Responses to Comments of Referee #1

In its present form, the paper mostly appears as a good piece of algebraic development, and along this line, corresponds typically to a technical note. However, a few concerns make me feel that the writing is not raised to something braking from previous attempts to model unconfined groundwater flow.

First. Contrary to the idea concealed in the paper, I am not convinced at all that analytical solutions are in general the reference tool for concrete case applications. Usually, analytical solutions drastically simplify the problem when concrete applications are faced with complex situations. Todays, concrete application turn toward (eventually simplified) numerical resolutions of groundwater flow simply because these approaches can handle complex geometry, medium heterogeneity, coupling vadose and saturated zone, etc. The point is that analytical solutions are still useful as reference for numerical model and/or to assess the relevance of some second-order mechanisms added to numerical models.

Response: We agree that analytical solutions are reference tools for applications with complicated situations or serve as the primary means for testing and benchmarking numerical models or assessing the relevance of some second-order mechanisms included in numerical models. In addition, they have advantages mentioned below as compared with the numerical methods:

1. They generally require less data than numerical schemes and are numerically stable, efficient, and easy to implement although they are limited to specific cases due to their simplifying assumptions.
2. They can easily be used to explore physical insights of the flow behavior affected by the aquifer properties, boundary, and/or surface recharge. Some new findings related to flow behavior are given below as example based on the present solution:

(1) A quantitative criterion is provided to assess the validity of neglecting the effect of the vertical flow. Such a practice of ignoring vertical flow was very commonly made in previous articles (e.g., Rao and Sarma, 1980; Rai et al., 1998; Chang and Yeh, 2007; Illas et al., 2008). Please refer to the 2nd conclusion in the previous manuscript for detail.

(2) The assumption of incompressibility is valid when the ratio of the specific yield to the storage coefficient is larger than 100. Otherwise, it leads to significant overestimation in predicting the hydraulic head. Please see the 3rd conclusion in the previous manuscript.

3. The sensitivity analysis based on the analytical solutions can determine which parameters are relatively critical to the success of a management plan (see, for example, Aguado and Sitar, 1977) or to investigate the source of inaccuracy in parameter estimation (e.g., Huang and Yeh, 2012).

4. If coupling with an optimization approach, they can identify the hydraulic parameters in aquifer test data analyses. For example, Yeh and Chen (2007) integrated a slug test solution for a well with a finite-thickness skin with the simulated annealing to determine the hydraulic parameters of the skin and formation zones.

Second. I doubt on the unconfined behavior of the aquifer modeled by the solutions of the authors. For the purpose of simplification, the analytical solution is based on two equations, namely, a diffusion equation corresponding to a confined system plus an additional equation for the free water surface only accounting for fluxes from the recharge over a limited area of the modeled domain. It is obvious that this dichotomy does not represent the continuity of flow between the vadose and the saturated zone that makes the unconfined systems so complicated. The authors would have been well advised to provide us with a comparison between their solution (and its simplifications) and a full three-dimensional resolution of the Richards equation for both the saturated and the vadose zone. The point is not to state that an analytical solution is able to deal with all the physics of flow, rather to show explicitly why the simplifications needed for building an analytical solution are reasonably acceptable.

Response: Thanks for the comment. Analysis of three-dimensional saturated and unsaturated flows based on Richards' equation and soil characteristic curve is indeed an interesting and challenging work, but this is obviously beyond the scope of this note. Tartakovsky and Neuman (2007) developed a semi-analytical solution for unsaturated and saturated flows toward a discharge well in an unconfined aquifer. Their solution is based on Richards' equation with the relative hydraulic conductivity defined as $k_0 = \exp(-\kappa z)$ where κ is a parameter and z is elevation from water table. The solution agrees well with the Neuman (1974) solution based on the same problem but neglecting the effect of unsaturated flow when $\kappa B = 10^3$ with the initial aquifer thickness B. To some extent, the present work is similar to the Neuman (1974) solution but differs from the aspect that our solution regards regional recharge as a plane source while Neuman's solution considers the pumping as a line sink. We may therefore infer from the work of Tartakovsky and Neuman (2007) that the present solution may also be valid if the condition of $\kappa B \geq 10^3$ is held. The following text is added in the revised manuscript (lines 180-187, page 9) to address the conditions of using the present solution.

"On the other hand, the effect of unsaturated flow on model's predictions can be ignored when $\kappa B \geq 10^3$ where κ is a parameter to define the relative hydraulic conductivity as $k_0 = \exp(-\kappa z)$ in the Richards' equation (Tartakovsky and Neuman, 2007). Tartakovsky and Neuman (2007) achieved agreement on aquifer drawdown evaluated by their analytical solution based on Eq. (1) for saturated flow and Richards' equation for unsaturated flow and by the Neuman (1974) solution based on Eqs. (1) and (8) with $I = 0$ when $\kappa B = 10^3$ (i.e., the case of $\kappa_D = 10^3$ in Fig. 2 in Tartakovsky and Neuman, 2007)."

Third. In association with the concern above, the provided analytical solution is compared with other analytical solutions grounded in the same theoretical framework. This way of doing usually goes with some self-satisfaction attitude because progresses are incremental and never work against the proposed methodology. Again, we would have been better informed if the proposed analytical solution had been faced with a (numerical) three-dimensional resolution of flow. It is now well known that solving a three-dimensional Richards equation with the problem of swapping between the unconfined non-saturated zone and the confined saturated zone is crux to model unconfined aquifers, especially when recharge is evoked as a condition triggering transient flow. Stated differently, one can be still interested in analytical solutions but it is mandatory to know when to apply them, what do they hide, and which (eventually useless) mechanism is overlooked. As an aside comment, we still seek for the usefulness of mixed boundary conditions when the paper only deals with the Dirichlet type of boundary condition.

Response: Unfortunately, it seems that the existing numerical solutions for 3D saturated and unsaturated flow were developed for some purposes (e.g., Dogan and Motz, 2005; Cey et al., 2006; Hunt et al., 2008; An et al., 2010; An et al., 2012) which were irrelevant to this study and therefore impossible to make comparison with the present solution. As regard to the use of the Robin boundary condition (RBC), we would like to mention that it is defined as a weighted combination of Dirichlet boundary condition and Neumann boundary condition while the mixed boundary condition (MBC) represents the boundary which changes its condition along a particular boundary, say from a Dirichlet condition to a Neumann condition (Duffy, 2008, page 1). Thus, the RBC and MBC are completely different types. In our study, we have adopted the RBC to describe flow across a boundary of a stratum having low permeability and investigate its effect on the hydraulic head at an observation point as described in section 3.1. It is clear that the Robin condition should be considered for the boundary under the condition $10^{-2} < K_1 d_1/(K_x b_1) < 10^2$ where K_1 and b_1 are the hydraulic conductivity and width of the medium at the boundary 1 illustrated in Figure 1(a), respectively, K_x is the aquifer hydraulic conductivity in the x direction normal to the boundary, and d_1 is a distance between the boundary and the edge of a recharge area. Note that the Robin condition reduces to Dirichlet condition when $K_1 d_1/(K_x b_1) \geq 10^2$ and the no-flow condition when $K_1 d_1/(K_x b_1) \leq 10^{-2}$.

Four. Technically speaking, the manuscript may appear unclear at some places. The first question raised by reading the mathematical development is the motivation to choose a distance from a well as the reference for building dimensionless coordinates in space. What if no well existed? Why not to build theses dimensionless variables by taking the size (along x and/or y directions) of the domain? Is there any incompatibility by doing so on the emergence of an analytical solution? A second concern is about the sensitivity of the solution to parameters. The authors delineate it as a first order-approximation (finite difference) of the derivatives of the solution with respect to (log) parameters. This calculation is de facto sensitive to the increment δp added to the parameter p when approximating dF/dp as [F(p+δp) – F(p)]/δp. My understanding is that the analytical solution is a double or a triple sum of elementary functions. Derivatives of a sum being sum of derivatives, why not to derive directly the analytical solution with respect to parameters? My first guess is that all the elementary functions enclosed in the solution are differentiable with respect to their parameters, with the consequence that an "exact" sensitivity evaluation could come out by directly differentiating the analytical solution. Notably, the sensitivity analysis performed in the paper is irrelevant. Calculating derivatives with respect to parameters is always local, with the meaning that the differentiation is performed in the vicinity of a prescribed value of the parameter. Conclusions on model sensitivity are thus local and only valid close to the prescribed values of the parameters. These values are not reported in the paper and a relevant way to analyze the sensitivity would be to duplicate calculations at several points in the parameter space. A third concern is about the appendix which is in my opinion hard to read when it should be limpid. The reader is continuously invited to swap between the writing in the appendix and the equations in the main text. This does not help to understand how the authors technically proceeded for building their analytical solution. My standpoint regarding this feature would be to either remove the appendix, or give it some flesh to document the reader and avoid him back and forth motions in the reading plus hard time to pass from eq. n to eq. n+1.

Response: Thanks for the comment. Our responses to those concerns are given below:

1. The term "observation well" is changed to "observation point" for avoiding confusion. The distance *d* between the edge of a recharge area and the observation point is chosen to define the dimensionless parameter $\kappa_z = K_z\, d^2/(K_x\, B^2)$ where *B* is the initial aquifer thickness and K_x and K_z are the aquifer hydraulic conductivities in the *x* and *z* directions, respectively. The parameter κ_z indicates that both K_z/K_x and d^2/B^2 are crucial factors in neglecting the effect of vertical flow on the hydraulic head. This parameter is similar to the one defined in Neuman (1975) as $\beta = K_z\, r^2/(K_r\, H^2)$ with K_r

representing the hydraulic conductivity in the radial direction and r denoting a radial distance measured from a pumping well to an observation point. He used this parameter to examine the validity of neglecting the effect of vertical flow on transient drawdown at the observation point (see Figure 1 in Neuman, 1975).

2. Direct differentiation of the solution with respect to each of the parameters is practically feasible. Yet, some of the results for parameters such as S_y, S_s, K_x, K_z, and K_1 are lengthy and in complicated forms. In addition, it is laborious to derive the sensitivity coefficients since we have seven parameters in total. The sensitivity coefficients based on the first-order finite differences give very good approximations to those obtained by direct differentiation. In addition, the curves of sensitivity coefficients show very clearly pictures exhibiting the relative strength and influential period of the impact of parameter change on the hydraulic head. The parameter values we choose and listed in Table 2 (in manuscript) are reasonable for sandy aquifers, which are suitable formations for groundwater exploitation. One might expect that different sets of parameter values for sandy aquifers should also provide similar sensitivity patterns to ours. The conclusions on the sensitivity analysis in section 3.5 should therefore be valid for different magnitudes of hydraulic parameters. It is worth noting that the patterns of sensitivity curves are somewhat different as shown in Figure 6 because Figure 6(a) is for three-dimensional flow while Figure 6(b) is for two-dimensional flow.
3. The derivation for the present solution mentioned in section 2.2 and given in Appendix A has been rewritten according to the comment and also given at the end of this reply.

Five. Even though I am not native speaker of English, I found a text riddled every ten lines with grammatical inconsistencies, awkward phrasings, etc. In any case, the manuscript would deserve pinpoint editing by a professional service. In its present form, the text is not completely clear and editing would probably improve readability.

Response: The manuscript has carefully been edited by a colleague who is good at English writing.

Finally, I found the paper interesting because the technique concealed in it is undoubtedly sound. The main concern is that the authors missed the target of showing us the added-value of their contribution. They partly kick in touch by comparing their results with those they inherit from. At least, the paper deserves a rigorous editing before publication. Nevertheless, my feeling is still that a relevant paper in a reputed journal such as HESS should argue on the advantages and drawbacks brought by the study. In its present form, the study only brings advantages by flawed comparisons between quite similar approaches. I would recommend to reject the paper in its present form but encourage the authors for a complete re-submission following the philosophy depicted above.

Response: Thank for the comment. We have addressed the issue of the restrictions (or drawbacks) of the present solution by inserting the following sentence in Conclusions of the revised manuscript.

"The present solution is applicable under the conditions of aquifer homogeneity and $|h|/B < 0.5$, $I/K_z < 0.2$ and $\kappa B \geq 10^3$ due to the neglect of unsaturated zone (Marino, 1967; Tartakovsky and Neuman, 2007)." (lines 445-447, page 20)

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Section 2.2 in the revised manuscript

2.2 Analytical solution

The mathematical model, Eqs. (10) and (12) – (17), can be solved by the methods of Laplace transform and double-integral transform. The former transform converts $\bar{h}(\bar{x},\bar{y},\bar{z},\bar{t})$ into $\tilde{h}(\bar{x},\bar{y},\bar{z},p)$, $\partial\bar{h}/\partial\bar{t}$ into $p\tilde{h}-\bar{h}|_{\bar{t}=0}$, and $\xi\bar{u}_x\bar{u}_y$ into $\xi\bar{u}_x\bar{u}_y/p$ where p is the Laplace parameter and $\bar{h}|_{\bar{t}=0}$ equals zero in Eq. (11). After taking the transform, the model become a boundary value problem expressed as

$$\frac{\partial^2\tilde{h}}{\partial\bar{x}^2}+\kappa_y\frac{\partial^2\tilde{h}}{\partial\bar{y}^2}+\kappa_z\frac{\partial^2\tilde{h}}{\partial\bar{z}^2}=p\tilde{h} \tag{18}$$

with boundary conditions $\partial\tilde{h}/\partial\bar{x}-\kappa_1\tilde{h}=0$ at $\bar{x}=0$, $\partial\tilde{h}/\partial\bar{x}+\kappa_2\tilde{h}=0$ at $\bar{x}=\bar{l}$, $\tilde{h}/\partial\bar{y}-\kappa_3\tilde{h}=0$ at $\bar{y}=0$, $\tilde{h}/\partial\bar{y}+\kappa_4\tilde{h}=0$ at $\bar{y}=\bar{w}$, $\partial\bar{h}/\partial\bar{z}=0$ at $\bar{z}=-1$, and $\partial\bar{h}/\partial\bar{z}+\varepsilon p\tilde{h}/\kappa_z=\xi\bar{u}_x\bar{u}_y/p$ at $\bar{z}=0$. We then apply the properties of the double-integral transform to the problem. One can refer to the definition in Latinopoulos (1985, Table I, aquifer type 1). The transform turns $\tilde{h}(\bar{x},\bar{y},\bar{z},p)$ into $\hat{h}(\alpha_m,\beta_n,\bar{z},p)$, $\partial^2\tilde{h}/\partial\bar{x}^2+\kappa_y(\partial^2\tilde{h}/\partial\bar{y}^2)$ into $-(\alpha_m^2+\kappa_y\beta_n^2)\hat{h}$ where $(m, n)\in 1, 2, 3, \dots\infty$, and eigenvalues α_m and β_n are the positive roots of the following equations that

$$\tan(\bar{l}\alpha_m) = \frac{\alpha_m(\kappa_1+\kappa_2)}{\alpha_m^2-\kappa_1\kappa_2} \tag{19}$$

and

$$\tan(\bar{w}\beta_n) = \frac{\beta_n(\kappa_3+\kappa_4)}{\beta_n^2-\kappa_3\kappa_4}. \tag{20}$$

In addition, $\bar{u}_x\bar{u}_y$ is transformed into U_mU_n given by

$$U_m = \frac{\sqrt{2}V_m}{\sqrt{\kappa_1+(\alpha_m^2+\kappa_1^2)[\bar{l}+\kappa_2/(\alpha_m^2+\kappa_2^2)]}} \tag{21}$$

$$U_n = \frac{\sqrt{2}V_n}{\sqrt{\kappa_3+(\beta_n^2+\kappa_3^2)[\bar{w}+\kappa_4/(\beta_n^2+\kappa_4^2)]}} \tag{22}$$

with

$$V_m = \{\kappa_1[\cos(\alpha_m\bar{x}_1) - \cos(\alpha_m\chi)] - \alpha_m[\sin(\alpha_m\bar{x}_1) - \sin(\alpha_m\chi)]\}/\alpha_m \tag{23}$$

$$V_n = \{\kappa_3[\cos(\beta_n\bar{y}_1) - \cos(\beta_n\psi)] - \beta_n[\sin(\beta_n\bar{y}_1) - \sin(\beta_n\psi)]\}/\beta_n \tag{24}$$

where $\chi = \bar{x}_1 + \bar{a}$ and $\psi = \bar{y}_1 + \bar{b}$.

Equation (18) then reduces to an ordinary differential equation as

$$\kappa_z \frac{\partial^2\hat{h}}{\partial\bar{z}^2} - (p + \alpha_m^2 + \kappa_y\beta_n^2)\hat{h} = 0 \tag{25}$$

Two boundary conditions are expressed, respectively, as

$$\partial\hat{h}/\partial\bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{26}$$

and

$$\frac{\partial\hat{h}}{\partial\bar{z}} + \frac{\varepsilon p}{\kappa_z}\hat{h} = \frac{\xi}{p}U_mU_n \quad \text{at} \quad \bar{z} = 0. \tag{27}$$

Solving Eq. (25) with Eqs. (26) and (27) results in

$$\hat{h}(\alpha_m, \beta_n, \bar{z}, p) = \frac{\xi U_m U_n \cosh[(1+\bar{z})\lambda]}{p[p\varepsilon\kappa_z\cosh\lambda+\kappa_z\lambda\sinh\lambda]} \tag{28}$$

where

$$\lambda = \sqrt{(p + \alpha_m^2 + \kappa_y\beta_n^2)/\kappa_z} \tag{29}$$

Inverting Eq. (28) to the space and time domains gives rise to the analytical solution that

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \xi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty}\phi_{j,m,n})F_mF_nU_mU_n \tag{30}$$

with

$$\phi_{m,n} = \frac{\cosh[(1+\bar{z})\lambda_{m,n}]}{\kappa_z\lambda_{m,n}\sinh\lambda_{m,n}} \tag{30a}$$

$$\phi_{0,m,n} = -2\lambda_{0,m,n}\cosh[(1+\bar{z})\lambda_{0,m,n}]\exp(-\gamma_{0,m,n}\bar{t})/\eta_{0,m,n} \tag{30b}$$

$$\phi_{j,m,n} = -2\lambda_{j,m,n}\cos[(1+\bar{z})\lambda_{j,m,n}]\exp(-\gamma_{j,m,n}\bar{t})/\eta_{j,m,n} \quad (30c)$$

$$\eta_{0,m,n} = \gamma_{0,m,n}[(1+2\varepsilon\kappa_z)\lambda_{0,m,n}\cosh\lambda_{0,m,n} + \left(1-\varepsilon\gamma_{0,m,n}\right)\sinh\lambda_{0,m,n}] \quad (30d)$$

$$\eta_{j,m,n} = \gamma_{j,m,n}[(1+2\varepsilon\kappa_z)\lambda_{j,m,n}\cos\lambda_{j,m,n} + \left(1-\varepsilon\gamma_{j,m,n}\right)\sin\lambda_{j,m,n}] \quad (30e)$$

$$\lambda_{m,n} = \sqrt{f_{m,n}/\kappa_z};\ \gamma_{0,m,n} = f_{m,n} - \kappa_z\lambda_{0,m,n}^2;\ \gamma_{j,m,n} = f_{m,n} + \kappa_z\lambda_{j,m,n}^2 \quad (30f)$$

$$f_{m,n} = \alpha_m^2 + \kappa_y\beta_n^2 \quad (30g)$$

$$F_m = \frac{\sqrt{2}[\alpha_m\cos(\alpha_m\bar{x})+\kappa_1\sin(\alpha_m\bar{x})]}{\sqrt{\kappa_1+(\alpha_m^2+\kappa_1^2)[\bar{l}+\kappa_2/(\alpha_m^2+\kappa_2^2)]}} \quad (30h)$$

$$F_n = \frac{\sqrt{2}[\beta_n\cos(\beta_n\bar{y})+\kappa_3\sin(\beta_n\bar{y})]}{\sqrt{\kappa_3+(\beta_n^2+\kappa_3^2)[\bar{w}+\kappa_4/(\beta_n^2+\kappa_4^2)]}} \quad (30i)$$

where $j \in 1, 2, 3, \ldots\infty$ and eigenvalues $\lambda_{0,m,n}$ and $\lambda_{j,m,n}$ are determined, respectively, by the following equations that

$$\frac{-\varepsilon\kappa_z\lambda_{0,m,n}^2+\lambda_{0,m,n}+\varepsilon f_{m,n}}{\varepsilon\kappa_z\lambda_{0,m,n}^2+\lambda_{0,m,n}-\varepsilon f_{m,n}} = \exp(2\lambda_{0,m,n}) \quad (31)$$

and

$$\tan\lambda_{j,m,n} = -\varepsilon(f_{m,n} + \kappa_z\lambda_{j,m,n}^2)/\lambda_{j,m,n}. \quad (32)$$

The method to find α_m, β_n, $\lambda_{0,m,n}$ and $\lambda_{j,m,n}$ is introduced in section 2.3. One can refer to Appendix A for the derivation of Eq. (30).

Appendix A: Derivation of Eq. (30)

Let us start with function $G(p)$ from Eq. (28) that

$$G(p) = \frac{\cosh[(1+\bar{z})\lambda]}{p(p\varepsilon\kappa_z\cosh\lambda+\kappa_z\lambda\sinh\lambda)} \quad (A1)$$

with

$$\lambda = \sqrt{(p+f_{m,n})/\kappa_z} \quad (A2)$$

where $f_{m,n} = \alpha_m^2 + \kappa_y\beta_n^2$. Equation (A1) is a single-value function to p in the complex plane because satisfying $G(p^+) = G(p^-)$ where p^+ and p^- are the polar coordinates defined, respectively, as

$$p^+ = r_a\exp(i\theta) - f_{m,n} \quad (A3)$$

and

$$p^- = r_a\exp[i(\theta-2\pi)] - f_{m,n} \quad (A4)$$

where r_a represents a radial distance from the origin at $p = -f_{m,n}$, $i = \sqrt{-1}$ is the imaginary unit, and θ is an argument between 0 and 2π. Substitute $p = p^+$ in Eq. (A3) into Eq. (A2), we have

$$\lambda = \sqrt{r_a/\kappa_z}\exp(i\theta/2) = \sqrt{r_a/\kappa_z}\,[\cos(\theta/2) + i\sin(\theta/2)] \quad \text{(A5)}$$

Similarly, we can have

$$\lambda = \sqrt{r_a/\kappa_z}\exp[i(\theta - 2\pi)/2] = -\sqrt{r_a/\kappa_z}\,[\cos(\theta/2) + i\sin(\theta/2)]. \quad \text{(A6)}$$

after p in Eq. (A2) is replaced by p^- in Eq. (A4). Substitution of Eqs. (A3) and (A5) into Eq. (A1) yields the same result as that obtained by substituting Eqs. (A4) and (A6) into Eq. (A1), indicating that Eq. (A1) is a single-value function without branch cut and its inverse Laplace transform equals the sum of residues for poles in the complex plane.

The residue for a simple pole can be formulated as

$$Res = \lim_{p\to\varphi} G(p)\exp(p\bar{t})\,(p - \varphi) \quad \text{(A7)}$$

where φ is the location of the pole of $G(p)$ in Eq. (A1). The $G(p)$ has infinite simple poles at the negative part of the real axis in the complex plane. The locations of these poles are the roots of equation that

$$p(p\varepsilon\kappa_z\cosh\lambda + \kappa_z\lambda\sinh\lambda) = 0 \quad \text{(A8)}$$

which is obtained by letting the denominator in Eq. (A1) to be zero. Obviously, one pole is at $p = 0$, and its residue based on Eqs. (A1) and (A7) with $\lambda_{m,n} = \sqrt{f_{m,n}/\kappa_z}$ can be expressed as

$$\phi_{m,n} = \cosh\left[(1 + \bar{z})\lambda_{m,n}\right]/(\kappa_z\lambda_{m,n}\sinh\lambda_{m,n}) \quad \text{(A9)}$$

The locations of other poles of $G(p)$ are the roots of the equation that

$$p\varepsilon\kappa_z\cosh\lambda + \kappa_z\lambda\sinh\lambda = 0 \quad \text{(A10)}$$

which is the expression in the parentheses in Eq. (A8). One pole is between $p = 0$ and $p = -f_{m,n}$. Let $\lambda = \lambda_{0,m,n}$, and Eq. (A2) becomes $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$. Substituting $\lambda = \lambda_{0,m,n}$, $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$, $\cosh\lambda_{0,m,n} = [\exp\lambda_{0,m,n} + \exp(-\lambda_{0,m,n})]/2$ and $\sinh\lambda_{0,m,n} = \left[\exp\lambda_{0,m,n} - \exp(-\lambda_{0,m,n})\right]/2$ into Eq. (A9) and rearranging the result leads to Eq. (31). The pole is at $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$ with a numerical value of $\lambda_{0,m,n}$. With Eq. (A1), Eq. (A7) equals

$$Res = \lim_{p\to\varphi}\frac{\cosh[(1+\bar{z})\lambda]}{p(p\varepsilon\kappa_z\cosh\lambda + \kappa_z\lambda\sinh\lambda)}\exp(p\bar{t})\,(p - \varphi) \quad \text{(A11)}$$

where $\lambda = \sqrt{(p + f_{m,n})/\kappa_z}$. Apply L'Hospital's Rule to Eq. (A11), and then we have

$$Res = \lim_{p\to\varphi}\frac{-2\lambda\cosh[(1+\bar{z})\lambda]}{p[(1+2\varepsilon\kappa_z)\lambda\cosh\lambda + (1-\varepsilon p)\sinh\lambda]}\exp(p\bar{t}) \quad \text{(A12)}$$

The residue for the pole at $p = -f_{m,n} + \kappa_z\lambda_{0,m,n}^2$ can be defined as

$$\phi_{0,m,n} = \frac{-2\lambda_{0,m,n}\cosh[(1+\bar{z})\lambda_{0,m,n}]\exp(-\gamma_{0,m,n}\bar{t})}{\gamma_{0,m,n}[(1+2\varepsilon\kappa_z)\lambda_{0,m,n}\cosh\lambda_{0,m,n} + (1-\varepsilon\gamma_{0,m,n})\sinh\lambda_{0,m,n}]} \quad \text{(A13)}$$

which is obtained by Eq. (A12) with $\lambda = \lambda_{0,m,n}$ and $p = -f_{m,n} + \kappa_z \lambda_{0,m,n}^2 = \gamma_{0,m,n}$. On the other hand, infinite poles behind $p = -f_{m,n}$ are at $p = \gamma_{j,m,n}$ where $j \in 1, 2, \ldots \infty$. Let $\lambda = \sqrt{-1}\lambda_{j,m,n}$, and Eq. (A2) yields $p = -f_{m,n} - \kappa_z \lambda_{j,m,n}^2$. Substitute $\lambda = \sqrt{-1}\lambda_{j,m,n}$, $p = -f_{m,n} - \kappa_z \lambda_{j,m,n}^2$, $\cosh\left(\sqrt{-1}\lambda_{j,m,n}\right) = \cos\lambda_{j,m,n}$ and $\sinh\left(\sqrt{-1}\lambda_{j,m,n}\right) = \sqrt{-1}\sin\lambda_{j,m,n}$ into Eq. (A9) and rearrange the result, and then we have Eq. (32). These poles are at $p = -f_{m,n} - \kappa_z \lambda_{j,m,n}^2$ with numerical values of $\lambda_{j,m,n}$. On the basis of Eq. (A12) with $\lambda = \sqrt{-1}\lambda_{j,m,n}$ and $p = -f_{m,n} - \kappa_z \lambda_{j,m,n}^2 = \gamma_{j,m,n}$, the residues for these poles at $p = -f_{m,n} - \kappa_z \lambda_{j,m,n}^2$ can be expressed as

$$\phi_{j,m,n} = \frac{-2\lambda_{j,m,n}\cos[(1+\bar{z})\lambda_{j,m,n}]\exp(-\gamma_{j,m,n}\bar{t})}{\gamma_{j,m,n}[(1+2\varepsilon\kappa_z)\lambda_{j,m,n}\cos\lambda_{j,m,n} + (1-\varepsilon\gamma_{j,m,n})\sin\lambda_{j,m,n}]} \tag{A14}$$

where $j \in 1, 2, \ldots \infty$. As a result, the inverse Laplace transform for Eq. (A1) is the sum of Eqs. (A9) and (A13) and a simple series expended in the RHS function in Eq. (A14) with $j \in 1, 2, \ldots \infty$ (i.e., $\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty}\phi_{j,m,n}$). Finally, Eq. (30) can be derived after taking the inverse double-integral transform for the result using the formula that (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \xi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\phi_{m,n} + \phi_{0,m,n} + \sum_{j=1}^{\infty}\phi_{j,m,n}\right)F_m F_n U_m U_n \tag{A15}$$

where ξ and $U_m U_n$ result from $\xi U_m U_n$ in Eq. (28).