Responses to Comments of Referee #2

1 General comments

The authors have treated a difficult and complicated hydrological problem. The solution methods are of a standard mathematical nature, but by no means trivial. Their final solution becomes a triple sum where zeros of transcedental equations have to be calculated. Moreover, the factors for the horizontal contributions $F_x(\alpha_m, \bar{x})$ and $F_y(\beta_n, \bar{y})$ are independent, but the term $\Phi(\alpha_m, \beta_n, \bar{z}, \bar{t})$ depends on α_m and β_n by means of the variable $f = \alpha_m^2 + \kappa_z \beta_m^2$. This analytical solution belongs to Class 2 according the classification in Veling and Maas (2009).

The style of the paper is straightforward and the derivation in the Appendix is intelligible.

In their sensitivity analysis the authors give useful dimensionless expressions with criteria when to use which approximation for given circumstances and when an approximation is not appropriate. Their sensitivity analysis could be extended even further by treating the boundary conditions in a different way.

The authors do not give much information about the numerical evaluation of the found analytical expression other than some details how the zeros of the transcedental equations have been found. A validation of solution has not been supplied other than comparisons with other published solutions of simpler problems. It is possible to make choices for the parameters such that this solution should be equal to earlier published ones (*e.g.* the recharge area is the whole aquifer). In that way an independent, partial check of this solution could be possible.

Can the authors give information about the performance of their code (calculation times, convergence properties of the triple sums) and about the availability?

The general impression is a good piece of technical work based on well-established equations and boundary conditions for such cases. This solution based on the inclusion of equation (8) (time dependent first order free surface equation) for the chosen finite aquifer with a finite recharge domain seems to be new.

Response: Thanks for the comment. It is indeed an interesting work to reduce the present solution to earlier published ones or to show their equivalency/equality. To the best of our knowledge, there have been four existing analytical solutions dealing with similar topics to this note (Zlotnik and Ledder, 1993; Ramana et al., 1995; Manglik et al., 1997; Manglik and Rai, 1998). Unfortunately, our solution cannot reduce to the Zlotnik and Ledder (1993) solution because their solution is based on aquifers of infinite extent in the horizontal direction while ours considers aquifers of finite extent. Neither, the present solution cannot reduce to any of the other three solutions due to different mathematical representations of regional recharge. Those solutions regard recharge as a source term in two-dimensional flow equation and are thus independent of elevation z. On the other hand, our solution considers regional recharge as a boundary condition specified on the top of the aquifer (Please refer to Yeh and Yeh (2007) for discussing the differences in point-source and boundary-source solutions), triggering the vertical flow below the recharge area and making the flow field three dimensional. Nevertheless, the present solution and those four solutions can give the same hydraulic head prediction at observation points under certain conditions discussed in sections 3.1 − 3.4 in the previous manuscript.

We add following text in the revised manuscript to address convergence of the series in the present solution:

"The first term on the right-hand side (RHS) of Eq. (30) is a double series expanded by α_m and β_n. The series converges within a few terms because the power of α_m (or β_n) in the denominator of $\phi_{m,n}$ in Eq. (30a) is two more than that in the nominator. The second term on the RHS of Eq. (30) is a double series expanded by α_m and β_n, and the third terms is a triple series expanded by α_m, β_n, and λ_j. They converge very fast due to exponential functions in Eqs. (30b) and (30c). Consider $(m, n) \in (1, 2, \ldots, N = 30)$ and $j \in (1, 2, \ldots, N_j = 15)$ for the default values of dimensionless parameters and variables for calculation in Table 2. The number of terms in one or the other double series is 30 × 30 = 900 and in the triple series is 30 × 30 × 15 = 13500. The total number is therefore 900 × 2 + 13500 = 15300. We apply Mathematica FindRoot routine to obtain the values of α_m, β_n, and λ_j and Sum routine to compute the double and triple series. It takes about 8 seconds to finish calculation for $\bar{t} = 10^5$ by a personal computer with Intel Core i5-4590 3.30 GHz processor and 8 GB RAM. In addition, the series is considered to be converged when the absolute value of the last term in the double series of $\phi_{m,n}$ is smaller than 10^{-20} (i.e., $10^{-50} < 10^{-20}$ in this case). That value in the other double or triple series may be even smaller than 10^{-50} due to exponential decay." (lines 286-299, page 14)

At the end of Acknowledgements, we add the sentence "The computer software used to generate the results in Figures 2−6 is available upon request."

2 Some specific remarks

Page 12249, l. 9: No mention is made of the work of Bruggeman (1999, 360 BIII-6, from p. 321) for comparable solutions in a finite strip.

Response: Thanks, we insert the following sentence in the revised manuscript:

"Bruggeman (1999) introduced an analytical solution for 2D steady-state flow induced by localized recharge into a vertical strip aquifer between two Robin boundaries." (lines 85-86, page 5)

Page 12252, l. 24: The introduction of the distance d is unclear in the case that the location of the observation well has coordinates (x_w, y_w) with $x_w > x_1 + a$, $y_w > y_1 + b$ or $x_w > x_1 + a$, $y_w < y_1$ or $x_w < x_1$, $y_w > y_1 + b$ or $x_w < x_1$, $y_w < y_1$. What should be the distance in such cases:

$$\mathrm{d} = \min(|x_w - x - a|, |y_w - y_1 - b|, |x_w - x_1|, |y_w - y_1|)$$

or

$$\mathrm{d} = \min\left(\begin{array}{c}\sqrt{\left(x_w - (x_1 + a)\right)^2 + \left(y_w - (y_1 + b)\right)^2}, \sqrt{\left(x_w - (x_1 + a)\right)^2 + (y_w - y_1)^2}, \\ \sqrt{(x_w - x_1)^2 + (y_w - (y_1 + b))^2}, \sqrt{(x_w - x_1)^2 + (y_w - y_1)^2}\end{array}\right) ?$$

Response: Thanks for the comment. Following sentence is added to give an explicit definition of d in the revised manuscript: "The shortest distance between the edge of the region and an observation point at (x, y) is defined as $d = \min(\sqrt{(x - x_e)^2 + (y - y_e)^2})$ where (x_e, y_e) is the coordinate of the edge closest to the point." (lines 149-151, page 8)

Page 12254, l. 4: The symbol l for the recharge rate has been introduced earlier for the width in the x-direction of the rectangular aquifer.

Response: Thanks, this is a typo by the typesetter of this journal. We will correct it.

Page 12254: l. 12: Remark the way of scaling: with d in the horizontal plane and with B in the vertical plane.

Response: We inserted the following sentence after the dimensionless definitions in equation (9):

"Notice that the variables in the horizontal and vertical directions are divided by d and B, respectively."

Page 12257, l. 1: It should be better to label f as $f_{m,n}$ to make clear the dependency on α_m and β_n. In fact, also λ_j should be better $\lambda_{j,m,n}$. In the current presentation the solution looks simpler that it is really!

Response: Thanks, they have been changed as suggested. Please refer to the new expression of the present solution at the end of this reply.

Page 12258, l. 20: More explanation is needed for formula (23); specify a reference here for the use of

Duhamel's Principle. Very likely, in the denominator ξ should be $\xi_t(0)$.

Response: We added the reference "Singh (2005)". It is $1/\xi$ rather than $1/\xi_t(0)$ so that coefficient ξ in equation (30) at the end of this reply can be eliminated.

Page 12258, after Section 3.2: Some information could be given about the way the authors have treated the triple sum numerically. Did they use convergence accelerators?

Response: No, we did not use accelerators because the present solution converges very fast (i.e., only a few terms are needed to achieve good accuracy). Please refer to the first response for the discussion on series convergence.

Page 12261, l. 5: The mention of "Fig. 2" does not seem to be correct.

Response: Thanks for the comment. It has been deleted.

Page 12264, l. 18: The sensitivity analysis w.r.t. a: have the authors taken in consideration that by changing a also the scaling variable d changes too by the chosen location of the observation points/wells A and B?

Response: Variable d equals a fixed value of 5 m for the case of observation point A and 250 m for the case of observation point B in Figure 6 in the manuscript.

3 Some minor remarks

Page 12248, l. 24: Change "the" into "a".

Response: As suggested.

Page 12257, l. 1, formula (18o): It is more natural to introduce variables before and not after the introduction of the formulas where they are used explicitly. The same applies to formulas (18k) and (18m). As exhibited here in this paper the distance between use and definition is rather great.

Response: Thanks for the comment. The order of these equations are rearranged. The associated text is given at the end of this reply.

Page 12257, l. 11: Change "first and second" into " second and third".

Page 12257, l. 12: Change "third" into "first".

Response: The associated text is revised according to new arrangement of equations.

Page 12260, l. 7: Very likely, the authors mean $10^{-3}P_c$ in stead of $10^{-3}\Delta P_c$.

Page 12264, l. 10: Change "squire" into "square".

Page 12271:, l. 3: Change "cauchy" into "Cauchy".

Responses: We thank reviewer's eyes in detail. They have been revised as suggested.

4 References

References

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Manglik, A., Rai, S. N., and Singh, R. N.: Response of an Unconfined Aquifer Induced by Time Varying Recharge from a Rectangular Basin, Water Resour Manag, 11, 185-196, 1997.

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Zlotnik, V., and Ledder, G.: Groundwater velocity in an unconfined aquifer with rectangular areal recharge, Water Resour Res, 29, 2827-2834, 1993.

Section 2.2 and Appendix A in the revised manuscript

2.2 Analytical solution

The mathematical model, Eqs. (10) and (12) – (17), can be solved by the methods of Laplace transform and double-integral transform. The former transform converts $\bar{h}(\bar{x},\bar{y},\bar{z},\bar{t})$ into $\tilde{h}(\bar{x},\bar{y},\bar{z},p)$, $\partial\bar{h}/\partial\bar{t}$ into $p\tilde{h}-\bar{h}|_{\bar{t}=0}$, and $\xi\bar{u}_x\bar{u}_y$ into $\xi\bar{u}_x\bar{u}_y/p$ where p is the Laplace parameter and $\bar{h}|_{\bar{t}=0}$ equals zero in Eq. (11). After taking the transform, the model become a boundary value problem expressed as

$$\frac{\partial^2\tilde{h}}{\partial\bar{x}^2}+\kappa_y\frac{\partial^2\tilde{h}}{\partial\bar{y}^2}+\kappa_z\frac{\partial^2\tilde{h}}{\partial\bar{z}^2}=p\tilde{h} \tag{18}$$

with boundary conditions $\partial\tilde{h}/\partial\bar{x}-\kappa_1\tilde{h}=0$ at $\bar{x}=0$, $\partial\tilde{h}/\partial\bar{x}+\kappa_2\tilde{h}=0$ at $\bar{x}=\bar{l}$, $\tilde{h}/\partial\bar{y}-\kappa_3\tilde{h}=0$ at $\bar{y}=0$, $\tilde{h}/\partial\bar{y}+\kappa_4\tilde{h}=0$ at $\bar{y}=\bar{w}$, $\partial\tilde{h}/\partial\bar{z}=0$ at $\bar{z}=-1$, and $\partial\tilde{h}/\partial\bar{z}+\varepsilon p\tilde{h}/\kappa_z=\xi\bar{u}_x\bar{u}_y/p$ at $\bar{z}=0$. We then apply the properties of the double-integral transform to the problem. One can refer to the definition in Latinopoulos (1985, Table I, aquifer type 1). The transform turns $\tilde{h}(\bar{x},\bar{y},\bar{z},p)$ into $\hat{h}(\alpha_m,\beta_n,\bar{z},p)$, $\partial^2\tilde{h}/\partial\bar{x}^2+\kappa_y(\partial^2\tilde{h}/\partial\bar{y}^2)$ into $-(\alpha_m^2+\kappa_y\beta_n^2)\hat{h}$ where $(m, n) \in 1, 2, 3, \ldots \infty$, and eigenvalues α_m and β_n are the positive roots of the following equations that

$$\tan(\bar{l}\alpha_m)=\frac{\alpha_m(\kappa_1+\kappa_2)}{\alpha_m^2-\kappa_1\kappa_2} \tag{19}$$

and

$$\tan(\bar{w}\beta_n)=\frac{\beta_n(\kappa_3+\kappa_4)}{\beta_n^2-\kappa_3\kappa_4}. \tag{20}$$

In addition, $\bar{u}_x\bar{u}_y$ is transformed into U_mU_n given by

$$U_m=\frac{\sqrt{2}V_m}{\sqrt{\kappa_1+(\alpha_m^2+\kappa_1^2)[\bar{l}+\kappa_2/(\alpha_m^2+\kappa_2^2)]}} \tag{21}$$

$$U_n=\frac{\sqrt{2}V_n}{\sqrt{\kappa_3+(\beta_n^2+\kappa_3^2)[\bar{w}+\kappa_4/(\beta_n^2+\kappa_4^2)]}} \tag{22}$$

with

$$V_m=\{\kappa_1[\cos(\alpha_m\bar{x}_1)-\cos(\alpha_m\chi)]-\alpha_m[\sin(\alpha_m\bar{x}_1)-\sin(\alpha_m\chi)]\}/\alpha_m \tag{23}$$

$$V_n=\{\kappa_3[\cos(\beta_n\bar{y}_1)-\cos(\beta_n\psi)]-\beta_n[\sin(\beta_n\bar{y}_1)-\sin(\beta_n\psi)]\}/\beta_n \tag{24}$$

where $\chi=\bar{x}_1+\bar{a}$ and $\psi=\bar{y}_1+\bar{b}$.

Equation (18) then reduces to an ordinary differential equation as

$$\kappa_z\frac{\partial^2\hat{h}}{\partial\bar{z}^2}-(p+\alpha_m^2+\kappa_y\beta_n^2)\hat{h}=0 \tag{25}$$

Two boundary conditions are expressed, respectively, as

$$\partial\hat{h}/\partial\bar{z}=0 \quad \text{at} \quad \bar{z}=-1 \tag{26}$$

and

$$\frac{\partial \hat{h}}{\partial \bar{z}} + \frac{\varepsilon p}{\kappa_z}\hat{h} = \frac{\xi}{p}U_m U_n \quad \text{at} \quad \bar{z} = 0. \tag{27}$$

Solving Eq. (25) with Eqs. (26) and (27) results in

$$\hat{h}(\alpha_m, \beta_n, \bar{z}, p) = \frac{\xi U_m U_n \cosh[(1+\bar{z})\lambda]}{p[p\varepsilon\kappa_z \cosh\lambda + \kappa_z \lambda \sinh\lambda]} \tag{28}$$

where

$$\lambda = \sqrt{(p + \alpha_m^2 + \kappa_y \beta_n^2)/\kappa_z} \tag{29}$$

Inverting Eq. (28) to the space and time domains gives rise to the analytical solution that

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \xi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty} \phi_{j,m,n}) F_m F_n U_m U_n \tag{30}$$

with

$$\phi_{m,n} = \frac{\cosh[(1+\bar{z})\lambda_{m,n}]}{\kappa_z \lambda_{m,n} \sinh\lambda_{m,n}} \tag{30a}$$

$$\phi_{0,m,n} = -2\lambda_{0,m,n} \cosh[(1+\bar{z})\lambda_{0,m,n}] \exp(-\gamma_{0,m,n}\bar{t})/\eta_{0,m,n} \tag{30b}$$

$$\phi_{j,m,n} = -2\lambda_{j,m,n} \cos[(1+\bar{z})\lambda_{j,m,n}] \exp(-\gamma_{j,m,n}\bar{t})/\eta_{j,m,n} \tag{30c}$$

$$\eta_{0,m,n} = \gamma_{0,m,n}[(1+2\varepsilon\kappa_z)\lambda_{0,m,n}\cosh\lambda_{0,m,n} + \left(1 - \varepsilon\gamma_{0,m,n}\right)\sinh\lambda_{0,m,n}] \tag{30d}$$

$$\eta_{j,m,n} = \gamma_{j,m,n}[(1+2\varepsilon\kappa_z)\lambda_{j,m,n}\cos\lambda_{j,m,n} + \left(1 - \varepsilon\gamma_{j,m,n}\right)\sin\lambda_{j,m,n}] \tag{30e}$$

$$\lambda_{m,n} = \sqrt{f_{m,n}/\kappa_z};\ \gamma_{0,m,n} = f_{m,n} - \kappa_z \lambda_{0,m,n}^2;\ \gamma_{j,m,n} = f_{m,n} + \kappa_z \lambda_{j,m,n}^2 \tag{30f}$$

$$f_{m,n} = \alpha_m^2 + \kappa_y \beta_n^2 \tag{30g}$$

$$F_m = \frac{\sqrt{2}[\alpha_m \cos(\alpha_m \bar{x}) + \kappa_1 \sin(\alpha_m \bar{x})]}{\sqrt{\kappa_1 + (\alpha_m^2 + \kappa_1^2)[\bar{l} + \kappa_2/(\alpha_m^2 + \kappa_2^2)]}} \tag{30h}$$

$$F_n = \frac{\sqrt{2}[\beta_n \cos(\beta_n \bar{y}) + \kappa_3 \sin(\beta_n \bar{y})]}{\sqrt{\kappa_3 + (\beta_n^2 + \kappa_3^2)[\bar{w} + \kappa_4/(\beta_n^2 + \kappa_4^2)]}} \tag{30i}$$

where $j \in 1, 2, 3, \ldots\infty$ and eigenvalues $\lambda_{0,m,n}$ and $\lambda_{j,m,n}$ are determined, respectively, by the following equations that

$$\frac{-\varepsilon\kappa_z \lambda_{0,m,n}^2 + \lambda_{0,m,n} + \varepsilon f_{m,n}}{\varepsilon\kappa_z \lambda_{0,m,n}^2 + \lambda_{0,m,n} - \varepsilon f_{m,n}} = \exp(2\lambda_{0,m,n}) \tag{31}$$

and

$$\tan\lambda_{j,m,n} = -\varepsilon(f_{m,n} + \kappa_z \lambda_{j,m,n}^2)/\lambda_{j,m,n}. \tag{32}$$

The method to find α_m, β_n, $\lambda_{0,m,n}$ and $\lambda_{j,m,n}$ is introduced in section 2.3. One can refer to Appendix A for the derivation of Eq. (30).

Appendix A: Derivation of Eq. (30)

Let us start with function $G(p)$ from Eq. (28) that

$$G(p) = \frac{\cosh[(1+\bar{z})\lambda]}{p(p\varepsilon\kappa_z \cosh\lambda + \kappa_z\lambda\sinh\lambda)} \quad \text{(A1)}$$

with

$$\lambda = \sqrt{(p + f_{m,n})/\kappa_z} \quad \text{(A2)}$$

where $f_{m,n} = \alpha_m^2 + \kappa_y\beta_n^2$. Equation (A1) is a single-value function to p in the complex plane because satisfying $G(p^+) = G(p^-)$ where p^+ and p^- are the polar coordinates defined, respectively, as

$$p^+ = r_a \exp(i\theta) - f_{m,n} \quad \text{(A3)}$$

and

$$p^- = r_a \exp[i(\theta - 2\pi)] - f_{m,n} \quad \text{(A4)}$$

where r_a represents a radial distance from the origin at $p = -f_{m,n}$, $i = \sqrt{-1}$ is the imaginary unit, and θ is an argument between 0 and 2π. Substitute $p = p^+$ in Eq. (A3) into Eq. (A2), we have

$$\lambda = \sqrt{r_a/\kappa_z}\exp(i\theta/2) = \sqrt{r_a/\kappa_z}\,[\cos(\theta/2) + i\sin(\theta/2)] \quad \text{(A5)}$$

Similarly, we can have

$$\lambda = \sqrt{r_a/\kappa_z}\exp[i(\theta - 2\pi)/2] = -\sqrt{r_a/\kappa_z}\,[\cos(\theta/2) + i\sin(\theta/2)]. \quad \text{(A6)}$$

after p in Eq. (A2) is replaced by p^- in Eq. (A4). Substitution of Eqs. (A3) and (A5) into Eq. (A1) yields the same result as that obtained by substituting Eqs. (A4) and (A6) into Eq. (A1), indicating that Eq. (A1) is a single-value function without branch cut and its inverse Laplace transform equals the sum of residues for poles in the complex plane.

The residue for a simple pole can be formulated as

$$Res = \lim_{p\to\varphi} G(p)\exp(p\bar{t})\,(p - \varphi) \quad \text{(A7)}$$

where φ is the location of the pole of $G(p)$ in Eq. (A1). The $G(p)$ has infinite simple poles at the negative part of the real axis in the complex plane. The locations of these poles are the roots of equation that

$$p(p\varepsilon\kappa_z\cosh\lambda + \kappa_z\lambda\sinh\lambda) = 0 \quad \text{(A8)}$$

which is obtained by letting the denominator in Eq. (A1) to be zero. Obviously, one pole is at $p = 0$, and its residue based on Eqs. (A1) and (A7) with $\lambda_{m,n} = \sqrt{f_{m,n}/\kappa_z}$ can be expressed as

$$\phi_{m,n} = \cosh\left[(1+\bar{z})\lambda_{m,n}\right]/(\kappa_z\lambda_{m,n}\sinh\lambda_{m,n}) \quad \text{(A9)}$$

The locations of other poles of $G(p)$ are the roots of the equation that

$$p\varepsilon\kappa_z \cosh\lambda + \kappa_z\lambda\sinh\lambda = 0 \quad \text{(A10)}$$

which is the expression in the parentheses in Eq. (A8). One pole is between $p = 0$ and $p = -f_{m,n}$. Let $\lambda = \lambda_{0,m,n}$, and Eq. (A2) becomes $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$. Substituting $\lambda = \lambda_{0,m,n}$, $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$, $\cosh\lambda_{0,m,n} = [\exp\lambda_{0,m,n} + \exp(-\lambda_{0,m,n})]/2$ and $\sinh\lambda_{0,m,n} = [\exp\lambda_{0,m,n} - \exp(-\lambda_{0,m,n})]/2$ into Eq. (A9) and rearranging the result leads to Eq. (31). The pole is at $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$ with a numerical value of $\lambda_{0,m,n}$. With Eq. (A1), Eq. (A7) equals

$$Res = \lim_{p\to\varphi} \frac{\cosh[(1+\bar{z})\lambda]}{p(p\varepsilon\kappa_z\cosh\lambda + \kappa_z\lambda\sinh\lambda)}\exp(p\bar{t})\,(p-\varphi) \quad \text{(A11)}$$

Apply L'Hospital's Rule to Eq. (A11), and then we have

$$Res = \lim_{p\to\varphi} \frac{-2\lambda\cosh[(1+\bar{z})\lambda]}{p[(1+2\varepsilon\kappa_z)\lambda\cosh\lambda + (1-\varepsilon p)\sinh\lambda]}\exp(p\bar{t}) \quad \text{(A12)}$$

The residue for the pole at $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$ can be defined as

$$\phi_{0,m,n} = \frac{-2\lambda_{0,m,n}\cosh[(1+\bar{z})\lambda_{0,m,n}]\exp(-\gamma_{0,m,n}\bar{t})}{\gamma_{0,m,n}[(1+2\varepsilon\kappa_z)\lambda_{0,m,n}\cosh\lambda_{0,m,n} + (1-\varepsilon\gamma_{0,m,n})\sinh\lambda_{0,m,n}]} \quad \text{(A13)}$$

which is obtained by Eq. (A12) with $\lambda = \lambda_{0,m,n}$ and $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2 = \gamma_{0,m,n}$. On the other hand, infinite poles behind $p = -f_{m,n}$ are at $p = \gamma_{j,m,n}$ where $j \in 1, 2, \dots \infty$. Let $\lambda = \sqrt{-1}\lambda_{j,m,n}$, and Eq. (A2) yields $p = -f_{m,n} - \kappa_z\lambda_{j,m,n}^2$. Substitute $\lambda = \sqrt{-1}\lambda_{j,m,n}$, $p = -f_{m,n} - \kappa_z\lambda_{j,m,n}^2$, $\cosh(\sqrt{-1}\lambda_{j,m,n}) = \cos\lambda_{j,m,n}$ and $\sinh(\sqrt{-1}\lambda_{j,m,n}) = \sqrt{-1}\sin\lambda_{j,m,n}$ into Eq. (A9) and rearrange the result, and then we have Eq. (32). These poles are at $p = -f_{m,n} - \kappa_z\lambda_{j,m,n}^2$ with numerical values of $\lambda_{j,m,n}$. On the basis of Eq. (A12) with $\lambda = \sqrt{-1}\lambda_{j,m,n}$ and $p = -f_{m,n} - \kappa_z\lambda_{j,m,n}^2 = \gamma_{j,m,n}$, the residues for these poles at $p = -f_{m,n} - \kappa_z\lambda_{j,m,n}^2$ can be expressed as

$$\phi_{j,m,n} = \frac{-2\lambda_{j,m,n}\cos[(1+\bar{z})\lambda_{j,m,n}]\exp(-\gamma_{j,m,n}\bar{t})}{\gamma_{j,m,n}[(1+2\varepsilon\kappa_z)\lambda_{j,m,n}\cos\lambda_{j,m,n} + (1-\varepsilon\gamma_{j,m,n})\sin\lambda_{j,m,n}]} \quad \text{(A14)}$$

where $j \in 1, 2, \dots \infty$. As a result, the inverse Laplace transform for Eq. (A1) is the sum of Eqs. (A9) and (A13) and a simple series expended in the RHS function in Eq. (A14) with $j \in 1, 2, \dots \infty$ (i.e., $\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty}\phi_{j,m,n}$). Finally, Eq. (30) can be derived after taking the inverse double-integral transform for the result using the formula that (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x},\bar{y},\bar{z},\bar{t}) = \xi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty}\phi_{j,m,n})F_m F_n U_m U_n \quad \text{(A15)}$$

where ξ and $U_m U_n$ result from $\xi U_m U_n$ in Eq. (28).