Referee #2: Comments and Responses

Analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer

The authors present a solution for transient flow toward a radial collector well. The title suggests that the solution covers transient flow in an unconfined aquifer, but the boundary conditions along the phreatic surface are simplified to such an extent that I doubt that the approximation is sufficiently close to the stated problem to be of much use. The phreatic surface is not only assumed to be a horizontal straight line, which in itself is a severe approximation, it is also assumed to remain in its original position at all times. The boundary along the moving phreatic surface, equation (7) in the paper, is simplified to equation (8), which implies that the vertical component of flow is equal to minus the specific yield multiplied by the rate of decrease in elevation of the phreatic surface, maintained at the original position ($z = 0$). Compressibility of the aquifer is included, but not in the sense of poro-elasticity, but using the Terzaghi approximation. I agree that this approximation is usually acceptable dealing with groundwater flow, but the authors should state their approximations carefully, including this one.

Response (1st): The simplification from Eq. (7) to Eq. (8) was first proposed by Boulton (1954) and later used to develop analytical solutions by, for example, Neuman (1972), Zhan and Zlotnik (2002), and Yeh et al. (2010). The simplification has been validated by agreement on drawdown measured by a field pumping test and predicted by Neuman (1972) solution based on Eq. (8) (e.g., Goldscheider and Drew, 2007, p. 88). We inserted the following sentence right below Eq. (8):

"Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test." (lines 198 – 199 of the revised manuscript)

We also inserted the following sentence to indicate the governing equation (i.e., Eq. (1)) is based on a concept proposed by Terzaghi: "The first term on the RHS of Eq. (1) depicts aquifer storage release based on the concept of effective stress proposed by Terzaghi (see, for example, Bear, 1979, p.84; Charbeneau, 2000, p.57)." (lines 171 – 173 of the revised manuscript)

The boundary conditions along the two streams are applied over the height of the aquifer (full penetration); this is not mentioned (referee 1 also mentions this point).

Response (2nd): We inserted following two sentences in Abstract and Introduction sections, respectively:

"The streams with low-permeability streambeds fully penetrate the aquifer thickness." (lines 22 – 23 of the revised manuscript) and "The streams fully penetrate the aquifer thickness and connect the aquifer with low-permeability streambeds." (lines 133 – 134 of the revised manuscript)

In addition, we also added a sentence shown below in Introduction section: "A stream of partial penetration can be considered as fully penetrating if the distance between the stream and well is larger than 1.5 times the aquifer thickness (Todd and Mays, 2005)." (lines 134 – 135 of the revised manuscript)

The authors integrate a point sink along the legs of the radial collector well, but fail to mention what boundary condition applies along the legs. The head should be maintained constant along the legs, whereas the condition applied by the authors is constant influx, as far as I have been able to gather from the description.

Response (3rd): Thanks for the suggestion. We add following sentence in the last paragraph of the Introduction section: "The flux across the well screen is assumed to be uniform along each of the laterals." (lines 132 – 133 of the revised manuscript).

The mathematical model resulting from the highly simplified boundary conditions and the application of the various transforms is not presented in sufficient detail for me to be able to verify the steps without re-deriving much of the work, which should not be necessary.

Response (4th): Please refer to the first response for the fact that the boundary condition is reasonably simplified. Regarding the application of those transforms, we added several intermediate equations and rewrote the associated text shown at the end of this reply.

The flow problem shown in Figure 2 is not clearly defined. The authors comment about existing models assuming 2-D flow with neglecting the vertical flow component; based on this comment, I assume that this figure applies to 3D flow, but this is not stated clearly. The sections shown in the figure do not mention whether these are horizontal or vertical; neither do they mention where the sections apply. If the flow considered is three-dimensional, then there does not exist a stream function, but the authors define one in equation (65). If the flow is transient ($\bar{t} = 10^7$), then the transient storage is yet another reason for the stream function not to exist; the divergence of the specific discharge vector is not zero. Perhaps the authors made the assumption that the time considered is so large that change in storage can be neglected, but this approximation must be stated. Furthermore, equation (65) is not obvious and, besides stating the approximation, the derivation should be presented.

Response (5th): Thanks for the comment. The derivation of the stream function is shown in Appendix C of the revised manuscript and also given at the end of this reply. In addition, we added the following sentence in section 3.1.

"$\bar{\psi} = K_y H \psi / Q$ is the dimensionless stream function describing 2-D streamlines at the vertical plane of $\bar{y} = 1$ based on $\bar{h}_w$ in Eq. (44) with $\bar{t} = 10^7$ for steady state." (lines 427 – 428 of the revised manuscript)

Summary

The authors present a very complex solution based on highly simplified boundary conditions and with insufficient detail. The authors do not present any comparison with existing solutions for simplified boundary conditions as a validation, both of their equations, and of their simplifying assumptions.

Response: Please refer to 1st response for the validation of the boundary condition.

The derivations are very difficult to follow and lack sufficient detail. The authors refer to equations further in the text, a procedure that violates standard approach in scientific work, and forces the reader to look ahead for equations that have not been digested yet.

Response: Please refer to 4th response for more detailed derivation.

I believe that the authors in their use of the stream function, violate basic principles; however, they may have made assumptions that are not stated clearly but if so, this needs to be rectified.

Response: Please refer to 5th response for the application of the stream function.

I suggest that the paper be shortened substantially and rewritten as follows:

- Remove the claim that the work applies to unconfined flow; it does not.
- Focus on one particular case, e.g., a radial collector well in a confined aquifer.

Response: Please refer to 1st response for the fact that the present solution is applicable to unconfined flow. In addition, we already demonstrated the application of the present solution to the well in confined aquifers in the second paragraph of section 3.4.

- State all boundary conditions clearly, including the ones along the legs of the radial collector well and the ones along the streams.

Response: Please refer to 2nd response for the statement of fully-penetrating streams and to 3rd response for the assumption of uniform flux on the laterals of the well.

- Make a comparison with an existing solution for at least one case.

Response: We already compared transient distributions of SDR predicted by the present solution and the Hunt (1999) solution in Fig. 6.

- Present the details of the analysis, taking into account that the reader should be able to follow the steps without the need to redo the analysis.

Response: Thanks for the comment. The text has been largely revised, and the new one is given at the end of this reply.

- If use is made of a stream function, make it clear that the flow is two-dimensional and steady. Otherwise, there does not exist a stream function at all.

Response: Please refer to 5th response for the statement of two-dimensional, steady-state flow.

References

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Text abstracted from lines 202 – 286 and 583 – 671 of the revised manuscript

Define dimensionless variables as $\bar{h} = (K_y\, H\, h)/Q$, $\bar{t} = (K_y\, t)/(S_s\, y_0^2)$, $\bar{x} = x/y_0$, $\bar{y} = y/y_0$, $\bar{z} = z/H$, $\bar{x}_0' = x_0'/y_0$, $\bar{y}_0' = y_0'/y_0$, $\bar{z}_0' = z_0'/H$, $\bar{w}_x = w_x/y_0$ and $\bar{w}_y = w_y/y_0$ where the overbar denotes a dimensionless symbol, and y_0, a distance between stream 1 and the center of the RCW, is chosen as a characteristic length. On the basis of the definitions, Eq. (1) can be written as

$$\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0') \tag{10}$$

where $\kappa_x = K_x/K_y$ and $\kappa_z = (K_z\, y_0^2)/(K_y\, H^2)$.

Similarly, the initial and boundary conditions are expressed as

$$\bar{h} = 0 \quad \text{at} \quad \bar{t} = 0 \tag{11}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = 0 \tag{12}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = \bar{w}_x \tag{13}$$

$$\partial \bar{h}/\partial \bar{y} - \kappa_1 \bar{h} = 0 \quad \text{at} \quad \bar{y} = 0 \tag{14}$$

$$\partial \bar{h}/\partial \bar{y} + \kappa_2 \bar{h} = 0 \quad \text{at} \quad \bar{y} = \bar{w}_y \tag{15}$$

$$\frac{\partial \bar{h}}{\partial \bar{z}} = -\frac{\gamma}{\kappa_z} \frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 0 \tag{16}$$

and

$$\partial \bar{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{17}$$

where $\kappa_1 = (K_1\, y_0)/(K_y\, b_1)$, $\kappa_2 = (K_2\, y_0)/(K_y\, b_2)$ and $\gamma = S_y/(S_s\, H)$.

2.2 Head solution for point sink

The model, Eqs. (10) – (17), reduces to an ordinary differential equation (ODE) with two boundary conditions in terms of $\bar{z}$ after taking Laplace transform and finite integral transform. The former transform converts $\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ in the model into $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$, $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')$ in Eq. (10) into $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')/p$, and $\partial \bar{h}/\partial \bar{t}$ in Eqs. (10) and (16) into $p\hat{h} - \bar{h}\big|_{\bar{t}=0}$ where p is the Laplace parameter, and the second term, initial condition in Eq. (11), equals zero (Kreyszig, 1999). The transformed model becomes a boundary value problem written as

$$\kappa_x \frac{\partial^2 \hat{h}}{\partial \bar{x}^2} + \frac{\partial^2 \hat{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} = p\hat{h} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0')/p \tag{18}$$

with boundary conditions $\partial \hat{h}/\partial \bar{x} = 0$ at $\bar{x} = 0$ and $\bar{x} = \bar{w}_x$, $\partial \hat{h}/\partial \bar{y} - \kappa_1 \hat{h} = 0$ at $\bar{y} = 0$, $\partial \hat{h}/\partial \bar{y} + \kappa_2 \hat{h} = 0$ at $\bar{y} = \bar{w}_y$, $\partial \hat{h}/\partial \bar{z} = -p\gamma \hat{h}/\kappa_z$ at $\bar{z} = 0$ and $\partial \bar{h}/\partial \bar{z} = 0$ at $\bar{z} = -1$. We then apply finite integral transform to the problem. One can refer to Appendix A for its detailed definition. The transform converts $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$ in the problem into $\tilde{h}(\alpha_m, \beta_n, \bar{z}, p)$, and $\delta(\bar{x}-\bar{x}_0')\,\delta(\bar{y}-\bar{y}_0')$ in Eq. (18) into $\cos(\alpha_m \bar{x}_0')K(\bar{y}_0')$ and $\kappa_x\,\partial^2\hat{h}/\partial\bar{x}^2 + \partial^2\hat{h}/\partial\bar{y}^2$ in Eq. (18) into $-(\kappa_x\alpha_m^2+\beta_n^2)\tilde{h}$ where $(m,\ n)\in 1,\ 2,\ 3,\ \dots\ \infty$, $\alpha_m = m\,\pi/\bar{w}_x$, $K(\bar{y}_0')$ is defined in Eq. (A2) with $\bar{y}=\bar{y}_0'$, and β_n are eigenvalues equaling the roots of the following equation as (Latinopoulos, 1985)

$$\tan\left(\beta_n\,\bar{w}_y\right)=\frac{\beta_n(\kappa_1+\kappa_2)}{\beta_n^2-\kappa_1\,\kappa_2} \tag{19}$$

The method to determine the roots is discussed in section 2.3. In turn, Eq. (18) becomes a second-order ODE defined by

$$\kappa_z\frac{\partial^2\tilde{h}}{\partial\bar{z}^2}-(\kappa_x\alpha_m^2+\beta_n^2+p)\tilde{h}=\cos(\alpha_m\,\bar{x}_0')\,K(\bar{y}_0')\,\delta(\bar{z}+\bar{z}_0')/\,p \tag{20}$$

with two boundary conditions denoted as

$$\frac{\partial\tilde{h}}{\partial\bar{z}}=-\frac{p\,\gamma}{\kappa_z}\tilde{h}\quad \text{at}\quad \bar{z}=0 \tag{21}$$

and

$$\partial\tilde{h}/\partial\bar{z}=0\quad \text{at}\quad \bar{z}=-1 \tag{22}$$

Eq. (20) can be separated into two homogeneous ODEs as

$$\kappa_z\frac{\partial^2\tilde{h}_a}{\partial\bar{z}^2}-(\kappa_x\alpha_m^2+\beta_n^2+p)\tilde{h}_a=0\quad \text{for}\quad -\bar{z}_0'\le\bar{z}\le 0 \tag{23}$$

and

$$\kappa_z\frac{\partial^2\tilde{h}_b}{\partial\bar{z}^2}-(\kappa_x\alpha_m^2+\beta_n^2+p)\tilde{h}_b=0\quad \text{for}\quad -1\le\bar{z}\le-\bar{z}_0' \tag{24}$$

where h_a and h_b, respectively, represent the heads above and below $\bar{z}=-\bar{z}_0'$ where the point sink is located. Two continuity requirements should be imposed at $\bar{z}=-\bar{z}_0'$. The first is the continuity of the hydraulic head denoted as

$$\tilde{h}_a=\tilde{h}_b\quad \text{at}\quad \bar{z}=-\bar{z}_0' \tag{25}$$

The second describes the discontinuity of the flux due to point pumping represented by the Dirac delta function in Eq. (20). It can be derived by integrating Eq. (20) from $\bar{z}=-\bar{z}_0'^{\,-}$ to $\bar{z}=-\bar{z}_0'^{\,+}$ as

$$\frac{\partial\tilde{h}_a}{\partial\bar{z}}-\frac{\partial\tilde{h}_b}{\partial\bar{z}}=\frac{\cos(\alpha_m\,\bar{x}_0')\,K(\bar{y}_0')}{p\,\kappa_z}\quad \text{at}\quad \bar{z}=-\bar{z}_0' \tag{26}$$

Solving Eqs. (23) and (24) simultaneously with Eqs. (21), (22), (25), and (26) yields the Laplace-domain head solution as

$$\tilde{h}_a(\alpha_m,\beta_n,\bar{z},p)=\Omega(-\bar{z}_0',\bar{z},1)\quad \text{for}\quad -\bar{z}_0'\le\bar{z}\le 0 \tag{27a}$$

and

$$\tilde{h}_b(\alpha_m, \beta_n, \bar{z}, p) = \Omega(\bar{z}, \bar{z}_0', -1) \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{27b}$$

with

$$\Omega(a,b,c) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + c\, p\gamma \sinh(b\lambda)]\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)}{p\kappa_z \lambda(p\gamma\cosh\lambda + \kappa_z \lambda \sinh\lambda)} \tag{28}$$

$$\lambda = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2 + p)/\kappa_z} \tag{29}$$

where a, b, and c are arguments. Taking the inverse Laplace transform and finite integral transform to Eq. (28) results in Eq. (31). One is referred to Appendix B for the detailed derivation. A time-domain head solution for a point sink is therefore written as

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \begin{cases} \Phi(-\bar{z}_0', \bar{z}, 1) \text{ for } -\bar{z}_0' \le \bar{z} \le 0 \\ \Phi(\bar{z}, \bar{z}_0', -1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0' \end{cases} \tag{30}$$

with

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n X_n + 2\sum_{m=1}^{\infty}\phi_{m,n} X_{m,n} \cos(\alpha_m \bar{x})\right] Y_n\right\} \tag{31}$$

$$\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i} \tag{32}$$

$$\psi_{m,n} = -\cosh[(1+a)\lambda_s] \cosh(b\,\lambda_s)/(\kappa_z \lambda_s \sinh\lambda_s) \tag{33}$$

$$\psi_{m,n,0} = \mu_{m,n,0} \cosh[(1+a)\lambda_0] \,[-\kappa_z \lambda_0 \cosh(b\,\lambda_0) + c\, p_0 \gamma \sinh(b\,\lambda_0)] \tag{34}$$

$$\psi_{m,n,i} = \nu_{m,n,i} \cos[(1+a)\lambda_i] \,[-\kappa_z \lambda_i \cos(b\,\lambda_i) + c\, p_i \gamma \sin(b\,\lambda_i)] \tag{35}$$

$$\mu_{m,n,0} = 2\exp(p_0 \bar{t})/\{p_0[(1+2\gamma)\kappa_z \lambda_0 \cosh\lambda_0 + (p_0\gamma + \kappa_z)\sinh\lambda_0]\} \tag{36}$$

$$\nu_{m,n,i} = 2\exp(p_i \bar{t})/\{p_i[(1+2\gamma)\kappa_z \lambda_i \cos\lambda_i + (p_i\gamma + \kappa_z)\sin\lambda_i]\} \tag{37}$$

$$Y_n = \frac{\beta_n \cos(\beta_n \bar{y}) + \kappa_1 \sin(\beta_n \bar{y})}{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2/(\beta_n^2 + \kappa_2^2)] + \kappa_1} \tag{38}$$

and

$$X_{m,n} = \cos(\alpha_m \bar{x}_0') \,[\beta_n \cos(\beta_n \bar{y}_0') + \kappa_1 \sin(\beta_n \bar{y}_0')] \tag{39}$$

where $\lambda_s = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2)/\kappa_z}$, $p_0 = \kappa_z \lambda_0^2 - \kappa_x \alpha_m^2 - \beta_n^2$, $p_i = -\kappa_z \lambda_i^2 - \kappa_x \alpha_m^2 - \beta_n^2$, ϕ_n and X_n equal $\phi_{m,n}$ and $X_{m,n}$ with $\alpha_m = 0$, respectively, and the eigenvalues λ_0 and λ_i are, respectively, the roots of the following equations:

$$e^{2\lambda_0} = \frac{-\gamma\kappa_z \lambda_0^2 + \kappa_z \lambda_0 + \gamma(\kappa_x \alpha_m^2 + \beta_n^2)}{\gamma\kappa_z \lambda_0^2 + \kappa_z \lambda_0 - \gamma(\kappa_x \alpha_m^2 + \beta_n^2)} \tag{40}$$

$$\tan\lambda_i = \frac{-\gamma(\kappa_z \lambda_i^2 + \kappa_x \alpha_m^2 + \beta_n^2)}{\kappa_z \lambda_i} \tag{41}$$

The determination for those eigenvalues is introduced in the next section. Notice that the solution consists of simple series expanded in β_n, double series expanded in β_n and λ_i (or α_m and β_n), and triple series expanded in α_m, β_n and λ_i.

Appendix A: Finite integral transform

Latinopoulos (1985) provided the finite integral transform for a rectangular aquifer domain where each side can be under either the Dirichlet, no-flow, or Robin condition. The transform associated with the boundary conditions, Eqs. (12) − (15), is defined as

$$\tilde{h}(\alpha_m, \beta_n) = \Im\{\bar{h}(\bar{x}, \bar{y})\} = \int_0^{\bar{w}_x} \int_0^{\bar{w}_y} \bar{h}(\bar{x}, \bar{y}) \cos(\alpha_m \bar{x}) K(\bar{y}) \, d\bar{y} \, d\bar{x} \quad \text{(A1)}$$

with

$$K(\bar{y}) = \sqrt{2} \frac{\beta_n \cos(\beta_n \bar{y}) + \kappa_1 \sin(\beta_n \bar{y})}{\sqrt{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2 /(\beta_n^2 + \kappa_2^2)] + \kappa_1}} \quad \text{(A2)}$$

where $\cos(\alpha_m \bar{x}) \, K(\bar{y})$ is the kernel function. According to Latinopoulos (1985, Eq. (9)), the transform has the property of

$$\Im\left\{\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2}\right\} = -(\kappa_x \alpha_m^2 + \beta_n^2) \tilde{h}(\alpha_m, \beta_n) \quad \text{(A3)}$$

The formula for the inverse finite integral transform can be written as (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x}, \bar{y}) = \Im^{-1}\{\tilde{h}(\alpha_m, \beta_n)\} = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty} \tilde{h}(0, \beta_n) K(\bar{y}) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{h}(\alpha_m, \beta_n)\cos(\alpha_m \bar{x}) K(\bar{y})\right] \quad \text{(A4)}$$

Appendix B: Derivation of equation (31)

The function of p in Eq. (28) is defined as

$$F(p) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{\kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \quad \text{(B1)}$$

Notice that the term $\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)$ in Eq. (28) is excluded because it is independent of p. $F(p)$ is a single-value function with respect to p. On the basis of the residue theorem, the inverse Laplace transform for $F(p)$ equals the summation of residues of poles in the complex plane. The residue of a simple pole can be derived according to the formula below:

$$\text{Res}|_{p=p_i} = \lim_{p \to p_i} F(p) \exp(p\bar{t}) \, (p - p_i) \quad \text{(B2)}$$

where p_i is the location of the pole in the complex plane.

The locations of poles are the roots of the equation obtained by letting the denominator in Eq. (B1) to be zero, denoted as

$$p \, \kappa_z \, \lambda(p \, \gamma \cosh \lambda + \kappa_z \, \lambda \sinh \lambda) = 0 \quad \text{(B3)}$$

where λ is defined in Eq. (29). Notice that $p = -\kappa_x \alpha_m^2 - \beta_n^2$ obtained by $\lambda = 0$ is not a pole in spite of being a root. Apparently, one pole is at p = 0, and the residue based on Eq. (B2) with $p_i = 0$ is expressed as

$$\text{Res}|_{p=0} = \lim_{p \to 0} \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{\kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \exp(p\bar{t}) \quad \text{(B4)}$$

Eq. (B4) with $p = 0$ and $\lambda = \lambda_s$ reduces to $\psi_{m,n}$ in Eq. (33).

Other poles are determined by the equation of

$$p\,\gamma\cosh\lambda + \kappa_z\,\lambda\sinh\lambda = 0 \tag{B5}$$

which comes from Eq. (B3). One pole is at $p = p_0$ between $p = 0$ and $p = -\kappa_x\,\alpha_m^2 - \beta_n^2$ in the negative part of the real axis. Newton's method can be used to obtain the value of p_0. In order to have proper initial guess for Newton's method, we let $\lambda = \lambda_0$ and then have $p = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \lambda_0$, $p = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$, $\cosh\lambda_0 = (e^{\lambda_0} + e^{-\lambda_0})/2$ and $\sinh\lambda_0 = (e^{\lambda_0} - e^{-\lambda_0})/2$ into Eq. (B5) and rearranging the result leads to Eq. (40). Initial guess for finding root λ_0 of Eq. (40) is discussed in section 2.3. With known value of λ_0, one can obtain $p_0 = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$. According to Eq. (B2), the residue of the simple pole at $p = p_0$ is written as

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0}\frac{\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p\kappa_z\lambda(p\gamma\cosh\lambda+\kappa_z\lambda\sinh\lambda)}\exp(p\bar{t})\,(p-p_0) \tag{B6}$$

where both the denominator and nominator equal zero when $p = p_0$. Applying L'Hospital's Rule to Eq. (B6) results in

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0}\frac{2\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p[(1+2\gamma)\kappa_z\lambda\cosh\lambda+(\gamma p+\kappa_z)\sinh\lambda]}\exp(p\bar{t}) \tag{B7}$$

Eq. (B7) with $p = p_0$ and $\lambda = \lambda_0$ reduces to $\psi_{m,n,0}$ in Eq. (34).

On the other hand, infinite poles are at $p = p_i$ behind $p = -\kappa_x\,\alpha_m^2 - \beta_n^2$. Similar to the derivation of Eq. (40), we let $\lambda = \sqrt{-1}\lambda_i$ and then have $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$ based on Eq. (29) for the absence of the imaginary unit. Substituting $\lambda = \sqrt{-1}\lambda_i$, $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B3) and rearranging the result yields Eq. (41). The determination of λ_i is discussed in section 2.3. With known value of λ_i, one can have $p_i = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$. The residues of those simple poles at $p=p_i$ can be expressed as $\psi_{m,n,i}$ in Eq. (35) by substituting $p_0 = p_i$, $p = p_i$, $\lambda = \sqrt{-1}\lambda_i$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B7). Eventually, the inverse Laplace transform for $F(p)$ equals the sum of those residues (i.e., $\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i}$). The time-domain result of $\Omega(a,b,c)$ in Eq. (28) is then obtained as $\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$. By substituting $\tilde{h}(\alpha_m,\beta_n) = \phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$ and $\tilde{h}(0,\beta_n) = \phi_n K(\bar{y}_0)$ into Eq. (A4) and letting $\bar{h}(\bar{x},\bar{y})$ to be $\Phi(a,b,c)$, the inverse finite integral transform for the result can be derived as

$$\Phi(a,b,c) = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty}\left(\phi_n\,K(\bar{y}_0)K(\bar{y}) + 2\sum_{m=1}^{\infty}\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\cos(\alpha_m\,\bar{x})\,K(\bar{y})\right)\right] \tag{B8}$$

Moreover, Eq. (B8) reduces to Eq. (31) when letting the terms of $K(\bar{y}_0)K(\bar{y})$ and $\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\,K(\bar{y})$ to be $2X_nY_n$ and $2X_{m,n}Y_n$, respectively.

Appendix C: Derivation of $\bar{\psi}$ in Eq. (65)

The dimensionless stream function $\overline{\psi}$ in Eq. (65) can be expressed as

$$\overline{\psi} = C - \sqrt{\kappa_z} \int \partial \bar{h}_w / \partial \bar{z}\, d\bar{x} \text{ at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \quad \text{(C1)}$$

where C is a coefficient resulting from the integration, and $\bar{h}$ is defined in Eq. (44). Substituting Eq. (44) into Eq. (C1) leads to

$$\overline{\psi}(\bar{x}, \bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \int \partial\Phi(-\bar{z}_0, \bar{z}, 1)/\partial\bar{z}\, d\bar{x} \text{ for } -\bar{z}_0 \le \bar{z} \le 0 \\ \int \partial\Phi(\bar{z}, \bar{z}_0, -1)/\partial\bar{z}\, d\bar{x} \text{ for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \text{ at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \quad \text{(C2)}$$

$$\Phi(a, b, c) = \frac{2}{\bar{w}_x} \left\{ \sum_{n=1}^{\infty} \left[\phi_n \hat{X}_{n,k} + 2 \sum_{m=1}^{\infty} \phi_{m,n} \hat{X}_{m,n,k} \cos(\alpha_m \bar{x}) \right] Y_n \right\} \quad \text{(C3)}$$

where $\phi_{m,n}, Y_n$, $\hat{X}_{n,k}$ and $\hat{X}_{m,n,k}$ are defined in Eqs. (32), (38), (45) and (46), respectively, and ϕ_n equals $\phi_{m,n}$ with $\alpha_m = 0$. In Eq. (C3), variable $\bar{x}$ appears only in $\cos(\alpha_m \bar{x})$, and variable $\bar{z}$ appears only in ϕ_n and $\phi_{m,n}$ in Eq. (32). Eq. (C2) therefore becomes

$$\overline{\psi}(\bar{x}, \bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \hat{\Phi}(-\bar{z}_0, \bar{z}, 1) \text{ for } -\bar{z}_0 \le \bar{z} \le 0 \\ \hat{\Phi}(-\bar{z}_0, \bar{z}, 1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \text{ at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \quad \text{(C4)}$$

$$\hat{\Phi}(a, b, c) = \frac{2}{\bar{w}_x} \left\{ \sum_{n=1}^{\infty} \left[\frac{\partial \phi_n}{\partial \bar{z}} \hat{X}_{n,k} \int d\bar{x} + 2 \sum_{m=1}^{\infty} \frac{\partial \phi_{m,n}}{\partial \bar{z}} \hat{X}_{m,n,k} \int \cos(\alpha_m \bar{x})\, d\bar{x} \right] Y_n \right\} \quad \text{(C5)}$$

Consider $\bar{t} = 10^7$ for steady-state flow that the exponential terms of $\exp(p_0\, \bar{t})$ and $\exp(p_i\, \bar{t})$ approach zero (i.e., $p_0 > 0$ and $p_i > 0$) for the default values of the parameters used to plot Figure 2. Then, we have $\phi_{m,n} = \psi_{m,n}$ defined in Eq. (33) because of $\psi_{m,n,0} \cong 0$, $\psi_{m,n,i} \cong 0$, $\mu_{m,n,0} \cong 0$ and $\nu_{m,n,i} \cong 0$. On the basis of $\phi_{m,n} = \psi_{m,n}$ and Eq. (33) with $a = -\bar{z}_0$ and $b = \bar{z}$ for $-\bar{z}_0 \le \bar{z} \le 0$ and $a = \bar{z}$ and $b = \bar{z}_0$ for $-1 \le \bar{z} \le -\bar{z}_0$, the result of differentiation, i.e., $\partial \phi_{m,n} / \partial \bar{z}$, in Eq. (C5) equals

$$\frac{\partial \phi_{m,n}}{\partial \bar{z}} = \begin{cases} -\lambda_s \cosh[(1 - \bar{z}_0)\lambda_s] \sinh(\bar{z}\, \lambda_s) / (\kappa_z\, \lambda_s \sinh \lambda_s) \text{ for } -\bar{z}_0 \le \bar{z} \le 0 \\ -\lambda_s \sinh[(1 + \bar{z})\lambda_s] \cosh(\bar{z}_0\, \lambda_s) / (\kappa_z\, \lambda_s \sinh \lambda_s) \text{ for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{(C6)}$$

Notice that $\partial \phi_n / \partial \bar{z}$ in Eq. (C5) equals Eq. (C6) with $\alpha_m = 0$. In addition, both integrations in Eq. (C5) can be done analytically as

$$\int \cos(\alpha_m \bar{x})\, d\bar{x} = \begin{cases} \sin(\alpha_m \bar{x}) / \alpha_m \text{ for } \alpha_m \neq 0 \\ \bar{x} \text{ for } \alpha_m = 0 \end{cases} \quad \text{(C7)}$$

On the other hand, coefficient C in Eq. (C4) is determined by the condition of $\overline{\psi} = 0$ at $\bar{x} = \bar{x}_0$ and results in

$$C = \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \hat{\Phi}(-\bar{z}_0, \bar{z}, 1) \text{ for } -\bar{z}_0 \le \bar{z} \le 0 \\ \hat{\Phi}(-\bar{z}_0, \bar{z}, 1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{(C8)}$$

where $\widehat{\Phi}$ is defined in Eq. (C5) with Eqs. (C6) and (C7), $\bar{x} = \bar{x}_0$ and $\bar{y} = 1$.