Hydrol. Earth Syst. Sci. Discuss., 12, C4266–C4284, 2015
www.hydrol-earth-syst-sci-discuss.net/12/C4266/2015/

Hydrology and Earth System Sciences
Open Access
Discussions

Interactive comment on "Technical note: Analytical solution for the mean drawdown of steady state pumping tests in two-dimensional isotropic heterogeneous aquifers" *by* A. Zech and S. Attinger

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Received and published: 15 October 2015

Response to Comment of S.P. Neuman on "Technical note: Analytical solution for the mean drawdown of steady state pumping tests in two-dimensional isotropic heterogeneous aquifers" by A. Zech and S. Attinger

We thank S.P. Neuman for the valuable comments. His response shows that the derivation of the Radial Coarse Graining (RCG) approach was not displayed sufficiently clear

in the manuscript and previous publications on that topic. Therefore, we will recapitulate the concept, including theoretical background, approximations as well as heuristic steps in more detail in this response. In order give the reader the possibility of having a closer look to the concept of RCG, a condensed version is planned to be added to the manuscript, as presented later.

The referee objects versus the publication of the work due to "fundamental inconsistencies" in the basic approach of RCG. In order to clarify that point, we start by a detailed description of the RCG approach in general. Afterwards we recapitulate the comments of the referee in detail (given in italic) and respond to the individual aspects he mentioned with reference to the detailed concept description. The final section is dedicated to the planned changes in the manuscript. We present a condensed description of the RCG approach to be added to section 2 of the manuscript.

Radial Coarse Graining Approach

The concept of Radial Coarse Graining can be best explained within five major steps:

1. Concept of Coarse Graining for uniform flow
2. Transfer of Coarse Graining to radial flow
3. Overcome non-locality for non-uniform flow
4. Derive effective hydraulic conductivity for well flow
5. Derivation of effective well flow head

Step 1: Coarse Graining for Uniform Flow

The basic concept of Coarse Graining for flow in porous media was introduced by *Attinger* [2003]. The author describes the upscaling procedure for uniform flow. The result is an upscaled log-normally distributed hydraulic conductivity field $K_\lambda^{\mathsf{CG}}(\vec{x})$, which is coarsened according to a cut-off value λ. Fluctuation smaller than λ are filtered out, fluctuation at a scale larger than the filter width λ are still resolved. The derivation included multiple points:

1. Starting point is the steady state head distribution for single-phase, incompressible flow through a heterogeneous medium: $-\nabla\left(K(\vec{x})\nabla\phi(\vec{x})\right) = \rho(\vec{x})$, with $K(\vec{x})$ being the hydraulic conductivity, $\phi(\vec{x})$ the hydraulic head and $\rho(\vec{x})$ the source/sink term in d-dimensional space.

2. Hydraulic conductivity is modelled as spatial random function with a log-normal distribution: $K(\vec{x}) = K_0 \exp f(\vec{x})$, with $f(\vec{x})$ being normally distributed. It is further assumed, that $K(\vec{x}) = \bar{K} + \tilde{K}(\vec{x})$ can be separated into a constant mean value $\bar{K}$ and a spatially depending fluctuation term $\tilde{K}(\vec{x})$ with zero mean.

3. A filter function $\langle . \rangle_\lambda$ is defined based on the parameter λ, which resolves all fluctuation larger than λ and filters out those smaller than λ.

4. The filter is applied to the head equation and thus to the hydraulic head distribution, which results from the stochastic description of the hydraulic conductivity. The filtered head equation is than solved in Fourier space.

5. After mathematical treatment, the author results in an expression whose "Fourier-back-transform [...] yields a non-local resolution dependent hydraulic conductivity tensor as found as well by *Neuman and Orr* [1993]". The expression is simplified by evaluating the hydraulic conductivity tensor at a specific point in Fourier space $\vec{q} = 0$, "which corresponds to localization in the work of *Neuman and Orr* [1993]."

6. The Fourier back-transformation of the filtered head equation reads in real space after localization:

$$-\nabla\left(\bar{K} + \langle\tilde{K}(\vec{x})\rangle_\lambda\right)\nabla\langle\phi(\vec{x})\rangle_\lambda + \delta\hat{K}^{\mathsf{eff}}(\vec{q}=0,\lambda)\nabla\langle\phi(\vec{x})\rangle_\lambda = \langle\rho(\vec{x})\rangle_\lambda \ , \quad (1)$$

   where $\delta\hat{K}^{\mathsf{eff}}$ is the scale-dependent effective hydraulic conductivity tensor which is induced by small scale heterogeneities varying on typical scales smaller than λ.

7. The upscaled hydraulic conductivity for the filtered head equation reads $K_\lambda^{\mathsf{CG}}(\vec{x}) = K^{\mathsf{eff}}(\lambda) + \langle\hat{K}(\vec{x})\rangle_\lambda$, where the effective mean value is $K^{\mathsf{eff}}(\lambda) = \bar{K} + \delta\hat{K}^{\mathsf{eff}}(\vec{q}=0,\lambda)$ in lowest order perturbation.

8. Explicit results for $\delta\hat{K}^{\mathsf{eff}}$ are first evaluated in lowest order perturbation. Then, renormalization group analysis is applied by extending the calculations to higher order perturbation theory. *Attinger* [2003] results in the closed form expression for the effective mean value

$$K^{\mathsf{eff}}(\lambda) = K_{\mathsf{G}} \exp\left(\sigma^2\left(\frac{1}{2} - \frac{1}{d}\right)\right)\exp\left(\frac{1}{d}\sigma^2\left(\frac{\ell^2}{\ell^2 + \lambda^2/4}\right)^{d/2}\right) \ , \quad (2)$$

   where σ^2 is the variance and ℓ is the correlation length of the unfiltered hydraulic conductivity distribution $K(\vec{x})$; d is the dimension of space.

9. The filtered hydraulic conductivity distribution $K_\lambda^{\mathsf{CG}}(\vec{x})$ is still a log-normal distributed quantity. *Attinger* [2003] showed, that the variance $\langle\sigma^2\rangle_\lambda$ and the correlation length $\langle\ell\rangle_\lambda$ of the filtered field $K_\lambda^{\mathsf{CG}}(\vec{x})$ can be expressed by the variance σ^2 and the correlation length ℓ of the unfiltered conductivity field $K(\vec{x})$:

$$\langle\sigma^2\rangle_\lambda \equiv \sigma^2\left(\frac{\ell^2}{\ell^2 + \lambda^2/4}\right)^{d/2} \quad \langle\ell\rangle_\lambda \equiv \left(\ell^2 + \lambda^2/4\right)^{1/2} \quad (3)$$

Step 2: Transfer of Coarse Graining Approach to Radial Flow and the Problem of Non-locality for Non-uniform Flow

The basic concept of Coarse Graining can be applied to non-uniform flow straight ahead. The critical step when transferring the results of *Attinger* [2003] to radial flow is the localization of the non-local resolution dependent hydraulic conductivity tensor $\delta\hat{K}^{\mathsf{eff}}$ (point 5, step 1).

For uniform flow, the assumption of constant pressure gradient $\nabla\phi(x)$ holds at least at the coarser scale. The gradient term and the non-local resolution dependent hydraulic conductivity tensor can be separated. And localization can be applied to $\delta\hat{K}^{\mathsf{eff}}$. For non-uniform flow, the basic assumption of constant flux is not valid any more.

Step 3: Heuristic Approach to Overcome Non-Locality

A heuristic approach is taken to overcome the limitation of non-locality. The basic idea it to ask, how to achieve a quasi-constant pressure gradient, in order to allow a localization. For well flow, the answer is given by adapting the size of the volume elements over which flow takes place, which corresponds to a change in the coordinate system.

The head gradient in well flow is proportional to the reciprocal of the distance r to the well: $\nabla h(r) = \frac{h(r)-h(r+\Delta r)}{\Delta r} \propto \frac{1}{r}$. The gradient is constant for volumes of size proportional to r. Thus, a new coordinate system with a constant head gradient must have cells with increasing volume relative to r. Figures 1 and 2 gives an illustration of the modified coordinate system.

The changed coordinate system impacts on the scaling procedure and the parameter λ. For uniform flow, λ is constant in a equidistant Cartesian coordinate system. Adapted to well flow, the scaling parameter needs to be proportional to r, because the

filter width increases with distance to the well.

Under the assumption of the adapted coordinate system the localization can be performed and so the following steps of the Coarse Graining procedure. The consequence for the upscaled hydraulic conductivity field is a change in the scaling parameter $\lambda = 2\zeta r$, where ζ is a factor of proportionality. The mean value $K^{\mathsf{eff}}(r)$ is radially depending and the filtered log-normal distributed field in d-dimensions reads

$$K_r^{\mathsf{RCG}}(\vec{x}) = K^{\mathsf{eff}}(r)+\langle\tilde{K}(\vec{x})\rangle_r = K_{\mathsf{G}}\exp\left(\sigma^2\left(\frac{1}{2}-\frac{1}{d}\right)\right)\exp\left(\frac{1}{d}\sigma^2\left(\frac{1}{1+\zeta^2 r^2/\ell^2}\right)^{d/2}\right)+\langle\tilde{K}(\vec{x}) \tag{4}$$

The filtered field $K_r^{\mathsf{RCG}}(\vec{x})$ can be understood as an upscaled hydraulic conductivity, which gives the same drawdown behaviour under well flow conditions as the originally unfiltered hydraulic conductivity field $K(\vec{x})$. The filtered field $K_r^{\mathsf{RCG}}(\vec{x})$ still contains spatial heterogeneity and local fluctuations, but reduced to the amount relevant to the pumping test. The coarsening is constructed to filter only those information out, which are not seen by the pumping test.

The step was presented by *Schneider and Attinger* [2008]. It is not performed in a mathematically straight way, but problem adapted to well flow conditions.

Step 4: From Spatially Variable towards Effective Hydraulic Conductivity

Spatial heterogeneity is still resolved in $K_r^{\mathsf{RCG}}(\vec{x})$, although reduced to the amount relevant to the pumping test. Aiming at deriving an effective description of the hydraulic conductivity for well flow condition, a further step of averaging is necessary. Thereby two different aspects are of interest: (i) an effective description of well flow conductivity for an ensemble and (ii) effective description of well flow conductivity for an individual field.

A result for an effective ensemble description can be derived by averaging $K_r^{\mathsf{RCG}}(\vec{x})$ appropriate to well flow condition. The averaging rule is determined by the boundary condition at the well.

In the following, we focus on 2D since the manuscript refers to two-dimensional well flow. Hydraulic conductivity is replaced by its depth average, the transmissivity T. The averaging rule at the well is given by the harmonic mean [*Dagan*, 1989]. The harmonic mean RCG-transmissivity can be calculated via the theoretical description making use of the variance of the coarsened transmissivity $\langle\sigma^2\rangle_r$ from Eq. (3) adapted to well flow with $\lambda = 2\zeta r$:

$$T_H^{\mathsf{RCG}}(r) = T_{\mathsf{G}} \exp\left(-\langle\sigma^2\rangle_r/2\right) = T_{\mathsf{G}} \exp\left(-\frac{1}{2}\frac{\sigma^2}{1+\zeta^2 r^2/\ell^2}\right) \ . \tag{5}$$

The general procedure is identical for three-dimensional anisotropic hydraulic conductivity as discussed in *Zech et al.* [2012].

An effective description of well flow transmissivity for an individual field can be derived from $T_H^{\mathsf{RCG}}(r)$. The behaviour of individual fields is different especially at the well due to a lack of ergodicity there. The local transmissivity at the well T_{well} is not identical to the harmonic mean T_{H} as expected for the ensemble, but refers to the specific value of transmissivity at the well location. An adapted radial coarse graining transmissivity accounts for local effects by replacing the harmonic mean $T_{\mathsf{H}} = T_{\mathsf{G}}\exp\left(-\frac{1}{2}\sigma^2\right)$ by T_{well}. In Eq. (5) this refers to substituting the variance by $-\frac{1}{2}\sigma^2 = \ln T_{\mathsf{well}} - \ln T_{\mathsf{G}}$.

Step 5: Effective Well Flow Head

The final step of the RCG approach is the derivation of the effective well flow head $h^{\mathsf{efw}}(r)$. The effective RCG-transmissivity (Eq. 5) is inserted to the determinis-

tic well head equation,

$$0 = \left(\frac{1}{r} + \frac{\mathrm{d}\ln T_H^{\mathsf{RCG}}(r)}{\mathrm{d}r}\right)\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2 h}{\mathrm{d}r^2} \ . \tag{6}$$

The analytical solution of Eq. (6) for two-dimensional flow is given by $\mathsf{h}_{\mathsf{efw}}(r) = -\frac{Q_w}{4\pi T_{\mathsf{G}}}\exp\left(\frac{\sigma^2}{2}\right)\left(\Gamma\left(\frac{\sigma^2}{2}\frac{-\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2}\frac{-\zeta^2 R^2/\ell^2}{1+\zeta^2 R^2/\ell^2}\right)\right)$ $+ \frac{Q_w}{4\pi T_{\mathsf{G}}}\left(\Gamma\left(\frac{\sigma^2}{2}\frac{1}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2}\frac{1}{1+\zeta^2 R^2/\ell^2}\right)\right) + h_R$, where Q_w is the pumping rate, T_{G} is the geometric mean, σ^2 is the log-transmissivity variance, and ℓ is the correlation length; ζ is the factor of proportionality determined to be 1.6 and R is an arbitrary distance to the well, where the hydraulic head $h(R) = h_R$ is known. Γ is the exponential integral $\Gamma(x) = \int_\infty^x \frac{\exp(z)}{z}\,\mathrm{d}z$.

The effective well flow head for the ensemble behaviour can be adapted to single realizations similar to the RCG transmissivity. The harmonic mean T_{H} is replaced by the local transmissivity at the well T_{well} by substituting $-\frac{1}{2}\sigma^2 = \ln T_{\mathsf{well}} - \ln T_{\mathsf{G}}$ in Eq. ().

Step 5 derived for two-dimensional well flow (Eq.) represents the achievement of the Technical Note under consideration (manuscript section 2.3, appendix). Steps 1 is published in *Attinger* [2003], steps 2-4 are presented in the work of *Schneider and Attinger* [2008] for 2D. Steps 4 and 5 are discussed in the work of *Zech et al.* [2012] for well flow in 3D. The latter were able to show the appropriateness of the approach by comparison with numerical pumping test simulations. The same for 2D well flow is one major issue of the current manuscript.

Detailed Response to Referee's Comments

In this technical note the authors (a) develop an analytical solution for mean steady state drawdown under horizontal flow to a well withdrawing water from a randomly heterogeneous aquifer at a constant rate and (b) suggest ways to evaluate properties of aquifer transmissivity on the basis of measured drawdowns. Their analysis is based on a Radial Coarse Graining (RCG) approach described in Schneider and Attinger [2008]. It considers two versions of coarse grained transmissivity, termed ensemble and local, given in parametric form as functions of radial distance to the well. The authors then propose ways to determine the corresponding parameters on the basis of measured drawdowns.

To properly review this note for HESS I found it necessary to study the above work of Schneider and Attinger (SA). Here I discovered what appear to be fundamental inconsistencies in their RCG approach. The development in SA starts with a stochastic representation of 2D steady flow in a random transmissivity field toward the well, subject to deterministic inner and outer boundary conditions. As we all know, this stochastic head equation embodies two physical principles, conservation of (incompressible) water volume and Darcy's law. RCG a la SA consists of upscaling transmissivities through weighted spatial averaging with a weight function that depends on radial distance. The resulting spatially averaged transmissivity is considered to be deterministic. Replacing transmissivity in the original stochastic equation with its upscaled version thus renders this equation deterministic in what the authors consider, and label, RCG drawdown. It is this "RCG" equation that Zech and Attinger rely on in the technical note under review.

Unfortunately, the latter RCG equation is not consistent with the two physical principles on which the original stochastic equation rests. To preserve these principles SA should have applied RCG to the original stochastic head equation, not just to transmissivity. Averaging the original equation would have resulted in a modified head equation, preserving the underlying physics, but including a new integro-differential term with an

integrand that contains both transmissivity and hydraulic gradient. This non-local cross term would be equivalent to the residual flux term in the probabilistically averaged stochastic head equation of Neuman and Orr [1993]. By (inadvertently?) dropping this mixed integro-differential term, SA have introduced a bias into their resulting RCG head equation the magnitude of which could be large or small, depending on circumstances. We know from subsequent numerical solutions of the Neuman and Orr stochastic moment equations that ignoring their residual flux, as has been common in the stochastic literature, may result in unjustifiably large biases.

With the detailed recapitalization of the concept of RCG we aim to elucidate the "fundamental inconsistencies" of the Radial Coarse Graining approach derived by *Schneider and Attinger* [2008] as mentioned by the referee. As the referee describes, the derivation of the RCG-transmissivity starts with a stochastic representation of transmissivity as spatial random function with a log-normal distribution. The major step consists of upscaling transmissivities through weighted spatial averaging with a weight function that depends on radial distance, resulting in a deterministic spatially averaged transmissivity.

As a first point, we want to specify that *"the spatially averaged transmissivity"* is not replaced in the original stochastic equation. The RCG-drawdown is the result of the deterministic head equation considering the transmissivity not constant, but radial depending. Thus, the *"RCG-equation"* as called by the referee, is an independent physical equation and thereby embodies the two physical principles, conservation of water volume and Darcy's law. The equation is not meant to replace the original stochastic equation and thereby does not refer to two physical principles of that equation.

The procedure of upscaling was applied to the head equation with spatially variable transmissivity (step 1). As mentioned by the referee, the averaging procedure leads to a new non-local integro-differential term (step 2). This term is not dropped, but treated in an heuristic way (step 3) to result in a filtered transmissivity field, which still resolves spatial heterogeneity, but at coarser scale. In order to gain an effective well flow transmissivity, spatial averaging is applied. The result is a mean RCG-transmissivity depending on the distance to the well, but not containing local fluctuation (step 4). The mean RCG transmissivity is not meant to fulfil the original head equation, but is constructed to reproduce the ensemble mean drawdown of pumping tests in heterogeneous media, in dependence of the statistical parameters of the underlying log-normally distributed hydraulic conductivity/transmissivity fields. This drawdown description (effective well flow solution) is derived by solving the head equation under well flow condition with the effective mean RCG-transmissivity (step 5 and section 2.3 in the manuscript). The closed form description of the effective well flow head enables to estimate the parameters of aquifer statistics by comparison with simulated and/or measured drawdowns.

We are aware that the adaption of Coarse Graining from uniform to radial flow conditions was not performed in a rigorous mathematical way, but includes heuristics steps, which are well-considered and adapted to the problem at hand. The assumptions made are verified by numerical proof. The effective well flow head is compared with mean drawdowns of simulated pumping tests in heterogeneous media. The very well match confirmed the appropriateness of the conjectures taken in the RCG-approach.

Since it appears that the derivation of the Radial Coarse Graining approach was not displayed sufficiently clear in the manuscript and previous publications, we aim to extend the manuscript with a condensed form of the concept description as given in the previous section. It is planned to expand section 2 with an additional subsection on the basic concept of coarse graining (then section 2.2) and a modification accordingly of the section "Radial coarse graining transmissivity" (than section 2.3). A draft version of the text is given in the next section. Minor adaption in the introduction and the other subsection of section 2 are further necessary.s

A lesser but not insignificant issue with RCG is the treatment of RCG transmissivity as deterministic: there is nothing in the SA approach to guarantee that weighted volume averaging of randomly varying transmissivity would itself not be random, albeit with a

lesser variance (but longer correlation scales).

We agree with the referee, that the weighted volume average of a randomly varying transmissivity is itself random, with a lesser variance but longer correlation scales. This is exactly what is stated by *Attinger* [2003] (step 1, point 9). The same is valid for Radial Coarse Graining (step 3). However, the derivation of an effective mean transmissivity for well flow includes averaging (step 4), which renders the RCG-transmissivity deterministic.

On a minor note, it would have been fair for Zech and Attinger to juxtapose their proposed pumping test interpretation method with that of Neuman et al. [2004].

Neuman et al. [2004] presented a graphical approach to estimate the statistical parameters of random transmissivity on the basis of steady state head data. The authors constructed a mathematical description for the apparent transmissivity $T_{\mathsf{a}}(r)$ as function of the radial distance to the well r. In line with theoretical findings of *Dagan* [1989] they defined $T(r) = T_{\mathsf{G}}$ for $r \geq 2\ell$ and $T(r) \to T_{\mathsf{H}} = T_{\mathsf{G}} \exp\left(-\sigma^2/2\right)$ for $r \to r_w$. In between $r_w \leq r \leq 2\ell\, T_{\mathsf{a}}(r)$ was approximated by a cubic polynomial. From $T_{\mathsf{a}}(r)$ the authors derived the hydraulic head drawdown and constructed type curves for the hydraulic head, depending on the variance σ^2 and ℓ. They further gave a multi-point strategy to analyze 'measured' drawdown data by type curve matching, including the estimation of T_{G}, ℓ and σ^2. Finally, *Neuman et al.* [2004] tested their proposed approach against head data corresponding to a series of synthetic pumping test in heterogeneous aquifers.

We thank the referee for the advise to recapitulate the results of *Neuman et al.* [2004] and juxtapose the work to ours. It helped to improve the manuscript significantly, especially with respect to the multi-point strategy to analyze 'measured' drawdown data. It inspired us to do a similar analysis, which is planned to be added to the manuscript. Details are provided in the response to the other referee.

Our approach is similar to that of *Neuman et al.* [2004] in the idea of deriving an ensemble mean drawdown for well flow depending on the statistics of the random transmissivity and testing it against simulated ensembles of pumping tests in heterogeneous media as synthetic 'measured' data. Starting point is in both cases a radial depending transmissivity, interpolating between the harmonic mean at the well and the geometric mean for the far field. The major differences between both approaches are:

1. The description of radial depending transmissivity in the RCG approach is not constructed based on results of Monte Carlo simulations. The RCG transmissivity is derived by upscaling, with physically motivated approximations.
2. The functional form of the RCG-transmissivity $T_H^{\mathsf{RCG}}(r)$ is different from the expression for T_{a} of *Neuman et al.* [2004].
3. The corresponding RCG-hydraulic head (effective well flow solution $h_{\mathsf{efw}}(r)$) is derived analytically by solving the head equation. Thus, $h_{\mathsf{efw}}(r)$ is given in a closed form mathematical expression.
4. 'Measured' drawdowns do not have to be matched to type curves, but are analyzed by minimizing the residuals. The analysis of the 'measured' data is straight ahead and parameters like T_{G}, σ^2 and ℓ are estimated directly.

The method of *Neuman et al.* [2004] will be juxtapose to our proposed pumping test interpretation in the introduction. A paragraph will be included, stating the approach of *Neuman et al.* [2004] and the differences to our approach, as given above in a condensed form. Furthermore, *Neuman et al.* [2004] stated similar regions of impact for variance and correlation length. Their results support our findings, that $T(r) = T_{\mathsf{G}}$ for $r \geq 2\ell$ and $T(r) \rightarrow T_{\mathsf{H}}$ for $r \rightarrow r_w$. A statement on that will be added to section 2.4.

I regret that, given the above fundamental inconsistencies, I cannot recommend publication of the technical note by Zech and Attinger in HESS.

We hope we could clarify the points mentioned by the referee as "fundamental inconsistencies". We are aware of the fact, that the RCG-approach is not a mathematical

rigorous one, but a problem-adapted. Since the comparison with numerical simulations showed an extremely good match between the RCG-hydraulic head solution and the simulated ensemble mean drawdown, we feel, that our approach has its value for 2D pumping test interpretation.

Additional Text for Manuscript

The following two subsection are drafts for a modified version of section 2.2 in the manuscript.

Concept of Radial Coarse Graining

A radially depending transmissivity for log-normally distributed media with Gaussian correlation structure was derived by *Schneider and Attinger* [2008], denoted as $T_{RCG}(r)$ based on an upscaling approach called Radial Coarse Graining. The basic idea the approach is to perform a spatial filtering of the flow equation which is appropriate to the non-uniform flow character of a pumping test. The concept of Radial Coarse Graining can be best explained within five major steps:

1. Coarse Graining for uniform flow
2. Transfer of Coarse Graining to radial flow conditions
3. Overcome non-locality of head equation for non-uniform flow
4. Effective Radial Coarse Graining transmissivity (section 2.3)
5. Derivation of effective well flow head (section 2.4)

The Coarse Graining approach for uniform flow (step 1) was introduced by *Attinger* [2003], including derivation, mathematical proof and numerical simulations. The author start at a spatially variable transmissivity field $T(\vec{x})$ and derive a filtered version $T_\lambda^{\mathsf{CG}}(\vec{x})$, where fluctuation smaller than a cut-off length λ are filtered out. The resulting upscaled Coarse Graining transmissivity field $T_\lambda^{\mathsf{CG}}(\vec{x})$ represents a log-normal distributed field with a smaller variance $\langle\sigma^2\rangle_\lambda$, but larger correlation length $\langle\ell\rangle_\lambda$. *Attinger* [2003] showed, that the statistical parameters relate to the parameter of the unfiltered field by $\langle\sigma^2\rangle_\lambda \equiv \sigma^2\frac{\ell^2}{\ell^2+\lambda^2/4}$ and $\langle\ell\rangle_\lambda \equiv \left(\ell^2+\lambda^2/4\right)^{1/2}$.

The concept of Coarse Graining can similarly be applied to non-uniform flow (step 2). The critical point when transferring the results of *Attinger* [2003] to radial flow is the Fourier back-transformation of the filtered head equation after localization. For uniform flow, this can be done due to the reasonable assumption of constant head gradient. For non-uniform flow, this assumption is not valid and thus, localization is not possible straight ahead.

A heuristic approach is taken to overcome the limitation of non-locality for well flow (step 3). Conditions of an quasi-constant head gradient are constructed by adapting the size of the volume elements over which flow takes place. The head gradient in well flow is proportional to the reciprocal of the distance r to the well: $\nabla h(r) = \frac{h(r)-h(r+\Delta r)}{\Delta r} \propto \frac{1}{r}$. The gradient is constant for volumes of size proportional to r. The step can be understood as a change from an equidistant Cartesian coordinate system to a polar coordinate system with cell sizes increasing with distance to the center, where the pumping well is located.

Under this adaption localization can be performed and so the following steps of the Coarse Graining procedure. The changed coordinate system impacts on the scaling procedure and the parameter λ. For uniform flow, λ is constant. Adapted to well flow, the scaling parameter needs to be proportional to r, because the filter width increases with distance to the well.

The result is an upscaled log-normal distributed field $T_r^{\mathsf{RCG}}(\vec{x})$ with an arithmetic mean $T_{\mathsf{A}}^{\mathsf{RCG}}(r)$ depending on r and a filtered fluctuation term. The variance of the coarsened field is similar to the ones derived by *Attinger* [2003] with $\lambda/2 = \zeta r$. The step was presented by *Schneider and Attinger* [2008]. It is not performed in a mathematically straight way, but problem adapted to well flow conditions.

Radial Coarse Graining Transmissivity

Spatial heterogeneity is still resolved in $T_r^{\mathsf{RCG}}(\vec{x})$, although reduced to the amount relevant to the pumping test. A further step of averaging is necessary to derive an effective description of the transmissivity for well flow conditions. Thereby, two different aspects are of interest: (i) an effective description of well flow transmissivity for an ensemble and (ii) effective description of well flow transmissivity for an individual field.

A result for an effective ensemble description is derived by averaging $T_r^{\mathsf{RCG}}(\vec{x})$ appropriate to well flow condition. The averaging rule is determined by the boundary condition at the well, which is the harmonic mean for two-dimensional well flow [*Dagan*, 1989]. The mean RCG-transmissivity, noticed by $T_{\mathsf{RCG}}(r)$, can be calculated via the theoretical description of the harmonic mean for log-normal distributed fields making use of the variance of the coarsened transmissivity $\langle\sigma^2\rangle_r = \frac{\sigma^2}{1+\zeta^2r^2/\ell^2}$:

$$T_{\mathsf{RCG}}(r) = T_{\mathsf{G}}\exp\left(-\langle\sigma^2\rangle_r/2\right) = T_{\mathsf{G}}\exp\left(-\frac{1}{2}\frac{\sigma^2}{(1+\zeta^2r^2/\ell^2)}\right) , \tag{7}$$

where r is the radial distance to the well, T_{G} is the geometric mean, σ^2 is the variance and ℓ is the correlation length of the log-normally distributed transmissivity $T(x)$. ζ is a factor of proportionality, which was determined to be $\zeta = 1.6$, as discussed in detail by *Zech et al.* [2012]. $T_{\mathsf{RCG}}(r)$ allows a transition from the representative transmissivity at the well $T_{\mathsf{H}} = T_{\mathsf{G}}\exp\left(-\frac{1}{2}\sigma^2\right)$ to far field value T_{G} depending on the radial distance r, controlled by the correlation length ℓ (Figure 1a).

An effective description of well flow transmissivity for an individual field can be derived from $T_{\mathsf{RCG}}(r)$ in Eq (7). The behaviour of individual fields is different especially at the well due to a lack of ergodicity there. The local transmissivity at the well T_{well} is not identical to the harmonic mean T_{H} as expected for the ensemble, but refers to the specific value of transmissivity at the well location. An adapted radial coarse graining transmissivity accounts for local effects by replacing the harmonic mean $T_{\mathsf{H}} = T_{\mathsf{G}} \exp\left(-\frac{1}{2}\sigma^2\right)$ by T_{well}. In Eq. (7) this refers to substituting the variance by $-\frac{1}{2}\sigma^2 = \ln T_{\mathsf{well}} - \ln T_{\mathsf{G}}$ and thus,

$$T_{\mathsf{RCG}}^{\mathsf{local}}(r) = T_{\mathsf{G}} \exp\left(\frac{\ln T_{\mathsf{well}} - \ln T_{\mathsf{G}}}{1+\zeta^2 r^2/\ell^2}\right) = T_{\mathsf{well}}^{\frac{1}{1+\zeta^2 r^2/\ell^2}} T_{\mathsf{G}}^{\frac{\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}} \quad . \tag{8}$$

$T_{\mathsf{RCG}}^{\mathsf{local}}(r)$ interpolates between the specific transmissivity at the well T_{well} and the far field value T_{G} depending on the radial distance r and the correlation length ℓ.

Interactive comment on Hydrol. Earth Syst. Sci. Discuss., 12, 6921, 2015.

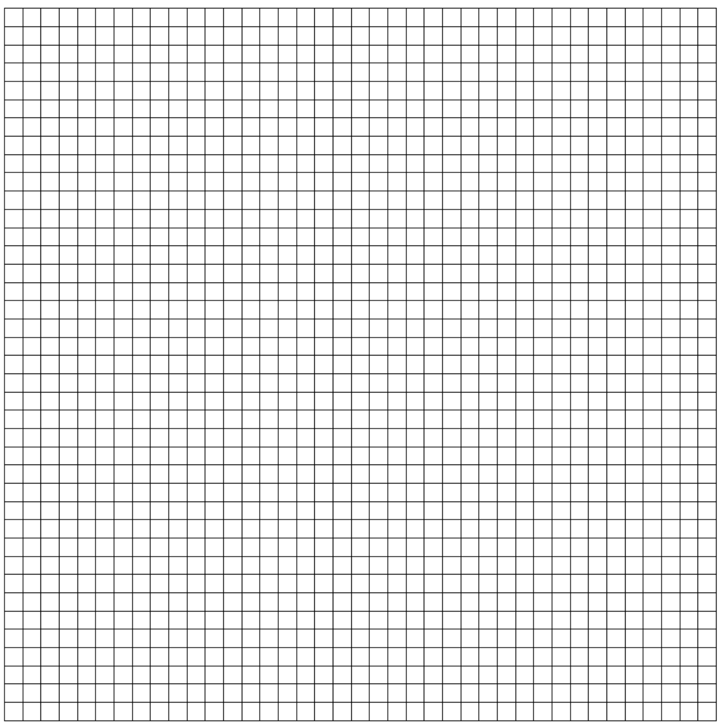

Fig. 1. Cartesian coordinate system representation; hydraulic head gradients can be assumed constant at every cell for uniform flow; scaling parameter λ is constant.

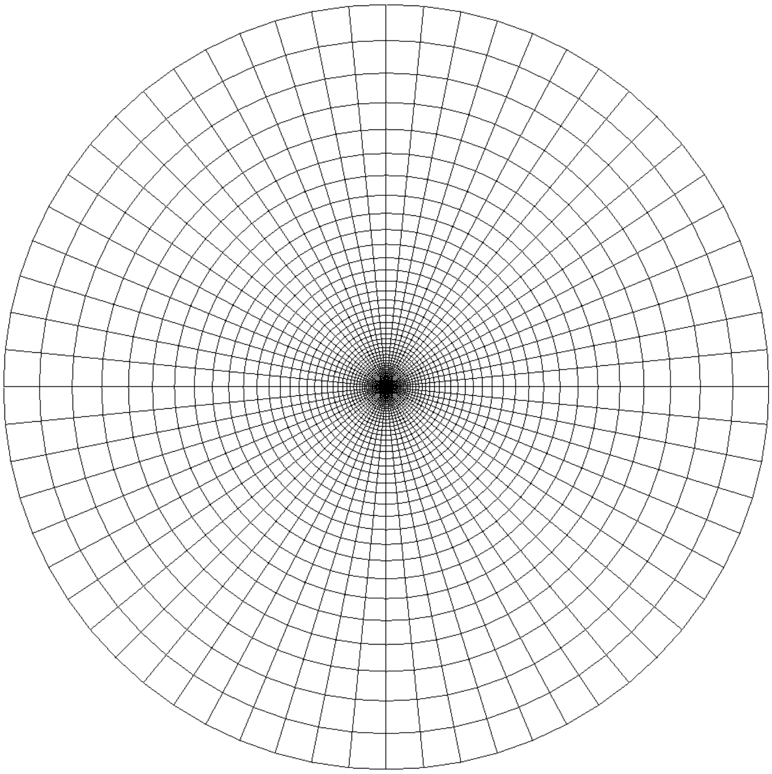

Fig. 2. Well flow adapted coordinate system representation with increasing cell size relative to the distance to the well r; hydraulic head gradients can be assumed constant at every cell for well flow.