

Supplementary material for: Temporal dynamics of hydrological threshold events.

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This material provides full solutions to the raw and central moments of the first passage time statistics detailed in the above mentioned article.

1 Raw Moments

1.1 First raw moment: $T_1 [s_0, s_\xi]$

$$T_1 [s_0, s_\xi] = \frac{\alpha(e^{(\alpha-\beta)s_\xi}\alpha - e^{(\alpha-\beta)s_0}\beta)}{(\alpha-\beta)^2\lambda} - \frac{\beta(\alpha(s_\xi - s_0) + 1)}{(\alpha-\beta)\lambda}$$

1.2 Second raw moment: $T_2 [s_0, s_\xi]$

$$T_2 [s_0, s_\xi] = (e^{-\beta s_\xi} (2e^{2\alpha s_\xi - \beta s_\xi} \alpha^4 - 2e^{\alpha s_0 - \beta s_0 + \alpha s_\xi} \alpha^3 \beta + e^{\beta s_\xi} \beta^2 (-\alpha + \beta) (2\beta - \alpha (4\beta + \alpha (-2 + (\alpha - \beta) (s_0 - s_\xi))) (s_0 - s_\xi))) + 2e^{(\alpha - \beta) s_0 + \beta s_\xi} \alpha \beta^2 (\alpha - 3\beta + \alpha \beta s_0 - \beta^2 s_0 + \alpha (\alpha - \beta) s_\xi) - 2e^{\alpha s_\xi} \alpha^2 \beta (-\alpha^2 (s_0 - 2s_\xi) + \alpha (3 + \beta s_0 - \beta s_\xi) - \beta (5 + \beta s_\xi)))) \div ((\alpha - \beta)^4 \lambda^2)$$

1.3 Third raw moment: $T_3 [s_0, s_\xi]$

$$T_3 [s_0, s_\xi] = (e^{-\beta(s_0 + 3s_\xi)} (6e^{\beta s_0 + 3\alpha s_\xi} \alpha^6 - 6e^{\alpha s_0 + 2\alpha s_\xi + \beta s_\xi} \alpha^5 \beta - 3e^{\alpha s_0 + 3\beta s_\xi} \alpha \beta^3 (\beta (\alpha^2 \beta s_0^2 - 2\alpha (1 + \beta s_0 (3 + \beta s_0)) + \beta (12 + \beta s_0 (6 + \beta s_0))) + 2\alpha (\alpha - \beta) (\alpha + \alpha \beta s_0 - \beta (4 + \beta s_0)) s_\xi + \alpha^2 (\alpha - \beta)^2 s_\xi^2) + 6e^{2\alpha s_\xi + \beta (s_0 + s_\xi)} \alpha^4 \beta (\alpha^2 (s_0 - 3s_\xi) + \alpha (-5 - \beta s_0 + \beta s_\xi) + \beta (9 + 2\beta s_\xi)) + e^{\beta (s_0 + 3s_\xi)} (\alpha - \beta) \beta^3 (-6\beta (\alpha + \beta) + \alpha^3 (\alpha - \beta)^2 s_0^3 + 6\alpha (\alpha - 3\beta) \beta s_\xi - 3\alpha^2 (\alpha - 3\beta) (\alpha - \beta) s_\xi^2 - \alpha^3 (\alpha - \beta)^2 s_\xi^3 - 3\alpha^2 (\alpha - \beta) s_0^2 (\alpha - 3\beta + \alpha (\alpha - \beta) s_\xi) + 3\alpha s_0 (-2(\alpha - 3\beta) \beta + 2\alpha (\alpha - 3\beta) (\alpha - \beta) s_\xi + \alpha^2 (\alpha - \beta)^2 s_\xi^2)) + 6e^{\alpha s_0 + \alpha s_\xi + 2\beta s_\xi} \alpha^3 \beta^2 (2\alpha^2 s_\xi + \alpha (3 + \beta s_0 - \beta s_\xi) - \beta (7 + \beta (s_0 + s_\xi))) + 3e^{\beta s_0 + \alpha s_\xi + 2\beta s_\xi} \alpha^2 \beta^2 (\alpha^4 (s_0 - 2s_\xi)^2 - 2\alpha^3 (s_0 - 2s_\xi) (3 + \beta s_0 - \beta s_\xi) + 2\alpha \beta (-13 + \beta s_\xi (5 + \beta s_\xi) - \beta s_0 (6 + \beta s_\xi)) + \beta^2 (30 + \beta s_\xi (10 + \beta s_\xi)) + \alpha^2 (6 + \beta (18s_0 + \beta s_0^2 - s_\xi (32 + 3\beta s_\xi)))))) \div ((\alpha - \beta)^6 \lambda^3)$$

1.4 Fourth raw moment: $T_4 [s_0, s_\xi]$

$$\begin{aligned}
T_4 [s_0, s_\xi] = & (e^{-\beta s_0 - \alpha s_\xi - 7\beta s_\xi} (24e^{\beta s_0 + 5\alpha s_\xi + 3\beta s_\xi} \alpha^8 - 24e^{\alpha s_0 + 4(\alpha + \beta)s_\xi} \alpha^7 \beta + \\
& 24e^{\alpha s_0 + 3\alpha s_\xi + 5\beta s_\xi} \alpha^5 \beta^2 (3\alpha^2 s_\xi + \alpha(5 + \beta s_0 - \beta s_\xi) - \beta(11 + \beta s_0 + 2\beta s_\xi)) + \\
& 24e^{\beta s_0 + 4(\alpha + \beta)s_\xi} \alpha^6 \beta (\alpha^2 (s_0 - 4s_\xi) + \alpha(-7 - \beta s_0 + \beta s_\xi) + \beta(13 + 3\beta s_\xi)) + \\
& e^{(\alpha + 7\beta)s_\xi} \beta^4 (4e^{\alpha s_0} \alpha (\beta (\alpha^3 \beta s_0^2 (-3 + \beta s_0) - 3\alpha^2 (-2 + \beta s_0 (-2 + \beta s_0) (3 + \beta s_0)) + \\
& 3\alpha \beta (-10 + \beta s_0 (2 + \beta s_0) (3 + \beta s_0)) - \beta^2 (60 + \beta s_0 (36 + \beta s_0 (9 + \beta s_0)))) + 3\alpha (\alpha - \\
& \beta) \beta (\alpha^2 \beta s_0^2 - 2\alpha (1 + \beta s_0 (4 + \beta s_0)) + \beta (20 + \beta s_0 (8 + \beta s_0))) s_\xi + 3\alpha^2 (\alpha - \beta)^2 (\alpha + \alpha \beta s_0 - \\
& \beta (5 + \beta s_0)) s_\xi^2 + \alpha^3 (\alpha - \beta)^3 s_\xi^3) + e^{\beta s_0} (\alpha - \beta) (-24\beta (\alpha^2 + 3\alpha \beta + \beta^2) + \alpha^4 (\alpha - \beta)^3 s_0^4 + \\
& 24\alpha \beta (\alpha^2 - 2\alpha \beta - 4\beta^2) s_\xi - 12\alpha^2 (\alpha - 6\beta) (\alpha - \beta) \beta s_\xi^2 + 4\alpha^3 (\alpha - 4\beta) (\alpha - \beta)^2 s_\xi^3 + \alpha^4 (\alpha - \\
& \beta)^3 s_\xi^4 - 4\alpha^3 (\alpha - \beta)^2 s_0^3 (\alpha - 4\beta + \alpha (\alpha - \beta) s_\xi) + 6\alpha^2 (\alpha - \beta) s_0^2 (-2(\alpha - 6\beta) \beta + 2\alpha (\alpha - \\
& 4\beta) (\alpha - \beta) s_\xi + \alpha^2 (\alpha - \beta)^2 s_\xi^2) + 4\alpha s_0 (6\beta (-\alpha^2 + 2\alpha \beta + 4\beta^2) + 6\alpha (\alpha - 6\beta) (\alpha - \beta) \beta s_\xi - \\
& 3\alpha^2 (\alpha - 4\beta) (\alpha - \beta)^2 s_\xi^2 - \alpha^3 (\alpha - \beta)^3 s_\xi^3)) - 12e^{\alpha s_0 + 2\alpha s_\xi + 6\beta s_\xi} \alpha^3 \beta^3 (4\alpha^4 s_\xi^2 + 4\alpha^3 s_\xi (3 + \\
& \beta s_0 - \beta s_\xi) + \beta^2 (56 + \beta (s_0 + s_\xi) (14 + \beta (s_0 + s_\xi))) - 2\alpha \beta (17 + \beta (9s_0 + \beta s_0^2 - s_\xi (7 + \\
& \beta s_\xi))) + \alpha^2 (6 + \beta (\beta s_0^2 + s_0 (4 - 6\beta s_\xi) - s_\xi (40 + 3\beta s_\xi)))) + 12e^{\beta s_0 + 3\alpha s_\xi + 5\beta s_\xi} \alpha^4 \beta^2 (\alpha^4 (s_0 - \\
& 3s_\xi)^2 - 2\alpha^3 (s_0 - 3s_\xi) (5 + \beta s_0 - \beta s_\xi) + 2\alpha \beta (-41 + 2\beta s_\xi (2 + \beta s_\xi) - 2\beta s_0 (5 + \beta s_\xi)) + \\
& 2\beta^2 (45 + 2\beta s_\xi (9 + \beta s_\xi)) + \alpha^2 (20 + \beta (\beta s_0^2 + 2s_0 (15 + \beta s_\xi) - s_\xi (74 + 11\beta s_\xi)))) + \\
& 4e^{\beta s_0 + 2\alpha s_\xi + 6\beta s_\xi} \alpha^2 \beta^3 (\alpha^6 (s_0 - 2s_\xi)^3 - 3\alpha^5 (s_0 - 2s_\xi)^2 (3 + \beta s_0 - \beta s_\xi) + 3\alpha \beta^2 (-56 + \\
& \beta s_\xi (5 + \beta s_\xi) (8 + \beta s_\xi) - \beta s_0 (42 + \beta s_\xi (12 + \beta s_\xi))) + \beta^3 (210 + \beta s_\xi (90 + \beta s_\xi (15 + \\
& \beta s_\xi))) + 3\alpha^4 (s_0 - 2s_\xi) (6 + \beta (\beta s_0^2 + s_0 (13 - \beta s_\xi) - s_\xi (22 + \beta s_\xi))) + 3\alpha^2 \beta (16 + \beta (\beta s_0^2 (7 + \\
& \beta s_\xi) + s_0 (72 - \beta^2 s_\xi^2) - s_\xi (128 + \beta s_\xi (21 + \beta s_\xi)))) - \alpha^3 (6 + \beta (\beta^2 s_0^3 + 3\beta s_0^2 (17 + \beta s_\xi) - \\
& 3s_0 (-36 + \beta s_\xi (48 + 5\beta s_\xi)) + s_\xi (-210 + \beta s_\xi (87 + 11\beta s_\xi)))))) \div (\lambda^4 (\alpha - \beta)^8)
\end{aligned}$$