borderline) type of distribution, while the LG and GEV are heavy-tailed.

#### 4 True hypothetical distribution

Since the true probability distribution of an observed peak flow series is not known, it would seem that a choice of a hypothetical distribution is the key point to the accurate estimation of high quantiles. However, discussed in this section the case where the assumed distribution is consistent with the real one, shows that the ranking of the methods in respect to the accuracy of large quantile estimate strongly depends on the type of the true distribution and its shape.

The issue is analyzed on the example of the quantile  $x_{F=0.99}$ , otherwise known as the quantile 1%. This is likely the most commonly estimated design value for the dimensioning of hydrological structures and it defines the flow values which is exceeded on average once every 100 years.

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## 4.1 Simulation experiment

Simulation experiments are performed for two-parameter distributions, LN2 and LG, the variation coefficient  $C_V (C_V = \sigma/\mu)$  varying from 0.2 to 1.0, with any mean  $\mu > 0$ . The  $N$ -element samples are considered for  $N=20(10)100$ . In each case, 20 000 random samples are generated. The value of  $x_{F=0.99}$  is calculated using four estimation methods under the right assumption that the population is log-normal and log-Gumbel distributed, respectively. The accuracy of the quantile  $x_{F=0.99}$  estimates is expressed by the relative root mean square error ( $\delta\text{RMSE}$ ) and the relative bias ( $\delta B$ ):

$$\delta\text{RMSE}(\hat{x}_{0.99}) = \frac{\sqrt{E(\hat{x}_{0.99} - x_{0.99})^2}}{x_{0.99}}, \quad \delta B(\hat{x}_{0.99}) = \frac{E(\hat{x}_{0.99} - x_{0.99})}{x_{0.99}} \quad (1)$$

The results of the experiment are presented in Tables 3 and 4 for LN2 and LG distributions, respectively. In the asymptotic case, i.e. for  $N \rightarrow \infty$ ,  $\delta\text{RMSE}(\hat{x}_{0.99})$  and  $\delta B(\hat{x}_{0.99})$  converge to zero. The quantile value in the first column is the true value.

For three-parameter distributions, LN3 and GEV, the mean equals to zero, the standard deviation equals to one and various values of skewness coefficient  $C_S (C_S = \mu_3/\mu_2^{3/2})$  are assumed for the Monte Carlo experiment. The results are shown in Tables 5 and 6. The range of  $C_S$  value considered here is conditioned by the existence of skewness coefficient for GEV distribution, which takes values greater than 1.1396 (e.g., Markiewicz et al., 2006, p. 394). Moreover, maximum likelihood estimation of GEV distribution is not always satisfactory and for some samples it appears that the likelihood function does not have a local maximum (Hosking et al., 1985). In our simulations of the GEV distribution this non-regularity of the likelihood function causes occasional non-convergence of the modified Powell hybrid algorithm (More et al., 1980; IMSL, 1997) that is used to maximize the log-likelihood. Last column in Table 6 shows the reliability of the MLM for the GEV distribution.

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## 4.2 Accuracy of upper quantile estimates for two-parameter distributions

For both distribution, LN2 and LG, for any value of variation coefficient, the method of moments gives the greatest bias and the higher  $C_V$  value the bias is greater. For small samples of 20 elements from LN2 distribution, the relative bias of quantile  $x_{0.99}$  estimated by MOM increases from  $-1.57\%$  for  $C_V=0.2$  to  $-11.41\%$  for  $C_V=1.0$  (Table 3), while and for LG distribution, these values are, respectively  $-3.41\%$  and  $-22.27\%$  (Table 4). For small values of  $C_V$  ( $C_V=0.2$ ) the difference between  $\delta B(\hat{x}_{0.99})$  from the MOM and from the second method in terms of high bias, i.e. the MLM, is not large, but the distance increases with increasing  $C_V$  value. For the two distributions, the output of the method of maximum likelihood converges to those of the MDM, LMM, which are the best among the four estimation methods studied, in most cases they give a relative bias lower than 1% in absolute value. A clear negative MOM detachment from other methods is observed for bias of LG. It remains large even for statistically large sample in particular for large  $C_V$  population value. It is also worth noting that the MDM, produces a competitive bias to LMM.

The relative root mean square error of quantile 0.99 estimate is the smallest for MLM both for LN2 and LG distribution except for LN2 with  $C_V=0.2$  and  $N=100$  where it is the largest one. Among the methods built on summary statistics, i.e., MOM, LMM, MDM, the method MOM produces the smallest  $\delta RMSE(\hat{x}_{0.99})$  for small samples ( $N=20$ ) of LN2 distribution, regardless of the value of  $C_V$  and for log-Gumbel distribution and  $C_V \geq 0.6$ , regardless of the sample size. While in other cases considered, the MOM is the method that yields the highest root mean square error among the four estimation methods studied.

It is worth noting that for the heavy-tailed distributions of large  $C_V$  value (i.e. with large skewness as well this time), the bias of classical measure of dispersion, i.e. the standard deviation, decreases very slowly with increasing sample size. This is clearly evident in Table 4.

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### 4.3 Accuracy of upper quantile estimates for three-parameter distributions

The strong inferiority of the MOM in respect to the relative bias of  $x_{0.99}$  quantile assessment, which has been observed for LN2 and LG distributions, has no place in the case of tree parameter LN3 and GEV. For 20-element samples, the absolute value of

- 5  $\delta B(\hat{x}_{0.99})$  obtained from MOM is similar to analogical value obtained from MLM both for LN3 distribution (Table 5) and GEV (Table 6). Then, with increasing sample size, the absolute  $\delta B(\hat{x}_{0.99})$  decreases significantly in the case of the MLM and slightly in the case of the MOM. For LN3 and the analyzed range of  $C_S$ , the LMM and MDM yield significantly smaller  $\delta B(\hat{x}_{0.99})$  than MOM and MLM, regardless of the sample size, while 10 in the case of GEV this regularity is not observed.

The  $\delta RMSE(\hat{x}_{0.99})$  of MLM is worth of special attention. Comparing the two-parameter distributions, the addition of the location parameter to the distribution characteristics effects in degradation of MLM position in the  $\delta RMSE$  ranking both for the LN3 and GEV distributions. The MLM losses its first place in all cases except the large 15 samples ( $N=100$ ) of the LN3. However even then, the superiority of the MLM over the three other methods is very small. For both distributions and small samples ( $N=20$ ) the ML-estimates of  $\hat{x}_{0.99}$  have the highest  $\delta RMSE$  of all four estimation methods considered and the differences between the quantile assessments obtained from the MLM and other three methods are considerable. For example, in the case of the LN3 distribution of  $C_S=2.0$  and  $N=20$ , the relative root mean square error of  $\hat{x}_{0.99}$  obtained 20 from the MLM is 21.13%, while  $\delta RMSE(\hat{x}_{0.99})$  from the MOM, LMM and MDM are only 11.68%, 14.18% and 14.12%, successively (Table 5). For GEV distribution the analogical values of  $\delta RMSE(\hat{x}_{0.99})$  are 82.61%, 36.23%, 53.69%, 45.23% for methods MLM, MOM, LMM and MDM, respectively (Table 6). The first place of MOM in  $\delta RMSE$  ranking is observed for LN3 and GEV with population  $C_S=2.0$ , while if  $C_S=4.0$ , the method based on mean deviation is the best in respect to  $\delta RMSE$  of quantile 0.99 estimate.

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## 5 False hypothetical distribution

Due to the fact that the true probability distribution of the annual maximum flow series is not detectable, the assumption of a false hypothetical distribution seems to be more realistic. The error of the estimate of quantile 0.99 differs significantly for particular 5 options of true and hypothetical distribution assumed, giving an evidence of strong influence of the type distribution, both true and hypothetical, on the accuracy of the estimators of large quantiles.

### 5.1 Simulation experiment

The Monte Carlo experiment is carried out similarly as in the case of true hypothetical 10 distribution, however, the hypothetical distribution is incorrectly assumed. Therefore two options for two-parameter distributions are considered, i.e.,  $T=LN2$ ,  $H=LG$  (Table 7) and  $T=LG$ ,  $H=LN2$  (Table 8), and two options for three-parameter PDFs, i.e.,  $T=LN3$ ,  $H=GEV$  (Table 9) and  $T=GEV$ ,  $H=LN3$  (Table 10). Note that the bias 15 for the asymptotic case ( $N \rightarrow \infty$ ) can be obtained analytically for two-parameter distributions, see Strupczewski et al. (2002a,b) and Weglarczyk et al. (2002), while for three-parameter PDFs the analogical values have not been derived yet. For the option of false hypothetical distribution, if the sample converges to infinity, then bias is the total error of quantile estimate, see Tables 7 and 8.

### 5.2 Accuracy of upper quantile estimates for two-parameter distributions

20 In the case of  $T=LN2$ ,  $H=LG$  the method of maximum likelihood is ranked the worst among four analyzed estimation methods. MLM generates the greatest both relative bias and relative root mean square error (Table 7). The MLM errors are large even for small values of the coefficient of variation and for  $C_V=0.2$ ,  $\delta B(\hat{x}_{0.99})$  varies 25 from 33.91% to 39.83%, depending on the size of the sample, and  $\delta RMSE(\hat{x}_{0.99})$  is 40.44% to 41.46%, reaching 42% for the asymptotic case. With increasing  $C_V$  of LN2

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distribution the errors increase significantly. For  $C_V=1.0$ ,  $\delta B(\hat{x}_{0.99})$  equals 312.7% for 20-element samples and 329.0% for 100-element samples, while  $\delta RMSE(\hat{x}_{0.99})$  is 490.3% and 366.5%, respectively. For  $N \rightarrow \infty$  both errors are 337.9%. Other methods of estimation, built on summary statistics, are much more accurate. In the considered range of  $C_V$ , MOM turns out to be the best with the smallest both  $\delta B$  (absolute values are compared) and  $\delta RMSE$ . The second place is occupied by MDM and next is LMM.

For the option  $T=LG$ ,  $H=LN2$ , as in the previous case, the relative bias of the  $x_{0.99}$  estimator are the greatest in case for MLM, but the differences compared with other methods are not so great (Table 8). The lowest  $\delta B(x_{0.99})$  values are produced by MOM. The method of moments is the only method for which the value of the relative bias strongly depends on the sample size. Other methods, MDM, LMM and MLM, already for  $N=20$  give values of  $\delta B(\hat{x}_{0.99})$  almost consistent with the asymptotic case. The methods MDM and LMM yield almost identical bias and they are classified between the best method MOM and the worst MLM. Regarding the relative root mean square error, for any  $C_V$  value the rank of estimation methods strongly depends on sample size. In general, for small  $C_V$  values of the size 0.2, the method MLM is the worst and MOM is the best. With the increase of  $C_V$ ,  $\delta RMSE(\hat{x}_{0.99})$  for MOM increases and decreases for MLM.

It is worth noting that in the case of false hypothetical distribution, the absolute total bias does not necessarily decrease with increasing sample size. This is due to the fact that the sampling and model biases may have the opposite signs. While in the case of true hypothetical distribution the absolute total bias does decrease with the sample size.

### 5.3 Accuracy of upper quantile estimates for three-parameter distributions

In the case of  $T=LN3$ ,  $H=GEV$  and  $C_S=2.0$ , the four analyzed estimation methods yield comparable values of both  $\delta B(\hat{x}_{0.99})$  (in absolute value) and  $\delta RMSE(\hat{x}_{0.99})$ , without clear superiority of one method over the other (Table 9). However, for  $C_S=4.0$ , the maximum likelihood method is strongly inferior to other three methods and the method

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based on mean deviation ranks very well. For 100-element samples, the relative bias of  $\hat{x}_{0.99}$  obtained from MLM is equal 35.48%, while  $\delta B(\hat{x}_{0.99})$  from the methods MOM, LMM and MDM are only -12.36%, 3.257% and -0.470%, respectively. The analogical values of  $\delta RMSE(\hat{x}_{0.99})$  are 55.85%, 28.67%, 27.29%, 23.75% for MLM, MOM, LMM and MDM, in turn.

For the option  $T=GEV$ ,  $H=LN3$ , the relative bias of the estimate of quantile 0.99 is the largest for MOM, then MLM is located and then MDM and LMM (Table 10). The rank of estimation method in respect of  $\delta RMSE$  value strongly depends on sample size. For the considered range of  $C_S$  and  $N=20$ , the sequence of methods from that which gives the smallest  $\delta RMSE(\hat{x}_{0.99})$  to this which gives the highest  $\delta RMSE(\hat{x}_{0.99})$  is as follows: MOM, MDM, LMM and MLM, while for  $N > 60$  the order is opposite.

## 6 Conclusions

Since the upper quantiles are design values for the dimensioning of hydrological structures, the accuracy of their estimates is a major and extremely important issue for flood frequency analysis. The studies presented in this paper show that the accuracy of the estimates of flood quantiles depends on the sample size, type of distributions, both real and hypothetical, and strongly depends on the method of estimation. Therefore, the properties of estimation methods can not be generalized in respect to distribution type or sample size, even if the hypothetical distribution is true. The correct identification of the distribution on the basis of short data series is not possible in hydrological reality. Therefore, the person making the choice of D/E procedure (explorer, hydrologist, designer) should be aware of the impact of the procedure selection on the value of desirable estimate. Presented in this paper a comparative analysis of large quantile estimates obtained by various methods of estimation under the assumption of true or false, but close to the true type of distribution, can be a source of information about the properties of selected D/E procedures. The studies on the estimation methods of flood quantiles when the hypothetical model is untrue should be continued. Despite a century of research, the problem of modeling of flood flows is still open.

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**Table 1.** Dispersion and skewness measures.

Estimation method	Dispersion measure	Skewness measure
MOM	Standard deviation $\sigma = \mu_2^{1/2} = \left[ \int_{-\infty}^{+\infty} (x - \mu)^2 dF(x) \right]^{1/2}$	Third moment $\mu_3 = \int_{-\infty}^{+\infty} (x - \mu)^3 dF(x)$
LMM	Second <i>L</i> -moment $\lambda_2 = \int_{-\infty}^{+\infty} 2(x - \mu)F(x)dF(x)$	Third <i>L</i> -moment $\lambda_3 = \mu + \int_{-\infty}^{+\infty} 6x[F^2(x) - F(x)]dF(x)$
MDM	Mean deviation $\delta_\mu = \int_{-\infty}^{+\infty}  x - \mu dF(x)$	Skewness measure $\delta_S = \mu - x_{0.5}$

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**Table 2.** Probability density functions of log-normal and GEV distributions.

Distribution	Probability density function (PDF)
Log-normal 3 (LN3) $\varepsilon=0$ : log-normal 2 (LN2)	$f(x) = \frac{1}{(x - \varepsilon)b\sqrt{2\pi}} \exp\left[-\frac{(\ln(x - \varepsilon) - m)^2}{2b^2}\right]$ $m$ – scale, $b > 0$ – shape; $\varepsilon < x < \infty$
Generalized extreme values (GEV) $\varepsilon=0$ : log-Gumbel (LG)	$f(x) = \frac{1}{\alpha} \left[ -\frac{\kappa}{\alpha}(x - \varepsilon) \right]^{1/\kappa-1} \exp\left\{ -\left[ -\frac{\kappa}{\alpha}(x - \varepsilon) \right]^{1/\kappa} \right\}$ $\alpha > 0$ – scale, $\kappa < 0$ – shape; $\varepsilon < x < \infty$

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**Table 3.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LN2, assuming LN2 model.

$T=\text{LN2}, H=\text{LN2}$ $\mu>0$	$N$	MOM		LMM		MDM		MLM	
		$\delta\text{RMSE}$	$\delta B$						
$C_V=0.2$ $x_{0.99}=1.554\mu$	20	8.690	-1.574	8.875	0.191	8.960	-0.850	8.634	-1.416
	60	5.089	-0.531	5.049	0.061	5.172	-0.276	4.985	-0.451
	100	3.963	-0.327	3.901	0.023	4.014	-0.179	4.070	-0.322
$C_V=0.6$ $x_{0.99}=3.115\mu$	20	25.80	-5.500	26.51	1.189	26.15	-1.130	23.98	-2.087
	60	16.40	-2.087	14.97	0.387	15.06	-0.355	13.85	-0.674
	100	13.21	-1.271	11.56	0.209	11.69	-0.244	10.70	-0.450
$C_V=1.0$ $x_{0.99}=4.905\mu$	20	38.64	-11.41	43.30	2.097	42.53	-0.661	37.17	-0.999
	60	26.38	-5.319	24.33	0.723	24.24	-0.131	21.01	-0.306
	100	22.14	-3.524	18.82	0.443	18.79	-0.087	16.16	-0.245

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**Table 4.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LG, assuming LG model.

$T=LG, H=LG$ $\mu>0$	$N$	MOM		LMM		MDM		MLM	
		$\delta RMSE$	$\delta B$						
$C_V=0.2$ $x_{0.99}=1.711\mu$	20	15.86	-3.409	15.30	0.387	15.35	-0.612	12.67	-1.403
	60	10.33	-1.371	8.693	0.131	8.785	-0.186	7.158	-0.468
	100	8.475	-0.848	6.733	0.070	6.816	-0.125	5.545	-0.292
$C_V=0.6$ $x_{0.99}=3.183\mu$	20	32.74	-14.19	40.78	1.196	41.37	-0.071	30.04	-0.817
	60	23.52	-8.713	23.69	0.489	24.05	0.182	16.62	-0.314
	100	20.40	-6.804	18.81	0.378	19.05	0.201	12.72	-0.220
$C_V=1.0$ $x_{0.99}=4.167\mu$	20	40.38	-22.27	52.30	0.363	52.47	-0.851	39.70	0.408
	60	28.37	-16.29	32.89	0.272	33.24	0.112	21.34	0.072
	100	24.66	-13.40	27.20	0.367	27.35	0.320	16.24	0.000



**Table 5.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LN3, assuming LN3 model.

$T=LN3, H=LN3$ $\mu=0, \sigma=1$		MOM		LMM		MDM		MLM	
	$N$	$\delta RMSE$	$\delta B$						
$C_S=2.0$ $x_{0.99}=3.519$	20	11.68	-4.87	14.18	1.29	14.12	1.366	21.13	4.288
	60	7.644	-2.46	7.844	0.252	8.326	0.132	7.954	0.937
	100	6.313	-1.74	6.011	0.198	6.628	0.144	5.847	0.517
$C_S=4.0$ $x_{0.99}=3.905$	20	13.40	-6.05	16.64	0.653	14.37	-0.54	27.08	7.338
	60	9.530	-3.34	9.411	0.382	9.198	0.045	9.194	1.712
	100	7.851	-2.51	7.126	0.035	7.058	0.024	6.604	0.971

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**Table 6.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from GEV, assuming GEV model.

$T = \text{GEV}, H = \text{GEV}$ $\mu = 0, \sigma = 1$	$N$	MOM		LMM		MDM		MLM		Reliability of MLM
		$\delta \text{RMSE}$	$\delta B$							
$C_S = 2.0$ $x_{0.99} = 3.479$	20	36.23	-12.05	53.69	22.37	45.23	17.06	82.61	14.14	93.78%
	60	25.02	-7.640	27.91	7.510	30.06	11.08	32.71	3.076	95.83%
	100	20.73	-6.009	21.25	3.857	23.91	7.212	23.71	1.725	96.01%
$C_S = 4.0$ $x_{0.99} = 3.696$	20	47.68	-19.47	59.07	13.52	43.33	11.86	111.1	21.54	95.17%
	60	34.35	-12.12	34.31	2.070	29.96	8.114	38.85	4.491	98.19%
	100	28.68	-9.514	27.14	0.319	24.37	4.958	28.00	2.165	98.67%



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**Table 7.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LN2, assuming LG model.

$T=\text{LN2}, H=\text{LN2}$ $\mu>0$	$N$	MOM		LMM		MDM		MLM	
		$\delta\text{RMSE}$	$\delta B$						
$C_V^{(LN)}=0.2$ $x_{0.99}=1.554\mu$	20	12.83	8.065	20.67	16.59	19.30	14.84	40.44	33.91
	60	11.10	9.399	17.83	16.39	17.23	15.68	41.20	38.61
	100	10.71	9.662	17.21	16.34	16.77	15.82	41.46	39.83
	$\infty$	10.08	10.08	16.31	16.31	16.09	16.09	42.09	42.09
$C_V^{(LN)}=0.6$ $x_{0.99}=3.115\mu$	20	21.21	-3.006	45.02	29.91	42.10	26.12	190.5	142.6
	60	13.26	0.169	35.24	29.45	33.77	27.70	174.9	156.4
	100	10.68	0.924	32.95	29.33	31.75	27.97	171.9	160.2
	$\infty$	2.194	2.194	29.22	29.22	28.46	28.46	167.4	167.4
$C_V^{(LN)}=1.0$ $x_{0.99}=4.905\mu$	20	32.26	-21.69	49.25	23.13	46.96	19.43	490.3	312.7
	60	23.73	-18.18	35.06	23.67	33.54	21.72	385.9	324.8
	100	21.19	-17.19	31.22	23.76	29.84	22.12	366.5	329.0
	$\infty$	15.04	-15.04	23.90	23.90	22.78	22.78	337.9	337.9



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**Table 8.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LG, assuming LN2 model.

$T=LG, H=LN2$ $\mu>0$	$N$	MOM		LMM		MDM		MLM	
		$\delta RMSE$	$\delta B$						
$C_V^{(LG)}=0.2$ $x_{0.99}=1.711\mu$	20	18.21	-11.75	16.94	-12.76	17.41	-13.31	17.73	-14.55
	60	13.58	-10.13	14.34	-12.88	14.45	-12.97	15.02	-13.74
	100	12.20	-9.767	13.80	-12.89	13.82	-12.90	14.44	-13.56
	$\infty$	9.157	-9.157	12.97	-12.97	12.87	-12.87	13.30	-13.30
$C_V^{(LG)}=0.6$ $x_{0.99}=3.183\mu$	20	45.48	-16.71	40.40	-19.84	43.01	-20.19	35.11	-28.46
	60	35.28	-10.63	28.26	-20.84	28.79	-20.67	30.27	-27.93
	100	31.16	-8.830	25.76	-21.08	25.99	-20.82	29.25	-27.84
	$\infty$	2.147	-2.147	21.55	-21.55	21.22	-21.22	27.70	-27.70
$C_V^{(LG)}=1.0$ $x_{0.99}=4.167\mu$	20	61.95	-17.71	63.39	-19.47	56.80	-20.20	42.12	-33.77
	60	50.09	-8.599	40.71	-21.20	38.16	-20.72	36.42	-33.72
	100	45.57	-5.411	35.97	-21.65	31.77	-21.24	35.34	-33.72
	$\infty$	17.70	17.70	22.65	-22.65	22.15	-22.15	33.70	-33.70

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**Table 9.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from LN3, assuming GEV model.

$T=LN3, H=GEV$ $\mu=1$	$N$	MOM		LMM		MDM		MLM		Reliability of MLM
		$\delta RMSE$	$\delta B$							
$C_s^{(LN3)}=2.0$ $x_{0.99}=3.519$	20	36.31	-12.45	53.79	21.57	45.39	16.15	45.11	10.13	88.03%
	60	24.89	-8.018	28.71	7.946	31.38	12.56	23.34	7.663	96.41%
	100	20.61	-6.086	22.40	5.369	26.10	10.13	17.30	6.638	96.94%
$C_s^{(LN3)}=4.0$ $x_{0.99}=3.905$	20	51.18	-22.86	57.41	7.624	40.11	-10.80	*	*	77.93%
	60	35.11	-15.09	35.06	3.333	28.22	-3.860	65.60	35.76	92.99%
	100	28.67	-12.36	27.29	3.257	23.75	-0.470	55.85	35.48	92.18%

\* Values are unreliable due to a low percentage of successful estimation



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**Table 10.** Relative accuracy [%] of  $\hat{x}_{0.99}$  for sample from GEV, assuming LN3 model.

$T=GEV, H=LN3$ $\mu=1$	$N$	MOM		LMM		MDM		MLM	
		$\delta RMSE$	$\delta B$						
$C_S^{(GEV)}=2.0$ $x_{0.99}=3.479$	20	41.29	-17.54	48.99	1.887	48.17	1.643	66.80	6.154
	60	28.10	-8.912	27.49	-2.329	28.91	-3.493	25.53	-4.279
	100	23.04	-6.317	21.25	-2.673	22.67	-4.055	19.44	-5.439
$C_S^{(GEV)}=4.0$ $x_{0.99}=3.696$	20	51.45	-21.70	61.93	-0.290	55.06	-4.047	75.95	5.476
	60	36.79	-11.01	34.95	-3.349	33.48	-6.472	28.89	-8.278
	100	30.70	-7.657	27.03	-3.733	26.30	-6.715	22.69	-10.01

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