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Scaling: a comment on Polsinelli and Kavvas (2015)

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Abstract. Polsinelli and Kavvas (2015) paper contributes to the understanding of scaling methods. I suggest to complement it, considering the wider perspective of Lie groups and the important problem of anomalous exponents that come from type-2 self similarity.

1 Introduction

As the title states, "A comparison of the modern Lie scaling method to classical scaling techniques", Polsinelli and Kavvas (2015) paper presents an important analysis of classical scaling methods with Lie group method illustrated by two main examples and other cases used to emphasize concepts. The first example is the Dupuit equation that describes the flow in a heterogeneous unconfined aquifer subject to a flux boundary condition. A second example is a linear 1-D contaminant transport problem. Among the cases they use, one can mention a confined aquifer to stress that a variable (velocity) is small and therefore advective effects are dominated by viscous effects. Also the cases of water infiltration into a hill slope during a rain event and of open channel flow are used to emphasize that the choice of variables in dimensional analysis depends on the possibility of measuring or controlling.

The main objective of the paper is clearly developed. Methods are fully described, examples are pertinent and well presented. Concepts are clear.

Of the three methods, I note that the inspection one is a minor variation of the classical scaling method. Moreover, the widespread attribution of the Π theorem to Buckingham should be straightened out as discussed here.

Beyond those two comments I have two suggestions. One is to acknowledge the generality of the Lie Group method. The second is to mention self similarity of the second kind in situations where the predictions based in classical dimensional analysis do not agree with observations.

It is true, as it is said in the paper, that an important goal of any scaling method is to predict information on one scale from known information to another scale. But scaling methods serve additional purposes. For instance, using similarity transformations it is possible to reduce the order of an equation, or to transform a partial differential equation into an ordinary one. In general, extracting the information from the scale symmetry of a problem can simplify a problem in various ways.

2 History of Scaling Methods

One of the greatest mathematicians in history (Arnold, 1998) tells us that his fellow Berry attributed to him a principle, the Arnold Principle, that states that "if a notion bears a personal name, then this name is not the name of the discoverer". Arnold's principle does apply to the Π theorem. It is true that Buckingham (1921) presented a version of the theorem and contributed to its popularization, but as Macagno (1971) has clarified, there are various precursors and probably three previous formulations of the theorem. Among the precursors one should mention Fourier (1878), Strutt-Lord-Rayleigh (1877-78) and Carvallo (1891). But clearly Vaschy (1892a,b) stated the theorem much earlier. Also Bertrand (1878) and Riabouchinsky (1911, 1915) probably independently arrived at some version of the theorem earlier than Buckingham. Despite the widespread use, I believe that Vaschy should be given due credit, for example, by calling it ".Vaschy-Buckingham" theorem.

3 Scaling Methods

The authors consider three methods: the classical scaling methodology based on the Π theorem, a modified inspectional analysis of scaling transformation and the Lie group method that considers the symmetries admitted by a system of equations. The second one is not more than a minor variation of the first one when considering a given system of equations. The result of this inspection analysis is an intuitive version of the Lie-group method. Presented this way, it may help with the clarification of a comparison between the other two methods.

4 Lie Group Analysis

The authors' presentation of the Lie method is correct, as it is what is needed for scaling analysis. This is, they present the so called local Lie group of one parameter stretching transformations (Logan, 1987, p. 447).

But the Lie group method is more general as it is not restricted to the scale symmetries but to all kind of symmetric transformations that leave a system of equations invariant (Lie, 1888). For instance, translation invariance, time invariance or rotational invariance. One of the important consequence of this is the Noether (1918) theorem that states that invariance of a system with respect to each symmetric transformation is equivalent to a conservation law. For instance, translation invariance is equivalent to linear moment conservation, rotational invariance to angular moment conservation, and time invariance to energy conservation.

Also, similarity transformations that leave an equation invariant have other important consequences with regard to the equation itself. That was the original motivation of Lie. The transformation may allow the solution of an otherwise difficult equation, or the reduction of its order, or the change from a partial to an ordinary differential equation (Bluman and Cole, 1974; Logan, 1987). The Lie-group method systematically considers all those difficult changes of variables that make an equation integrable. Of course, scaling is one of the important symmetries, but it is not the only one.

5 Types of Self-Similarity

Successful examples of application of scaling share a very important property that is not always emphasized. For those problems there is a clear way of separating the important variables from the ones that do not play a significant role because they are either too small or too large. Polsinelli and Kavvas (2015) made this point clear in various places, for instance in discussing the confined aquifer case, or the open channel case.

In those successful cases, one of the dimensionless numbers, say π_1 is small and therefore don't play a significant role in the problem. Mathematically this corresponds to the case that the limit of the function

$$\pi = \phi(\pi_1, \pi_2, \ldots, \pi_k)$$

that relates the non dimensional numbers goes to a non trivial limit when $\pi_1 \to 0$. Non trivial in this case is a limit different from 0 or ∞. These cases are classified as type-1 self similarity. For those cases it is possible to reduce the number of arguments of ϕ and to simplify the problem to

$$\pi = \phi(\pi_2, \ldots, \pi_k).$$

But there are cases in which π_1 is small but it continues to play a significant role in the limit. In such cases, it cannot be removed from the problem. It is possible to save the self similarity concept if one assumes the existence of a limit to 0 or ∞ in the form of a power function. Stated another way,, if for some real θ, the following asymptotic relation is true

$$\pi = \pi_1^\theta \phi(\pi_2, \ldots, \pi_k) + o(\pi_1^\theta).$$

Depending on the sign of θ the limit is to 0 or to ∞ as $\pi_1 \to 0$. But the exponent θ cannot be determined from dimensional analysis or from Lie group methods. It is called an anomalous exponent, for example, the fractal dimension. They lead to type-2 self similarity (Barenblatt, 1996). These exponents are common in empirical relations in Hydrology and Hydraulics, but they require theoretical explanations (Gupta and Mesa, 2014).

Among the areas of science with complete theories of anomalous exponents one should consider as a model the Statistical Mechanics of phase transitions, including the renormalization group technique (Goldenfeld, 1992).

6 Conclusions

Scaling is a very important problem in Science in general and in Hydrology in particular. Scaling comes from the simple idea that the laws of nature are invariant under scale contraction or amplification.. In particular they are independent of the arbitrarily chosen basic units of measurements. From this essential symmetry it is possible to derive the Π theorem and the classical scaling methods.

The comparison between classical methods of scale analysis and Lie group method presented in Polsinelli and Kavvas (2015) contributes to the understanding of the scale issue, but two important complements are necessary:

1. The Lie Group method is a general method for obtaining consequences from symmetries of a differential equation. Scaling is a particular type of symmetry that correspond to the local Lie group of one parameter stretching transformations. But this scaling symmetry is just one among the general symmetries. All the symmetries have general important consequences, for instance in the form of conservation laws.

2. Type-2 self similarity corresponds to scaling with anomalous exponents that require determination from considerations beyond dimensional analysis. Among the successful methods, the renormalization group method holds promise in Hydrology to solve long-standing open problems(Gupta and Mesa, 2014).

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