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Scaling: a comment on Polsinelli and Kavvas (2015)

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Abstract. Polsinelli and Kavvas (2015) paper contributes to the understanding of scaling methods. I suggest to complement it, considering the wider perspective of Lie groups and the important problem of anomalous exponents that come from type-2 self similarity.

1 Introduction

As the title states, "A comparison of the modern Lie scaling method to classical scaling techniques", Polsinelli and Kavvas (2015) paper presents an important analysis of classical scaling methods with Lie group method illustrated by two main examples and other cases used to emphasize concepts. The first example is the Dupuit equation that describes the flow in a heterogeneous unconfined aquifer subject to a flux boundary condition. A second example is a linear 1-D contaminant transport problem. Among the cases they use, one can mention a confined aquifer to stress that a variable (velocity) is small and therefore advective effects are dominated by viscous effects. Also the cases of water infiltration into a hill slope during a rain event and of open channel flow are used to emphasize that the choice of variables in dimensional analysis depends on the possibility of measuring or controlling.

The main objective of the paper is clearly developed. Methods are fully described, examples are pertinent and well presented. Concepts are clear.

My comments are about minor points. I believe that the widespread attribution of the Π theorem to Buckingham should be straighten. Of the three methods I find that the inspection one is a minor variation of the classical scaling method.

Beyond those comments I have two suggestions. One is to acknowledge the generality of the Lie Group method. The second is to mention self similarity of the second kind.

It is true, as it is said in the paper, that an important goal of any scaling method is to predict information on one scale from known information at another scale. But scaling methods serve additional purposes. For instance, using similarity transformations it is possible to reduce the order of an equation, or to transform a partial differential equation into an ordinary one. In general, extracting the information from the scale symmetry of a problem can simplify a problem in various ways.

2 History of Scaling Methods

One of the greatest mathematicians in history (Arnold, 1998) tell us that his fellow Berry attributed to him a principle, the Arnold Principle, that states that "if a notion bears a personal name, then this name is not the name of the discoverer". He immediately added that "The Arnold Principle is applicable to itself". Arnold's principle does apply to the Π theorem. It is true that Buckingham (1921) presented a version of the theorem and contributed to its popularization, but as Macagno (1971) has clarified, there are various precursors and probably three previous formulations of the theorem. Among the precursors one should mention Fourier (1878), Strutt-Lord-Rayleigh (1877-78) and Carvallo (1891). But clearly Vaschy (1892a,b) stated the theorem much earlier. Also Bertrand (1878) and Riabouchinsky (1911, 1915) probably independently arrived at some version of the teorem earlier than Buckingham. Despite the widespread use, I believe one should give due credit to Vaschy.

3 Scaling Methods

The authors consider three methods: the classical scaling methodology based on the Π theorem, a modified inspectional analysis of scaling transformation and the Lie group method that considers the symmetries admitted by a system of equations. The second one is not more than a minor variation of the first one when considering a given system of equations. The result of this inspection analysis is an intuitive version of the Lie method. I believe that presented this way, it may help clarification of the comparison between the other two methods.

4 Lie Group Analysis

The authors' presentation of the Lie method is correct, as it is what is needed for scaling analysis. This is, they present the so called local Lie group of one parameter stretching transformations (Logan, 1987, p. 447).

But the Lie group method is more general as it is not restricted to the scale symmetries but to all kind of symmetric transformations that leave invariant an equation system (Lie, 1888). For instance, translation invariance, time invariance or rotational invariance. One of the important consequence of this is the Noether (1918) theorem that states that invariance of a system with respect to each symmetric transformation is equivalent to a conservation law. For instance translation invariance is equivalent to linear moment conservation, rotational invariance to angular moment conservation, and time invariance to energy conservation.

Also, similarity transformations that leave invariant an equation have other important consequences with regard to the equation itself. That was the original motivation of Lie. The transformation may allow the solution of an otherwise difficult equation, or the reduction of its order or the change from a partial to an ordinary differential equation (Bluman and Cole, 1974; Logan, 1987). The Lie method systematically considers all those difficult and magic changes of variables that made integrable an equation.

Of course, scaling is one of the important symmetries, but it is not the only one.

5 Types of Self-Similarity

Successful examples of application of scaling share a ver important property that is not always emphasized. For those problems there is a clear way of separating the important variables from the ones that do not play a significant role because they are either too small or too large. Polsinelli and Kavvas (2015) made clear this point in various places, for instance discussing the confined aquifer case, or the open channel case.

In those successful cases, one of the dimensionless numbers, say π_1 is small and therefore don't play a significant role in the problem. Mathematically this corresponds to the case that the limit of the function

$$\pi = \phi(\pi_1, \pi_2, \ldots, \pi_k)$$

that relates the non dimensional numbers goes to a non trivial limit when $\pi_1 \to 0$. Non trivial in this case is a limit different from 0 or ∞. These cases are classified as type-1 self similarity. For those cases it is possible to reduce the number of arguments of ϕ and to simplify the problem to

$$\pi = \phi(\pi_2, \ldots, \pi_k).$$

But there are cases with trivial or non-existent limits. In the trivial case, it is possible to save the self similarity concept if one assumes the existence of a limit to 0 or ∞ in the form of a power function. This is, if for some real θ, the following asymptotic relation is true

$$\pi = \pi_1^\theta \phi(\pi_2, \ldots, \pi_k) + o(\pi_1^\theta).$$

Depending on the sign of θ the limit is to 0 or to ∞ as $\pi_1 \to 0$. But the exponent θ cannot be determined from dimensional analysis or from Lie group methods. It is called an anomalous exponents. These exponents are common in empirical relations in Hydrology and Hydraulics, but we still need to develop a theoretical explanations for those observations (Barenblatt, 1996; Gupta and Mesa, 2014). The meaning of type-2 self similarity is that one cannot discard a particular variable present in π_1. Despite being too small or too large it plays a significant role in the problem.

Among the areas of science with complete theories of anomalous exponents one should consider as a model the Statistical Mechanics of phase transitions, including the renormalization group technique (Goldenfeld, 1992).

6 Conclusions

Scaling is a very important problem in Science in general and in Hydrology in particular. Scaling comes from the simple idea that the laws of nature are independent of the observers. In particular they are independent of the arbitrarily chosen basic units of measurements. From this essential symmetry it is possible to derive the Π theorem and the classical scaling methods.

The comparison between classical methods of scale analysis and Lie group method presented in Polsinelli and Kavvas (2015) contributes to the understanding of the scale issue, but two important complements are necessary:

1. The Lie Group method is a general method for obtaining consequences from symmetries of a differential equation. Scaling is a particular type of symmetry that correspond to the local Lie group of one parameter stretching transformations. But this scaling symmetry is just one among the general symmetries. All the symmetries have general important consequences, for instance in the form of conservation laws.

2. Type-2 self similarity corresponds to scaling with anomalous exponents that require determination from considerations beyond dimensional analysis. Among the successful methods, the renormalization group method is worth exploring in Hydrology (Gupta and Mesa, 2014).

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