Responses to Comments of Referee #1

The authors have addressed my comments fully and clearly. I have a very minor point that came out while re-reading the manuscript and author responses.

In the response to my comment about the non-dimensionalization (my comment 6 -- covered on pages 3-4 of the author rebuttal document) the authors' mention they selected y_0 as a characteristic length, but actually they used 2 characteristic lengths (they used H in the vertical direction). This leads to κ_z including not just the permeability ratio K_y/K_z (similar to κ_x), but also the squared length scale ratio y_0^2/H^2. Because of this choice, one cannot disentangle the effects of changing κ_z that are due to permeability changes, or due to the square root of length scaling changes.

I understand it is too late in the process to change this sort of detail (and it does not change the results or impact of the manuscript), but a consistent single characteristic length scale would have been a bit simpler.

Response: Thanks for the comment. The definition of $\kappa_z = K_z\, y_0^2/(K_y\, H^2)$ can explicitly demonstrate that both K_z/K_y and y_0^2/H^2 are crucial factors in neglecting the effect of the vertical flow on stream filtration/depletion rate (SDR) (Huang et al., 2014). Such definition is similar to the work of Neuman (1975) in which he defined dimensionless parameter $\beta = K_z\, r^2/(K_r\, H^2)$ with K_r representing the hydraulic conductivity in the radial direction and r denoting a radial distance measured from a pumping well to an observation point. He used the parameter to examine the validity of neglecting the effect of the vertical flow on time-dependent drawdown at the observation point (see Figure 1 in Neuman, 1975).

In our definition, the effect of K_z/K_y on SDR can be clearly explored once the value of y_0 is selected. On the other hand, the effect of y_0^2/H^2 on SDR can also be assessed if the value of K_z/K_y is known.

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Responses to Comments of Referee #2

Analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer-second review

The authors have addressed most of my comments. In my original review, I mentioned that the stream function presented for the case of section 3.1 does not exist. The answer of the authors to my objection is unsatisfactory. I understand from the problem description that the well is placed along z = 0.5, 9.5 ≤ x ≤ 10.5, y = 1, while the for domain has width W_y = 20. This implies that the flow is three-dimensional. Although the authors present the flow net for y = 1 and claim that the flow in y = 0 is two-dimensional, this does not imply that the stream function exists. The presence of non-zero second order derivatives prevents the stream function, as presented by the authors, from being single-valued.

Response: Thanks for the valuable comment on the fact that the stream function does not exist for three-dimensional flow. We realize that the stream function exists only for a few cases. For the cases of no stream function, one should rely on numerical approaches to determine the flow path lines. The path lines can be regarded as the streamlines for steady flow but not for transient flow.

We have removed the text associated with the stream function from section 3.1. The appendix C showing the derivation of the stream function is also removed. The section 3.1 is now slightly revised as:

"**3.1 Identification of 3-D or 2-D flow at observation point**

Most existing models assume 2-D flow with neglecting the vertical flow for pumping at a horizontal well (e.g., Mohamed and Rushton, 2006; Haitjema et al., 2010). The head distributions predicted by those models are inaccurate if an observation point is close to the region where the vertical flow prevails. Figure 2 demonstrates the equipotential lines predicted by the present solution for a horizontal well in an unconfined aquifer for $\bar{x}_0 = 10$, $\bar{w}_x = \bar{w}_y = 20$ and κ_z = 0.1, 1, and 10. The well is located at $9.5 \le \bar{x} \le 10.5$, $\bar{y} = 1$ and $\bar{z} = 0.5$ as illustrated in the figure. The equipotential lines are based on steady-state head distributions plotted by Eq. (44) with $\bar{y} = 1$ and $\bar{t} = 10^7$. When κ_z = 0.1, in the range of 10 ≤ $\bar{x}$ ≤ 13.66, the contours of the hydraulic head are in a curved path, and the flow toward the well is thus slanted. Moreover, the range decreases to 10 ≤ $\bar{x}$ ≤ 11.5 when κ_z = 1 and to 10 ≤ $\bar{x}$ ≤10.82 when κ_z = 10. Beyond these ranges, the head contours are nearly vertical, and the flow is essentially horizontal. Define $\bar{d}$ = d/y_0 as a shortest dimensionless horizontal distance between the well and a nearest location of only horizontal flow. The $\bar{d}$ is therefore chosen as 3.16, 1 and 0.32 for the cases of κ_z = 0.1, 1 and 10, respectively. Substituting (κ_z, $\bar{d}$) = (0.1, 3.16), (1, 1) and (10, 0.32) into $\kappa_z\bar{d}^2$ leads to about unity. We may therefore conclude that the vertical flow at an observation point is negligible if its location is beyond the range of $\bar{d} < \sqrt{1/\kappa_z}$ (i.e., $d < H\sqrt{K_y/K_z}$) for thin aquifers, an observation point far from the well, and/or a small ratio of K_y/K_z."

As a second comment, I remain of the opinion that the the paper would much benefit from a more thorough validation, especially in view of my preceding statement. The comparison with Hunt's solution is given, but holds only for a very limited case.

Response: Please refer to the first response to the statement. The results in Figures 2−6 are plotted on the basis of the same computer program but with different parameters. We achieve agreement on stream depletion rate (SDR) predicted by the present solution and Hunt (1999) solution when adjusting parameters for the situation used to develop the latter solution. This indicates that the program and the derivation of the present solution are correct. In addition, Huang et al. (2012) verified their solution in comparison with the Hantush and Papadopoulos (1962) solution describing short-time and long-time drawdown distributions due to a radial collector well. The solution of Huang et al. (2012) is a special case of the present solution as discussed in section 2.6.4 of the revised manuscript. We would not repeat it in this revised manuscript; it has been discussed in detail in Huang et al. (2012).

Summary

In my view, the use of the stream function in example 3.1 is in error. If I misunderstood their example as being three-dimensional, whereas in reality is two-dimensional, then the authors have to make this much more clear than in the present form of the paper. If the problem is indeed three-dimensional, then the use of the stream function must be removed from the paper. The reader may easily verify the truth of my statement by considering the case of a point sink at the origin in three-dimensional space. The discharge potential for that case has the form

$$\Phi = -\frac{Q}{4\pi}\frac{1}{r} \tag{1}$$

where

$$r^2 = x_1^2 + x_2^2 + x_3^2 \tag{2}$$

where x_i, i = 1, 2, 3, are the Cartesian coordinates. Set $x_2 = 0$. Then

$$\Phi = \frac{Q}{4\pi}\frac{1}{\sqrt{x_1^2+x_2^2}} \tag{3}$$

Introduce complex variables $z = x_1 + iy_1$, $\bar{z}$, $x_1 - iy_1$, so that (3) becomes

$$\Phi = -\frac{Q}{4\pi}\frac{1}{\sqrt{z\bar{z}}} \tag{4}$$

This function is not holomorphic, and thus the stream function does not exist for this case.

Response: We appreciate reviewer's explanation in detail. The text associated with the stream function has been removed.

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Figure

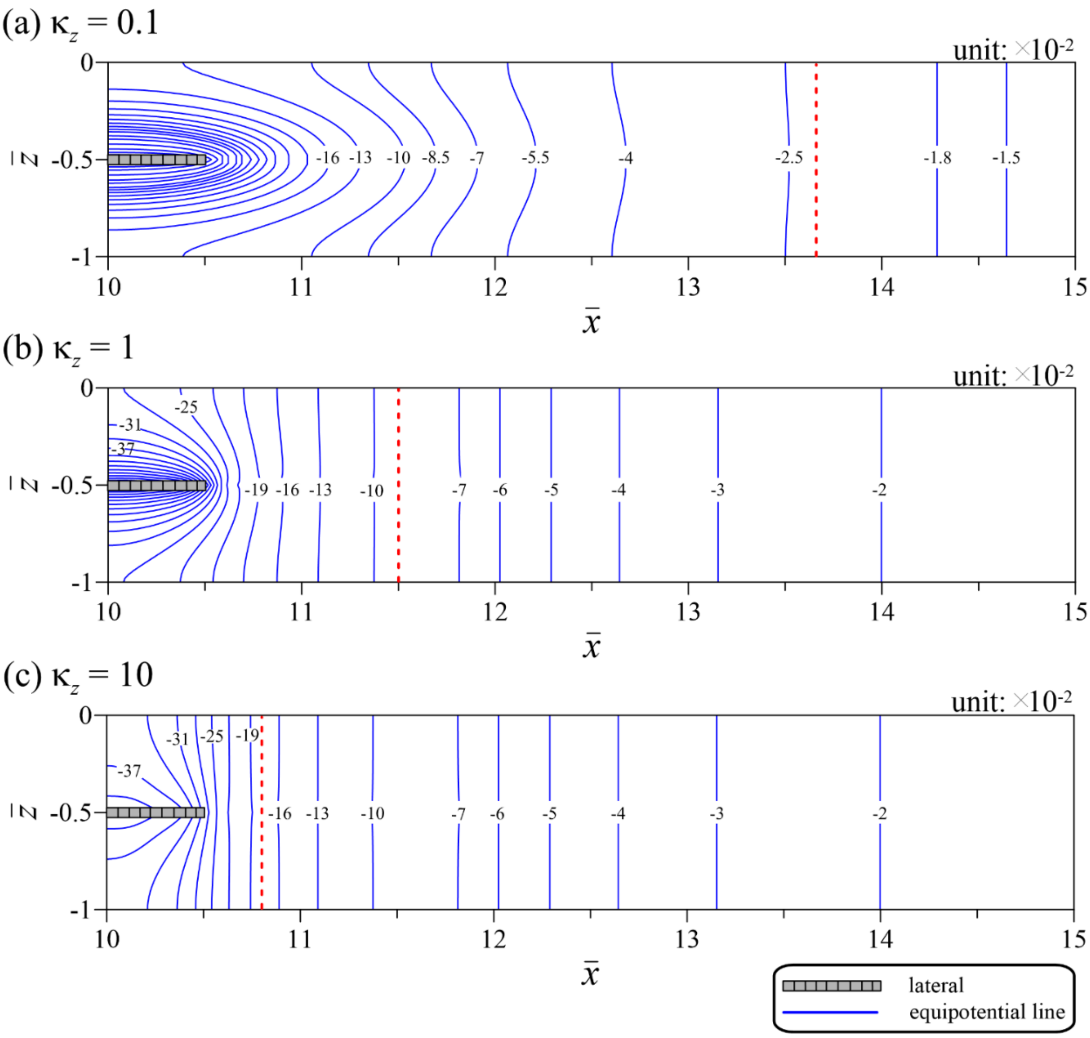

Figure 2. Equipotential lines predicted by the present solution for κ_z = (a) 0.1, (b) 1 and (c) 10.

Approximate analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer

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Abstract

This study develops a three-dimensional mathematical model for describing transient hydraulic head distributions due to pumping at a radial collector well (RCW) in a rectangular confined or unconfined aquifer bounded by two parallel streams and no-flow boundaries. The streams with low-permeability streambeds fully penetrate the aquifer ~~thickness~~. The governing equation with a point-sink term is employed. A first-order free surface equation delineating the water table decline induced by the well is considered. Robin boundary conditions are adopted to describe fluxes across the streambeds. The head solution for the point sink is derived by applying the methods of finite integral transform and Laplace transform. The head solution for a RCW is obtained by integrating the point-sink solution along the laterals of the RCW and then dividing the integration result by the sum of lateral lengths. On the basis of Darcy's law and head distributions along the streams, the solution for the stream depletion rate (SDR) can also be developed. With the aid of the head and SDR solutions, the sensitivity analysis can then be performed to explore the response of the hydraulic head to the change in a specific parameter such as the horizontal and vertical hydraulic conductivities, streambed permeability, specific storage, specific yield, lateral length and well depth. Spatial head distributions subject to the anisotropy of aquifer hydraulic conductivities are analyzed. A quantitative criterion is provided to identify whether groundwater flow at a specific region is 3-D or 2-D without the vertical component. In addition, another criterion is also given to allow the neglect of vertical flow effect on SDR. Conventional 2-D flow models can be used to provide accurate head and SDR predictions if satisfying these two criteria.

Keywords: Robin boundary condition, sensitivity analysis, stream depletion rate, first-order free surface equation, analytical solution

1. Introduction

The applications of a radial collector well (RCW) have received much attention in the aspects of water resource supply and groundwater remediation since rapid advances in drilling technology. An average yield for the well approximates 27,000 m^3/day (Todd and Mays, 2005). As compared to vertical wells, RCWs require less operating cost, produce smaller drawdown, and have better efficiency of withdrawing water from thin aquifers. In addition, RCWs can extract water from an aquifer underlying obstacles such as buildings, but vertical wells cannot. Recently, Huang et al. (2012) reviewed semi-analytical and analytical solutions associated with RCWs. Since then, Yeh and Chang (2013) provided a valuable overview of articles associated with RCWs.

A variety of analytical models involving a horizontal well, a specific case of a RCW with a single lateral, in aquifers were developed (e.g., Park and Zhan, 2003; Hunt, 2005; Anderson, 2013). The flux along the well screen is commonly assumed to be uniform. The equation describing three-dimensional (3-D) flow is used. Kawecki (2000) developed analytical solutions of the hydraulic heads for the early linear flow perpendicular to a horizontal well and late pseudo-radial flow toward the middle of the well in confined aquifers. They also developed an approximate solution for unconfined aquifers on the basis of the head solution and an unconfined flow modification. The applicability of the approximate solution was later evaluated in comparison with a finite difference solution developed by Kawecki and Al-Subaikhy (2005). Zhan et al. (2001) presented an analytical solution for drawdown induced by a horizontal well in confined aquifers and compared the difference in the type curves based on the well and a vertical well. Zhan and Zlotnik (2002) developed a semi-analytical solution of drawdown due to pumping from a nonvertical well in an unconfined aquifer accounting for the effect of instantaneous drainage or delayed yield when the free surface declines. They discussed the influences of the length, depth, and inclination of the well on temporal drawdown distributions. Park and Zhan (2002) developed a semi-analytical drawdown solution considering the effects of a finite diameter, the wellbore storage, and a skin zone around a horizontal well in anisotropic leaky aquifers. They found that those effects cause significant change in drawdown at an early pumping period. Zhan and Park (2003) provided a general semi-analytical solution for pumping-induced drawdown in a confined aquifer, an unconfined aquifer on a leaky bottom, or a leaky aquifer below a water reservoir. Temporal drawdown distributions subject to the aquitard storage effect were compared with those without that effect. Sun and Zhan (2006) derived a semi-analytical solution of drawdown due to pumping at a horizontal well in a leaky aquifer. A transient one-dimensional flow equation describing the vertical flow across the aquitard was considered. The derived solution was used to evaluate the Zhan and Park (2003) solution which assumed steady-state vertical flow in the aquitard.

Sophisticated numerical models involved in RCWs or horizontal wells were also reported. Steward (1999) applied the analytic element method to approximate 3-D steady-state flow induced by horizontal wells in contaminated aquifers. They discussed the relation between a pumping rate and the size of a polluted area. Chen et al. (2003) utilized the polygon finite difference method to deal with three kinds of seepage-pipe flows including laminar, turbulent, and transitional flows within a finite-diameter horizontal well. A sandbox experiment was also carried out to verify the prediction made by the method. Mohamad and Rushton (2006) used MODFLOW to predict flows inside an aquifer, from the aquifer to a horizontal well, and within the well. The predicted head distributions were compared with field data measured in Sarawak, Malaysia. Su et al. (2007) used software TOUGH2 based on the integrated finite difference method to handle irregular configurations of several laterals of two RCWs installed beside the Russian River, Forestville, California and analyzed pumping-induced unsaturated regions beneath the river. Lee et al. (2012) developed a finite element solution with triangle elements to assess whether the operation of a RCW near Nakdong River in South Korea can induce riverbank filtration. They concluded that the well can be used for sustainable water supply at the study site. In addition, Rushton and Brassington (2013a) extended Mohamad and Rushton (2006) study by enhancing the Darcy-Weisbach formula to describe frictional head lose inside a horizontal well. The spatial distributions of predicted flux along the well revealed that the flux at the pumping end is four times of the magnitude of that at the far end. Later, Rushton and Brassington (2013b) applied the same model to a field experiment at the Seton Coast, northwest England.

Well pumping in aquifers near streams may cause groundwater−surface water interactions (e.g., Rodriguez et al., 2013; Chen et al., 2013; Zhou et al., 2013; Exner-Kittridge et al., 2014; Flipo et al., 2014; Unland et al., 2014). The stream depletion rate (SDR), commonly used to quantify stream water filtration into the adjacent aquifer, is defined as the ratio of the filtration rate to a pumping rate. The SDR ranges from zero to a certain value which could be equal to or less than unity (Zlotnik, 2004). Tsou et al. (2010) developed an analytical solution of SDR for a slanted well in confined aquifers adjacent to a stream treated as a constant-head boundary. They indicated that a horizontal well parallel to the stream induces the steady-state SDR of unity more quickly than a slanted well. Huang et al. (2011) developed an analytical SDR solution for a horizontal well in unconfined aquifers near a stream regarded as a constant-head boundary. Huang et al. (2012) provided an analytical solution for SDR induced by a RCW in unconfined aquifers adjacent to a stream with a low-permeability streambed ~~treated as~~under the Robin condition. The influence of the configuration of the laterals on temporal SDR and spatial drawdown distributions was analyzed. Recently, Huang et al. (2014) gave an exhaustive review on analytical and semi-analytical SDR solutions and classified these solutions into two categories. One group involved two-dimensional (2-D) flow toward a fully-penetrating vertical well according to aquifer types and stream treatments. The other group included the solutions involving 3-D and quasi 3-D flows in the lights of aquifer types, well types, and stream treatments.

At present, existing analytical solutions associated with flow toward a RCW in unconfined aquifers have involved laborious calculation (Huang et al., 2012) and predicted approximate results (Hantush and Papadopoulos, 1962). The Huang et al. (2012) solution involves numerical integration of a triple integral in predicting the hydraulic head and a quintuple integral in predicting SDR. The integrand is expressed in terms of an infinite series expanded by roots of nonlinear equations. The integration variables are related to those roots. The application of their solution is therefore limited to those who are familiar with numerical methods. In addition, the accuracy of the Hantush and Papadopoulos (1962) solution is limited to some parts of a pumping period; that is, it gives accurate drawdown predictions at early and late times but divergent ones at middle time.

The objective of this study is to present new analytical solutions of the head and SDR, which overcome the above−mentioned limitations, for 3-D flow toward a RCW. A mathematical model is built to describe 3-D spatiotemporal hydraulic head distributions in a rectangular unconfined aquifer bounded by two parallel streams and by the no-flow stratums in the other two sides. The flux across the well screen is assumed to be uniform along each of the laterals. The assumption is valid for a short lateral within 150 m verified by agreement on drawdown observed in field experiments and predicted by existing analytical solutions (Huang et al., 2011; 2012). The streams fully penetrate the aquifer ~~thickness~~ and connect the aquifer with low-permeability streambeds. The model for the aquifer system with two parallel streams can be used to determine the fraction of water filtration from the streams and solve the associated water right problem (Sun and Zhan, 2007). The transient 3-D groundwater flow equation with a point-sink term is considered. The first-order free surface equation is used to describe water table decline due to pumping. Robin boundary conditions are adopted to describe fluxes across the streambeds. The head solution for a point sink is derived by the methods of Laplace transform and finite integral transform. The analytical head solution for a RCW is then obtained by integrating the point-sink solution along the well and dividing the integration result by the total lateral length. The RCW head solution is expressed in terms of a triple series expanded by eigenvalues which can be obtained by a numerical algorithm such as Newton's method. On the basis of Darcy's law and the RCW head solution, the SDR solution can then be obtained in terms of a double series with fast convergence. With the aid of both solutions, the sensitivity analysis is performed to investigate the response of the hydraulic head to the change in each of aquifer parameters. ~~The spatial distributions of the head and streamline are discussed.~~ Spatial head distributions subject to the anisotropy of aquifer hydraulic conductivities are analyzed. The influences of the vertical flow and well depth on temporal SDR distributions are investigated. Moreover, temporal SDR distributions induced by a RCW and a fully penetrating vertical well in confined aquifers are also compared. A quantitative criterion is provided to identify whether groundwater flow at a specific region is 3-D or 2-D without the vertical component. In addition, another criterion is also given to judge the suitability of neglecting the vertical flow effect on SDR.

2. Methodology

2.1. Mathematical model

Consider a RCW in a rectangular unconfined aquifer bounded by two parallel streams and no-flow stratums as illustrated in Figure 1. The symbols for variables and parameters are defined in Table 1. The origin of the Cartesian coordinate is located at the lower left corner. The aquifer domain falls in the range of $0 \le x \le w_x$, $0 \le y \le w_y$, and $-H \le z \le 0$. The RCW consists of a caisson and several laterals, each of which extends ~~finitely~~ with length L_k and counterclockwise with angle θ_k where $k \in 1, 2, \ldots N$ and N is the number of laterals. The caisson is located at (x_0, y_0), and the surrounding laterals are at $z = -z_0$.

First of all, a mathematical model describing 3-D flow toward a point sink in the aquifer is proposed. The equation describing 3-D hydraulic head distribution $h(x, y, z, t)$ is expressed as

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} + Q\,\delta(x - x_0')\delta(y - y_0')\delta(z + z_0') \tag{1}$$

where $\delta(\ )$ is the Dirac delta function, the second term on the right-hand side (RHS) indicates the point sink, and Q is positive for pumping and negative for injection. The first term on the RHS of Eq. (1) depicts aquifer storage release based on the concept of effective stress proposed by Terzaghi (see, for example, Bear, 1979; Charbeneau, 2000), which is valid under the assumption of constant total stress. By choosing water table as a reference datum where the elevation head is set to zero, the initial condition can therefore be denoted as

$$h = 0 \quad \text{at} \quad t = 0 \tag{2}$$

Note that equation (2) introduces negative hydraulic head for pumping, and the absolute value of the head equals drawdown.

The aquifer boundaries at $x = 0$ and $x = w_x$ are considered to be impermeable and thus expressed as

$$\partial h / \partial x = 0 \quad \text{at} \quad x = 0 \tag{3}$$

and

$$\partial h / \partial x = 0 \quad \text{at} \quad x = w_x \tag{4}$$

Streambed permeability is usually less than the adjacent aquifer formation. The fluxes across the streambeds can be described by Robin boundary conditions as

$$K_y \frac{\partial h}{\partial y} - \frac{K_1}{b_1} h = 0 \quad \text{at} \quad y = 0 \tag{5}$$

and

$$K_y \frac{\partial h}{\partial y} + \frac{K_2}{b_2} h = 0 \quad \text{at} \quad y = w_y \tag{6}$$

The free surface equation describing water table decline is written as

$$K_x\left(\frac{\partial h}{\partial x}\right)^2+K_y\left(\frac{\partial h}{\partial y}\right)^2+K_z\left(\frac{\partial h}{\partial z}\right)^2-K_z\frac{\partial h}{\partial z}=S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z=h \tag{7}$$

Neuman (1972) indicated that the effect of the second-order terms in Eq. (7) is generally ignorable to develop analytical solutions. Eq. (7) is thus linearized by neglecting the quadratic terms, and the position of the water table is fixed at the initial condition (i.e., z = 0). The result is written as

$$K_z\frac{\partial h}{\partial z}=-S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z=0 \tag{8}$$

Notice that Eq. (8) is applicable when the conditions $|h|/H \le 0.1$ and $|\partial h/\partial x| + |\partial h/\partial y| \le 0.01$ are satisfied. These two conditions had been studied and verified by simulations in, for example, Nyholm et al. (2002), Goldscheider and Drew (2007) and Yeh et al. (2010). Nyholm et al. (2002) achieved agreement on drawdown measured in a field pumping test and predicted by MODFLOW which models flow in the study site as confined behavior because of $|h|/H \le 0.1$ in the pumping well. Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test. In addition, Yeh et al. (2010) also achieved agreement on the hydraulic head predicted by their analytical solution based on Eq. (8), their finite difference solution based on Eq. (7) with $\partial h/\partial y = 0$ (referring to Eq. (7a)), and Teo et al. (2003) solution derived by applying the perturbation technique to deal with Eq. (7a) when $|h|/H = 0.1$ and $|\partial h/\partial x| = 0.01$ (i.e., $\alpha = 0.1$ and $|\partial \phi/\partial x| = 0.01$ at $x = 0$ in Yeh et al. (2010, Fig. 5(a)). On the other hand, the bottom of the aquifer is considered as a no-flow boundary condition denoted as

$$\partial h/\partial z = 0 \quad \text{at} \quad z=-H \tag{9}$$

Define dimensionless variables as $\bar{h} = (K_y\, H\, h)/Q$, $\bar{t} = (K_y\, t)/(S_s\, y_0^2)$, $\bar{x} = x/y_0$,

$\bar{y} = y/y_0$, $\bar{z} = z/H$, $\bar{x}_0' = x_0'/y_0$, $\bar{y}_0' = y_0'/y_0$, $\bar{z}_0' = z_0'/H$, $\bar{w}_x = w_x/y_0$ and $\bar{w}_y = w_y/y_0$ where the overbar denotes a dimensionless symbol, *H* is the initial aquifer thickness, and y_0~~,~~ is a distance between stream 1 and the center of the RCW~~, is chosen as a characteristic length~~. On the basis of the definitions, Eq. (1) can be written as

$$\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0') \tag{10}$$

where $\kappa_x = K_x/K_y$ and $\kappa_z = (K_z\, y_0^2)/(K_y\, H^2)$.

Similarly, the initial and boundary conditions are expressed as

$$\bar{h} = 0 \quad \text{at} \quad \bar{t} = 0 \tag{11}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = 0 \tag{12}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = \bar{w}_x \tag{13}$$

$$\partial \bar{h}/\partial \bar{y} - \kappa_1 \bar{h} = 0 \quad \text{at} \quad \bar{y} = 0 \tag{14}$$

$$\partial \bar{h}/\partial \bar{y} + \kappa_2 \bar{h} = 0 \quad \text{at} \quad \bar{y} = \bar{w}_y \tag{15}$$

$$\frac{\partial \bar{h}}{\partial \bar{z}} = -\frac{\gamma}{\kappa_z} \frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 0 \tag{16}$$

and

$$\partial \bar{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{17}$$

where $\kappa_1 = (K_1\, y_0)/(K_y\, b_1)$, $\kappa_2 = (K_2\, y_0)/(K_y\, b_2)$ and $\gamma = S_y/(S_s\, H)$.

2.2 Head solution for point sink

The model, Eqs. (10) – (17), reduces to an ordinary differential equation (ODE) with two boundary conditions in terms of $\bar{z}$ after taking Laplace transform and finite integral transform. The former transform converts $\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ into $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$, $\delta(\bar{x} - \bar{x}_0')\, \delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')$ in Eq. (10) into $\delta(\bar{x} - \bar{x}_0')\, \delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')/p$, and $\partial \bar{h}/\partial \bar{t}$ in Eqs. (10) and (16) into $p\hat{h} - \bar{h}|_{\bar{t}=0}$ where *p* is the Laplace parameter, and the second term, initial condition in Eq. (11), equals zero (Kreyszig, 1999). The transformed model becomes a boundary value problem written as

$$\kappa_x \frac{\partial^2 \hat{h}}{\partial \bar{x}^2} + \frac{\partial^2 \hat{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} = p\hat{h} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0')/p \tag{18}$$

with boundary conditions $\partial\hat{h}/\partial\bar{x} = 0$ at $\bar{x} = 0$ and $\bar{x} = \bar{w}_x$, $\partial\hat{h}/\partial\bar{y} - \kappa_1 \hat{h} = 0$ at $\bar{y} = 0$, $\partial\hat{h}/\partial\bar{y} + \kappa_2 \hat{h} = 0$ at $\bar{y} = \bar{w}_y$, $\partial\hat{h}/\partial\bar{z} = -p\gamma\hat{h}/\kappa_z$ at $\bar{z} = 0$ and $\partial\bar{h}/\partial\bar{z} = 0$ at $\bar{z} = -1$. We then apply finite integral transform to the problem. One can refer to Appendix A for its detailed definition. The transform converts $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$ in the problem into $\tilde{h}(\alpha_m, \beta_n, \bar{z}, p)$, and $\delta(\bar{x} - \bar{x}_0')\ \delta(\bar{y} - \bar{y}_0')$ in Eq. (18) into $\cos(\alpha_m \bar{x}_0')K(\bar{y}_0')$ and $\kappa_x \partial^2\hat{h}/\partial\bar{x}^2 + \partial^2\hat{h}/\partial\bar{y}^2$ in Eq. (18) into $-(\kappa_x\alpha_m^2 + \beta_n^2)\tilde{h}$ where $(m, n) \in 1, 2, 3, \ldots \infty$, $\alpha_m = m\pi/\bar{w}_x$, $K(\bar{y}_0')$ is defined in Eq. (A2) with $\bar{y} = \bar{y}_0'$, and β_n ~~are~~ represents eigenvalues equaling the roots of the following equation as (Latinopoulos, 1985)

$$\tan(\beta_n \bar{w}_y) = \frac{\beta_n(\kappa_1 + \kappa_2)}{\beta_n^2 - \kappa_1 \kappa_2} \tag{19}$$

The method to determine the roots is discussed in section 2.3. In turn, Eq. (18) becomes a second-order ODE defined by

$$\kappa_z \frac{\partial^2 \tilde{h}}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h} = \cos(\alpha_m \bar{x}_0') K(\bar{y}_0') \delta(\bar{z} + \bar{z}_0')/p \tag{20}$$

with two boundary conditions denoted as

$$\frac{\partial \tilde{h}}{\partial \bar{z}} = -\frac{p\gamma}{\kappa_z}\tilde{h} \quad \text{at} \quad \bar{z} = 0 \tag{21}$$

and

$$\partial\tilde{h}/\partial\bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{22}$$

Eq. (20) can be separated into two homogeneous ODEs as

$$\kappa_z \frac{\partial^2 \tilde{h}_a}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_a = 0 \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \tag{23}$$

and

$$\kappa_z \frac{\partial^2 \tilde{h}_b}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_b = 0 \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{24}$$

where h_a and h_b, respectively, represent the heads above and below $\bar{z} = -\bar{z}_0'$ where the point sink is located. Two continuity requirements should be imposed at $\bar{z} = -\bar{z}_0'$. The first is the continuity of the hydraulic head denoted as

$$\tilde{h}_a = \tilde{h}_b \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{25}$$

The second describes the discontinuity of the flux due to point pumping represented by the Dirac delta function in Eq. (20). It can be derived by integrating Eq. (20) from $\bar{z} = -\bar{z}_0'^{\,-}$ to $\bar{z} = -\bar{z}_0'^{\,+}$ as

$$\frac{\partial \tilde{h}_a}{\partial \bar{z}} - \frac{\partial \tilde{h}_b}{\partial \bar{z}} = \frac{\cos(\alpha_m \bar{x}_0')K(\bar{y}_0')}{p\kappa_z} \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{26}$$

Solving Eqs. (23) and (24) simultaneously with Eqs. (21), (22), (25), and (26) yields the Laplace-domain head solution as

$$\tilde{h}_a(\alpha_m, \beta_n, \bar{z}, p) = \Omega(-\bar{z}_0', \bar{z}, 1) \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \tag{27a}$$

and

$$\tilde{h}_b(\alpha_m, \beta_n, \bar{z}, p) = \Omega(\bar{z}, \bar{z}_0', -1) \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{27b}$$

with

$$\Omega(a,b,c) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + c\, p\gamma \sinh(b\lambda)]\cos(\alpha_m \bar{x}_0)K(\bar{y}_0)}{p\kappa_z \lambda (p\gamma \cosh\lambda + \kappa_z \lambda \sinh\lambda)} \tag{28}$$

$$\lambda = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2 + p)/\kappa_z} \tag{29}$$

where *a*, *b*, and *c* are arguments. Taking the inverse Laplace transform and finite integral transform to Eq. (28) results in Eq. (31). One is referred to Appendix B for the detailed derivation. A time-domain head solution for a point sink is therefore written as

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \begin{cases} \Phi(-\bar{z}_0', \bar{z}, 1) \text{ for } -\bar{z}_0' \le \bar{z} \le 0 \\ \Phi(\bar{z}, \bar{z}_0', -1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0' \end{cases} \tag{30}$$

with

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n\, X_n + 2\sum_{m=1}^{\infty}\phi_{m,n}\, X_{m,n}\ \cos(\alpha_m\,\bar{x})\right]\ Y_n\right\} \tag{31}$$

$$\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i} \tag{32}$$

$$\psi_{m,n} = -\cosh[(1+a)\lambda_s]\cosh(b\,\lambda_s)/(\kappa_z\,\lambda_s\ \sinh\lambda_s) \tag{33}$$

$$\psi_{m,n,0} = \mu_{m,n,0}\ \cosh[(1+a)\lambda_0]\ [-\kappa_z\,\lambda_0\ \cosh(b\,\lambda_0) + c\,p_0\,\gamma\ \sinh(b\,\lambda_0)] \tag{34}$$

$$\psi_{m,n,i} = \nu_{m,n,i}\ \cos[(1+a)\lambda_i]\ [-\kappa_z\,\lambda_i\ \cos(b\,\lambda_i) + c\,p_i\,\gamma\ \sin(b\,\lambda_i)] \tag{35}$$

$$\mu_{m,n,0} = 2\exp(p_0\,\bar{t})/\{p_0[(1+2\,\gamma)\,\kappa_z\,\lambda_0\ \cosh\lambda_0 + (p_0\,\gamma+\kappa_z)\sinh\lambda_0]\} \tag{36}$$

$$\nu_{m,n,i} = 2\exp(p_i\,\bar{t})/\{p_i[(1+2\,\gamma)\,\kappa_z\,\lambda_i\ \cos\lambda_i + (p_i\,\gamma+\kappa_z)\sin\lambda_i]\} \tag{37}$$

$$Y_n = \frac{\beta_n\cos(\beta_n\,\bar{y}) + \kappa_1\sin(\beta_n\,\bar{y})}{(\beta_n^2+\kappa_1^2)[\overline{w}_y + \kappa_2/(\beta_n^2+\kappa_2^2)] + \kappa_1} \tag{38}$$

and

$$X_{m,n} = \cos(\alpha_m\,\bar{x}_0')\ [\beta_n\ \cos(\beta_n\,\bar{y}_0') + \kappa_1\ \sin(\beta_n\bar{y}_0')] \tag{39}$$

where $\lambda_s = \sqrt{(\kappa_x\alpha_m^2+\beta_n^2)/\kappa_z}$, $p_0 = \kappa_z\lambda_0^2 - \kappa_x\alpha_m^2 - \beta_n^2$, $p_i = -\kappa_z\lambda_i^2 - \kappa_x\alpha_m^2 - \beta_n^2$, ϕ_n and X_n equal $\phi_{m,n}$ and $X_{m,n}$ with $\alpha_m = 0$, respectively, and the eigenvalues λ_0 and λ_i are, respectively, the roots of the following equations:

$$e^{2\lambda_0} = \frac{-\gamma\,\kappa_z\,\lambda_0^2 + \kappa_z\,\lambda_0 + \gamma\,(\kappa_x\,\alpha_m^2+\beta_n^2)}{\gamma\,\kappa_z\,\lambda_0^2 + \kappa_z\,\lambda_0 - \gamma\,(\kappa_x\,\alpha_m^2+\beta_n^2)} \tag{40}$$

$$\tan\lambda_i = \frac{-\gamma(\kappa_z\,\lambda_i^2 + \kappa_x\,\alpha_m^2 + \beta_n^2)}{\kappa_z\,\lambda_i} \tag{41}$$

The determination for those eigenvalues is introduced in the next section. Notice that the solution consists of simple series expanded in β_n, double series expanded in β_n and λ_i (or α_m and β_n), and triple series expanded in α_m, β_n and λ_i.

2.3 Evaluations for β_n, λ_0 and λ_i

Application of Newton's method with proper initial guesses to determine the eigenvalues β_n, λ_0 and λ_i has been proposed by Huang et al. (2014) and is briefly introduced herein. The eigenvalues are situated at the intersection points of the left-hand side (LHS) and RHS functions of Eq. (19) for β_n, Eq. (40) for λ_0, and Eq. (41) for λ_i. Hence, the initial guesses for β_n are considered as $\beta_v - \delta$ if $\beta_v > (\kappa_1\,\kappa_2)^{0.5}$ and as $\beta_v + \delta$ if $\beta_v < (\kappa_1\,\kappa_2)^{0.5}$ where $\beta_v = (2n-1)\pi/(2\,\bar{w}_y)$ and δ is a chosen small value such as 10^{-8} for avoiding being right at the vertical asymptote. In addition, the guess for λ_0 can be formulated as

$$\lambda_{0\,initial} = \delta + \left\{-\kappa_z - \sqrt{\kappa_z[\kappa_z + 4\,\gamma^2(\kappa_x\alpha_m^2 + \beta_n^2)]}\right\}/(2\gamma\kappa_z) \tag{42}$$

where the RHS second term represents the location of the vertical asymptote derived by letting the denominator of the RHS function in Eq. (40) to be zero and solving λ_0 in the resultant equation. Moreover, the guessed value for λ_i is $(2\,i-1)\pi/2 + \delta$.

2.4 Head solution for radial collector well

The lateral of RCW is approximately represented by a line sink composed of a series of adjoining point sinks. The locations of these point sinks are expressed in terms of ($\bar{x}_0 + \bar{l}\cos\theta$, $\bar{y}_0 + \bar{l}\sin\theta$, $\bar{z}_0$) where ($\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$) = (x_0/y_0, 1, z_0/H) is the central of the lateral, and $\bar{l}$ is a variable to define different locations of the point sink. The solution of head $\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t})$ for a lateral can therefore be derived by substituting $\bar{x}_0' = \bar{x}_0 + \bar{l}\cos\theta$, $\bar{y}_0' = 1 + \bar{l}\sin\theta$ and $\bar{z}_0' = \bar{z}_0$ into the point-sink solution, Eq. (30), then by integrating the resultant solution to $\bar{l}$, and finally by dividing the integration result into the sum of lateral lengths. The derivation can be denoted as

$$\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t}) = \left(\textstyle\sum_{k=1}^{N}\bar{L}_k\right)^{-1}\textstyle\sum_{k=1}^{N}\int_0^{\bar{L}_k}\bar{h}(\bar{x},\bar{y},\bar{z},\bar{t})\,d\bar{l} \tag{43}$$

where $\bar{L}_k = L_k/y_0$ is the *k*-th dimensionless lateral length. Note that the integration variable $\bar{l}$ (i.e., $\bar{x}_0'$ and $\bar{y}_0'$) appears only in X_n and $X_{m,n}$ in Eq. (31). The integral in Eq. (43) can thus be done analytically by integrating X_n and $X_{m,n}$ with respect to $\bar{l}$. After the integration, Eq. (43) can be expressed as

$$\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t}) = \left(\textstyle\sum_{k=1}^{N}\bar{L}_k\right)^{-1}\textstyle\sum_{k=1}^{N}\begin{cases}\Phi(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ \Phi(\bar{z},\bar{z}_0,-1) & \text{for } -1\le\bar{z}\le-\bar{z}_0\end{cases} \tag{44}$$

where Φ is defined by Eqs. (31) – (38), and X_n and $X_{m,n}$ in Eq. (31) are replaced, respectively, by

$$X_{n,k} = -G_k/(\beta_n \sin\theta_k) \tag{45}$$

and

$$X_{m,n,k} = \frac{\alpha_m F_k \cos\theta_k + \beta_n G_k \sin\theta_k}{\alpha_m^2 \cos^2\theta_k - \beta_n^2 \sin^2\theta_k} \tag{46}$$

with

$$F_k = \sin(X\alpha_m)[\beta_n \cos(Y\beta_n) + \kappa_1 \sin(Y\beta_n)] - \sin(\bar{x}_0\alpha_m)(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \tag{47}$$

$$G_k = \cos(X\alpha_m)[\kappa_1 \cos(Y\beta_n) - \beta_n \sin(Y\beta_n)] - \cos(\bar{x}_0\alpha_m)(\kappa_1 \cos\beta_n - \beta_n \sin\beta_n) \tag{48}$$

where $X = \bar{x}_0 + \bar{L}_k \cos\theta_k$ and $Y = 1 + \bar{L}_k \sin\theta_k$. Notice that Eq. (45) is obtained by substituting $\alpha_m = 0$ into Eq. (46). When $\theta_k = 0$ or π, Eq. (45) reduces to Eq. (49) by applying L'Hospital's rule.

$$X_{n,k} = \bar{L}_k(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \tag{49}$$

2.5 SDR solution for radial collector well

On the basis of Darcy's law and the head solution for a RCW, the SDR from streams 1 and 2 can be defined, respectively, as

$$SDR_1(\bar{t}) = -\int_{\bar{x}=0}^{\bar{x}=\bar{w}_x}\left(\int_{\bar{z}=-\bar{z}_0}^{\bar{z}=0}\frac{\partial\bar{h}_w}{\partial\bar{y}}d\bar{z} + \int_{\bar{z}=-1}^{\bar{z}=-\bar{z}_0}\frac{\partial\bar{h}_w}{\partial\bar{y}}d\bar{z}\right)d\bar{x} \quad \text{at} \quad \bar{y} = 0 \tag{50}$$

and

$$SDR_2(\bar{t}) = \int_{\bar{x}=0}^{\bar{x}=\bar{w}_x}\left(\int_{\bar{z}=-\bar{z}_0}^{\bar{z}=0}\frac{\partial\bar{h}_w}{\partial\bar{y}}d\bar{z} + \int_{\bar{z}=-1}^{\bar{z}=-\bar{z}_0}\frac{\partial\bar{h}_w}{\partial\bar{y}}d\bar{z}\right)d\bar{x} \quad \text{at} \quad \bar{y} = \bar{w}_y \tag{51}$$

Again, the double integrals in both equations can be done analytically. Notice that the series term of $2\sum_{m=1}^{\infty}\phi_{m,n} X_{m,n} \cos(\alpha_m \bar{x})$ in Eq. (31) disappears due to the consideration of Eqs. (3) and (4) and the integration with respect to $\bar{x}$ in Eqs. (50) and (51) when deriving the SDR solution. The SDR_1 and SDR_2 are therefore expressed in terms of double series and given below:

$$SDR_1(\bar{t}) = -\frac{2}{\sum_{k=1}^{N}\bar{L}_k}\sum_{k=1}^{N}\sum_{n=1}^{\infty}\left(\psi_n' + \psi_{n,0}' + \sum_{i=1}^{\infty}\psi_{n,i}'\right)X_{n,k}\, Y_n'(0) \tag{52}$$

and

$$SDR_2(\bar{t}) = \frac{2}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \left(\psi_n' + \psi_{n,0}' + \sum_{i=1}^{\infty} \psi_{n,i}' \right) X_{n,k}\, Y_n'(\bar{w}_y) \quad (53)$$

with

$$Y_n'(\bar{y}) = \frac{\kappa_1 \beta_n \cos(\beta_n\, \bar{y}) - \beta_n^2 \sin(\beta_n\, \bar{y})}{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2 /(\beta_n^2 + \kappa_2^2)] + \kappa_1} \quad (54)$$

$$\psi_n' = -\{\sinh(\bar{z}_0 \lambda_s') \cosh[(1 - \bar{z}_0\,)\lambda_s'] + \sinh[(1 - \bar{z}_0\,)\lambda_s'] \cosh(\bar{z}_0\, \lambda_s')\}/(\kappa_z \lambda_s'^2 \sinh \lambda_s') \quad (55)$$

$$\psi_{n,0}' = -\mu_{n,0}(\theta_{n,0} + \vartheta_{n,0})/\lambda_0 \quad (56)$$

$$\theta_{n,0} = \cosh[(1 - \bar{z}_0)\lambda_0]\, \{p_0'\, \gamma[-1 + \cosh(\bar{z}_0\, \lambda_0) + \kappa_z\, \lambda_0 \sinh(\bar{z}_0\, \lambda_0)]\} \quad (57)$$

$$\vartheta_{n,0} = \sinh[(1 - \bar{z}_0)\lambda_0]\, [\kappa_z\, \lambda_0 \cosh(\bar{z}_0\, \lambda_0) + p_0'\, \gamma \sinh(\bar{z}_0\, \lambda_0)] \quad (58)$$

$$\psi_{n,i}' = \nu_{n,i}(\sigma_{n,i} - \eta_{n,i})/\lambda_i \quad (59)$$

$$\sigma_{n,i} = \cos[(1 - \bar{z}_0)\lambda_i]\, \{p_i'\, \gamma[-1 + \cos(\bar{z}_0\, \lambda_i)] - \kappa_z\, \lambda_i \sin(\bar{z}_0\, \lambda_i)\} \quad (60)$$

$$\eta_{n,i} = \sin[(1 - \bar{z}_0)\lambda_i]\, [\kappa_z\, \lambda_i \cos(\bar{z}_0\, \lambda_i) + p_i'\, \gamma \sin(\bar{z}_0\, \lambda_i)] \quad (61)$$

where $\lambda_s' = \beta_n/\sqrt{\kappa_z}$; $p_0' = \kappa_z\, \lambda_0^2 - \beta_n^2$; $p_i' = -\kappa_z\, \lambda_i^2 - \beta_n^2$; $\mu_{n,0}$ equals $\mu_{m,n,0}$ in Eq. (36) with $\alpha_m = 0$; $\nu_{n,i}$ equals $\nu_{m,n,i}$ in Eq. (37) with $\alpha_m = 0$; $X_{n,k}$ is defined in Eq. (45) for $\theta_k \neq 0$ or π and Eq. (49) for $\theta_k = 0$ or π; and λ_0 and λ_i are the roots of Eqs. (40) and (41) with $\alpha_m = 0$, respectively.

2.6 Special cases of the present solution

2.6.1 Confined aquifer of finite extent

If $\gamma = 0$ (i.e., $S_y = 0$ in Eq. (8)), the top boundary is regarded as an impermeable stratum. The aquifer is then a confined system. Under this circumstance, Eq. (40) reduces to $e^{2\,\lambda_0} = 1$ having the root of $\lambda_0 = 0$, and Eq. (41) yields $\tan \lambda_i = 0$ having the roots of $\lambda_i = i\,\pi$ where $i \in 1, 2, 3, \ldots\ \infty$. With $\gamma = 0$, $\lambda_0 = 0$ and $\lambda_i = i\,\pi$, the head solution for a confined aquifer can be expressed as Eq. (44) with Eqs. (31) – (38) and (45) – (49) where $\psi_{m,n,0}$ in Eq. (32)

is replaced by

$$\psi_{m,n,0} = -\exp(p_0\,\bar{t})/p_0 \tag{62}$$

Similarly, the SDR solution for a confined aquifer can be written as Eqs. (52) and (53) where the RHS function in Eq. (56) reduces to that in Eq. (62) by applying L'Hospital's rule with $\gamma = 0$ and $\lambda_0 = 0$.

2.6.2 Confined aquifer of infinite extent

The head solution introduced in section 2.6.1 is applicable to spatiotemporal head distributions in confined aquifers of infinite extent before the lateral boundary effect comes. Wang and Yeh (2008) indicated that the time can be quantified, in our notation, as $t = R^2S_s/(16K_y)$ (i.e., $\bar{t} = R^2/(16y_0^2)$ for dimensionless time) where R is the shortest distance between a RCW and aquifer lateral boundary. Prior to the time, the present head solution with N = 1 for a horizontal well in a confined aquifer gives very close results given in Zhan et al. (2001).

2.6.3 Unconfined aquifer of infinite extent

Prior to the beginning time mentioned in section 2.6.2, the absolute value calculated by the present head solution, Eqs. (44) with N = 1, represents drawdown induced by a horizontal well in unconfined aquifers of infinite extent. The calculated drawdown should be close to that from Zhan and Zlotnik (2002) solution for the case of the instantaneous drainage from water table decline.

2.6.4 Unconfined aquifer of semi-infinite extent

When $\kappa_1 \to \infty$ (i.e., $b_1 = 0$), Eq. (14) reduces to the Dirichlet condition of $\bar{h} = 0$ for stream 1 in the absence from a low-permeability streambed, and Eq. (19) becomes $\tan(\beta_n \bar{w}_y) = -\beta_n/\kappa_2$. In addition, the boundary effect occurring at the other three sides of the aquifer can be neglected prior to the beginning time. Moreover, when N = 1 and $\theta_1 = 0$, a RCW can be regarded as a horizontal well parallel to stream 1. Under these three conditions, the present head and SDR predictions are close to those in Huang et al. (2011), the head solution of which agrees well with measured data from a field experiment executed by Mohamed and Rushton (2006). On the other hand, before the time when the boundary effect occurs at $\bar{x} = 0$, $\bar{x} = \bar{w}_x$ and $\bar{y} = \bar{w}_y$, the present head and SDR solutions for a RCW give close predictions to those in Huang et al. (2012), the head and SDR solutions of which agree well with observation data taken from two field experiments carried out by Schafer (2006) and Jasperse (2009), respectively.

2.7 Sensitivity analysis

The hydraulic parameters determined from field observed data are inevitably subject to measurement errors. Consequently, head predictions from the analytical model have uncertainty due to the propagation of measurement errors. Sensitivity analysis can be considered as a tool of exploring the response of the head to the change in a specific parameter (Zheng and Bennett, 2002). One may define the normalized sensitivity coefficient as

$$S_{i,t} = \frac{P_i}{H}\frac{\partial h}{\partial P_i} \tag{63}$$

where $S_{i,t}$ is the normalized sensitivity coefficient for the ith parameter at time t, and P_i represents the magnitude of the ith parameter. Eq. (63) can be approximated as

$$S_{i,t} = \frac{h(P_i + \Delta P_i) - h(P_i)}{\Delta P_i} \times \frac{P_i}{H} \tag{64}$$

where ΔP_i is an increment chosen as $10^{-3} P_i$ (Yeh et al., 2008).

3. Results and discussion

This section demonstrates head and SDR predictions and explores some physical insights regarding flow behavior. In section 3.1, ~~groundwater flow and~~ equipotential lines ~~induced by pumping~~ are drawn ~~discussed~~to identify 3-D or 2-D flow without the vertical flow at a specific region. In section 3.2, the influence of anisotropy on spatial head and temporal SDR distributions is studied. In section 3.3, the sensitivity analysis is performed to investigate the response of the head to the change in each hydraulic parameter. In section 3.4, the effects of the vertical flow and well depth on temporal SDR distributions for confined and unconfined aquifers are investigated. For conciseness, we consider a RCW with two laterals with N = 2, $\bar{L}_1 = \bar{L}_2 = 0.5$, θ_1 = 0 and θ_2 = π. The well can be viewed as a horizontal well parallel to streams 1 and 2. The default values for the other dimensionless parameters are $\bar{w}_x = \bar{w}_y = 2$, γ = 100, $\bar{x}_0 = 1$, $\bar{y}_0 = 1$, $\bar{z}_0 = 0.5$, $\kappa_x = \kappa_z = 1$, and $\kappa_1 = \kappa_2 = 20$.

3.1 Identification of 3-D or 2-D flow at observation point~~Groundwater flow and hydraulic head~~

Most existing models assume 2-D flow with neglecting the vertical flow for pumping at a horizontal well (e.g., Mohamed and Rushton, 2006; Haitjema et al., 2010). The head distributions predicted by those models are inaccurate if an observation point~~well~~ is close to the region where the vertical flow prevails. Figure 2 demonstrates the ~~streamlines and~~ equipotential lines predicted by the present solution for a horizontal well in an unconfined aquifer for $\bar{x}_0 = 10$, $\bar{w}_x = \bar{w}_y = 20$ and κ_z = 0.1, 1, and 10. The well is located at $9.5 \leq \bar{x} \leq 10.5$, $\bar{y} = 1$ and $\bar{z} = 0.5$ as illustrated in the figure. The equipotential lines are based on steady-state head distributions plotted by Eq. (44) with $\bar{y} = 1$ and $\bar{t} = 10^7$. ~~The stream function ψ can be derived via the Cauchy-Riemann equation, in our notation, as~~

$$\cancel{\frac{\partial \bar{\psi}}{\partial \bar{x}} = -\sqrt{\kappa_z}\frac{\partial \bar{h}_w}{\partial \bar{z}} \qquad (65)}$$

~~where $\bar{\psi} = K_y H \psi / Q$ is the dimensionless stream function describing 2-D streamlines at the vertical plane of $\bar{y} = 1$ based on $\bar{h}_w$ in Eq. (44) with $\bar{t} = 10^7$ for steady state. The function $\bar{\psi}$ is obtained firstly by substituting Eq. (44) into Eq. (65), then by differentiating the result with respect to $\bar{z}$, and eventually by integrating the differentiation result to $\bar{x}$. The coefficient arising from the integration is determined by the condition of $\bar{\psi} = 0$ at $\bar{x} = \bar{x}_0$. The detailed derivation of the stream function is shown in Appendix C.~~ When κ_z = 0.1, in the range of 10 ≤ $\bar{x}$ ≤ 13.66, the contours of the hydraulic head are in a curved path, and the flow

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toward the well is thus slanted. Moreover, the range decreases to 10 ≤ $\bar{x}$ ≤ 11.5 when κ_z = 1 and to 10 ≤ $\bar{x}$ ≤10.82 when κ_z = 10. Beyond these ranges, the head contours are nearly vertical, and the flow is essentially horizontal. Define $\bar{d}$ = d/y_0 as a shortest dimensionless horizontal distance between the well and a nearest location of only horizontal flow. The $\bar{d}$ is therefore chosen as 3.16, 1 and 0.32 for the cases of κ_z = 0.1, 1 and 10, respectively. Substituting (κ_z, $\bar{d}$) = (0.1, 3.16), (1, 1) and (10, 0.32) into $\kappa_z \bar{d}^2$ leads to about unity. We may therefore conclude that the vertical flow at an observation ~~location~~ point is negligible if its location is beyond the range of ~~*a shortest dimensionless horizontal distance between the location and a RCW is less than*~~ $\bar{d}$ ~~=~~ ≤ $\sqrt{1/\kappa_z}$ (i.e., d ~~=~~ $<$ $H\sqrt{K_y/K_z}$) for thin aquifers, an observation ~~locations~~ point far from the well, and/or a small ratio of K_y/K_z.

3.2 Anisotropy analysis of hydraulic head and stream depletion rate

Previous articles have seldom analyzed flow behavior for anisotropic aquifers, i.e., κ_x (K_x/K_y) ≠ 1. Head predictions based on the models, developed for isotropic aquifers, will be inaccurate if κ_x ≠ 1. Consider $\bar{w}_x$ = $\bar{w}_y$ = 2, $\bar{t}$ = 10^7 for steady-state head distributions, and a RCW with $\bar{L}_1 = \bar{L}_2$ = 0.25, θ_1 = 0, θ_2 = π, and $(\bar{x}_0, \bar{y}_0, \bar{z}_0)$ = (1, 1, -0.5) for symmetry. The contours of the dimensionless head at $\bar{z} = -0.5$ are shown in Figures 3(a) – 3(d) for κ_x = 1, 10 and 50, 10^{-3}, and 10^{-4}, respectively. The figure indicates that the anisotropy causes a significant effect on the head distributions in comparison with the case of κ_x = 1. In Figure 3(b), the contours exhibit smooth curves in the strip regions of 1 ≤ $\bar{y}$ ≤ 1.45 for the case of κ_x = 10 and 1 ≤ $\bar{y}$ ≤ 1.2 for the case of κ_x = 50. For the region of $\bar{y}$ ≥ 1.45, the predicted heads for both cases agree well, and all the contour lines are parallel, indicating that the flow is essentially unidirectional. Substituting (κ_x, $\bar{y}$) = (10, 1.45) and (50, 1.2) into $\kappa_x\,(\bar{y} - 1)^2$ results in a value about 2. Accordingly, we may draw the conclusion that plots from the inequality of $\kappa_x(\bar{y}-1)^2 \le 2$ indicate the strip region for κ_x being greater than 10. Some existing models assuming 2-D flow in a vertical plane with neglecting the flow component along a horizontal well give accurate head predictions beyond the region (e.g., Anderson, 2000; Anderson, 2003; Kompani-Zare et al., 2005).

Aquifers with $K_yH \ge 10^3$ m^2/day can efficiently produce plenty of water from a well. RCWs usually operate with $Q \le 10^5$ m^3/day for field experiments (e.g., Schafer, 2006; Jasperse, 2009). We therefore define significant dimensionless head drop as $|\bar{h}| > 10^{-5}$ (i.e., $|h| > 1$ mm). The anisotropy of κ_x <1 produces the drop in the strip areas of $1 \le \bar{x} \le 1.48$ for the case of $\kappa_x = 10^{-3}$ in Figure 3(c) and $1 \le \cancel{\bar{y}}\bar{x} \le 1.32$ for the case of $\kappa_x = 10^{-4}$ in Figure 3(d). Substituting (κ_x, $\bar{x}$) = (10^{-3}, 1.48) and (10^{-4}, 1.32) into $(\bar{x}-\bar{x}_0-\bar{L}_1)^2/\kappa_x$ approximates 52.9. This result leads to the conclusion that the area can be determined by the inequalities of $(\bar{x}-\bar{x}_0-\bar{L}_1)^2 \le 52.9\kappa_x$ and $(\bar{x}-\bar{x}_0+\bar{L}_2)^2 \le 52.9\kappa_x$ for any value of κ_x in the range κ_x <1. For a RCW with irregular lateral configurations, the inequalities become $(\bar{x}-\max\bar{x}_k)^2 \le 52.9\kappa_x$ and $(\bar{x}-\min\bar{x}_k)^2 \le 52.9\kappa_x$ where $\bar{x}_k$ is coordinate $\bar{x}$ of the far end of the k-th lateral. The conclusion applies in principle to reduction in grid points for numerical solutions based on finite difference methods or finite element methods. On the other hand, we have found that Eq. (52) or (53) with various κ_x predicts the same temporal SDR distribution (not shown), indicating that the SDR is independent of κ_x.

3.3 Sensitivity analysis of hydraulic head

Consider an unconfined aquifer of H = 20 m and $w_x = w_y$ = 800 m with a RCW having two laterals of $L_1 = L_2$ = 50 m, $\theta_1 = 0$ and $\theta_2 = \pi$ and two piezometers installed at point A of (400 m, 340 m, −10 m) and point B of (400 m, 80 m, −10 m) illustrated in Figure 4. As discussed in section 3.1, the temporal head distribution at point A exhibits the unconfined behavior in Figure 4(a) because of $\kappa_z\bar{d}^2$<1 while at point B displays the confined one in Figure

4(b) due to $\kappa_z\bar{d}^2$>1. The sensitivity analysis is conducted with the aid of equation (64) to observe head responses at these two piezometers to the change in each of K_x, K_y, K_z, S_s, S_y, K_1, L_1 and z_0. The temporal distribution curves of the normalized sensitivity coefficients for those eight parameters are shown in Figures 4(a) for point A and 4(b) for point B when $K_x = K_y = 1$ m/day, $K_z = 0.1$ m/day, $S_s = 10^{-5}$ m^{-1}, $S_y = 0.2$, $K_1 = K_2 = 0.1$ m/day, $b_1 = b_2 = 1$ m, $Q = 100$ m^3/day, $x_0 = y_0 = 400$ m, and $z_0 = 10$ m. The figure demonstrates that the hydraulic heads at both piezometers are most sensitive to the change in K_y, second sensitive to the change in K_x and thirdly sensitive to the change in S_y, indicating that K_y, K_x and S_y are the most crucial factors in designing a pumping system. This figure also shows that the heads at point A is sensitive to the change in S_s at the early period of 4×10^{-3} day < t < 10^{-1} day but at point B is insensitive to the change over the entire period. In addition, the head at point A is sensitive to the changes in K_z and z_0 due to 3-D flow (i.e., $\kappa_z\bar{d}^2 < 1$) as discussed in section 3.1. In contrast, the head at point B is insensitive to the changes in K_z and z_0 because the vertical flow diminishes (i.e., $\kappa_z\bar{d}^2 > 1$). Moreover, the head at point A is sensitive to the change in L_1 but the head at point B is not because its location is far away from the well. Furthermore, the normalized sensitivity coefficient of K_1 for point A away from stream 1 approaches zero but for point B in the vicinity of stream 1 increases with time and finally maintains a certain value at the steady state. Regarding the sensitivity analysis of SDR, Huang et al. (2014) has performed the sensitivity analysis of normalized coefficients of SDR_1 to the changes in K_y, K_1 and S_s for a confined aquifer and in K_y, K_z, K_1, S_s and S_y for an unconfined aquifer.

3.4 Effects of vertical flow and well depth on stream depletion rate

Huang et al. (2014) reveals that the effect of the vertical flow on SDR induced by a vertical well is dominated by the magnitude of the key factor κ_z (i.e., $K_z y_0^2/(K_y H^2)$) where y_0 herein is a distance between stream 1 and the vertical well. They concluded that the effect is negligible when $\kappa_z \geq 10$ for a leaky aquifer. The factor should be replaced by $\kappa_z\bar{a}^2$ (i.e., $K_z a^2/(K_y H^2)$)

where a is a shortest distance measured from stream 1 to the end of a lateral of a RCW, and $\bar{a} = a/y_0 = 1$ in this study due to N = 2, θ_1 = 0 and θ_2 = π. We investigate SDR in response to various $\bar{z}_0$ and $\kappa_z \bar{a}$ for unconfined and confined aquifers. The temporal SDR_1 distributions predicted by Eq. (52) for stream 1 adjacent to an unconfined aquifer are shown in Fig. 5(a) for $\bar{z}_0$ = 0.5 and $\kappa_z \bar{a}^2$ = 0.01, 0.1, 1, 10, 20 and 30 and Fig. 5(b) for $\kappa_z \bar{a}^2$ = 1 and 30 when $\bar{z}_0$ = 0.1, 0.3, 0.5, 0.7 and 0.9. The curves of SDR_1 versus $\bar{t}$ is plotted in both panels by the present SDR solution for a confined aquifer. In Fig. 5(a), the present solution for an unconfined aquifer predicts a close SDR_1 to that for the confined aquifer when $\kappa_z \bar{a}^2$ = 0.01, indicating that the vertical flow in the unconfined aquifer is ignorable. The SDR_1 for the unconfined aquifer with $\kappa_z \bar{a}^2$ = 30 behaves like that for a confined one, indicating the vertical flow is also ignorable. The SDR_1 is therefore independent of well depths $\bar{z}_0$ when $\kappa_z \bar{a}^2$ = 30 as shown in Fig. 5(b). We may therefore conclude that, under the condition of $\kappa_z \bar{a}^2 \leq 0.01$ or $\kappa_z \bar{a}^2 \geq 30$, a 2-D horizontal flow model can give good predictions in SDR_1 for unconfined aquifers. In contrast, SDR_1 increases with decreasing $\kappa_z \bar{a}^2$ when $0.01 < \kappa_z \bar{a}^2 < 30$ in Fig. 5(a), indicating that the vertical flow component induced by pumping in unconfined aquifers significantly affects SDR_1. The effect of well depth $\bar{z}_0$ on SDR_1 is also significant as shown in Fig. 5(b) when $\kappa_z \bar{a}^2$ = 1. Obviously, the vertical flow effect should be considered in a model when $0.01 < \kappa_z \bar{a}^2 < 30$ for unconfined aquifers.

It is interesting to note that the SDR_1 or SDR_2 induced by two laterals (i.e., θ_1 = 0 and θ_2 = π) parallel to the streams adjacent to a confined aquifer is independent of $\kappa_z \bar{a}^2$ and $\bar{z}_0$ but depends on aquifer width of $\bar{w}_y$. The temporal SDR distribution curves based on Eqs. (52) and (53) with γ = 0 for a confined aquifer with $\bar{w}_y$ = 2, 4, 6, 10 and 20 are plotted in Fig. 6. The dimensionless distance between the well and stream 1 is set to unity (i.e., $\bar{y}_0 = 1$) for each case. The SDR_1 predicted by Hunt (1999) solution based on a vertical well in a confined aquifer extending infinitely is considered. The present solution for each $\bar{w}_y$ gives the same SDR_1 as the Hunt solution before the time when stream 2 contributes filtration water to the aquifer and influences the supply of SDR_1. It is interesting to note that the sum of steady-state SDR_1 and SDR_2 is always unity for a fixed $\bar{w}_y$. The former and latter can be estimated by $(\bar{w}_y - 1)/\bar{w}_y$ and $1/\bar{w}_y$, respectively. Such a result corresponds with that in Sun and Zhan (2007) which investigates the distribution of steady-state SDR_1 and SDR_2 induced by a vertical well.

4. Concluding remarks

This study develops a new analytical model describing 3-D flow induced by a RCW in a rectangular confined or unconfined aquifer bounded by two parallel streams and no-flow stratums in the other two sides. The flow equation in terms of the hydraulic head with a point sink term is employed. Both streams fully penetrate the aquifer and are under the Robin condition in the presence of low-permeability streambeds. A first-order free surface equation (8) describing the water table decline gives good predictions when the conditions $|h|/H \leq 0.1$ and $|\partial h/\partial x| + |\partial h/\partial y| \leq 0.01$ are satisfied. The flux across the well screen might be uniform on a lateral within 150 m. The head solution for the point sink is expressed in terms of a triple series derived by the methods of Laplace transform and finite integral transform. The head solution for a RCW is then obtained by integrating the point-sink solution along the laterals and dividing the integration result by the sum of lateral lengths. The integration can be done analytically due to the aquifer of finite extent with Eqs. (3) − (6). On the basis of Darcy's law and the head solution, the SDR solution for two streams can also be acquired. The double integrals of defining the SDR in Eqs. (50) and (51) can also be done analytically due to considerations of Eqs. (3) − (6). The sensitivity analysis is performed to explore the response of the head to the change in each of the hydraulic parameters and variables. New findings regarding the responses of flow and SDR to pumping at a RCW are summarized below:

1. Groundwater flow in a region based on $\bar{d}$ $\leq$ $\sqrt{1/\kappa_z}$ is 3-D, and temporal head distributions exhibit the unconfined behavior. A mathematical model should consider 3-D flow when predicting the hydraulic head in the region. Beyond this region, groundwater flow is horizontal, and temporal head distributions display the confined behavior. A 2-D flow model can predict accurate hydraulic head.
2. The aquifer anisotropy of κ_x >10 causes unidirectional flow in the strip region determined based on $\kappa_x\,(\bar{y}-1)^2 > 2$ for a horizontal well. Existing models assuming 2-D flow in a vertical plane with neglecting the flow component along the well give accurate head predictions in the region.
3. The aquifer anisotropy of κ_x <1 produces significant change in the head (i.e., $|\bar{h}| > 10^{-5}$ or $|h| > 1$ mm) in the strip area determined by $(\bar{x} - \max \bar{x}_k)^2 \leq 52.9\,\kappa_x$ and $(\bar{x} - \min \bar{x}_k)^2 \leq 52.9\kappa_x$ for a RCW with irregular lateral configurations.
4. The hydraulic head in the whole domain is most sensitive to the change in K_y, second sensitive to the change in K_x, and thirdly sensitive to the change in S_y. They are thus the most crucial factors in designing a pumping system.
5. The hydraulic head is sensitive to changes in K_z, S_s, z_0 and L_k in the region of $\bar{d}$ < $\sqrt{1/\kappa_z}$ and is insensitive to the changes of them beyond the region.
6. The hydraulic head at observation ~~locations~~ points near stream 1 is sensitive to the change in K_1 but away from the stream isn't.
7. The effect of the vertical flow on SDR is ignorable when $\kappa_z\bar{a}^2$ ≤ 0.01 or $\kappa_z\bar{a}^2$ ≥ 30 for unconfined aquifers. In contrast, neglecting the effect will underestimate SDR when 0.01 < $\kappa_z\bar{a}^2$ < 30.
8. For unconfined aquifers, SDR increases with dimensionless well depth $\bar{z}_0$ when 0.01 < κ_z < 30 and is independent of $\bar{z}_0$ when κ_z ≤ 0.01 or κ_z ≥ 30. For confined aquifers, SDR is

independent of $\bar{z}_0$ and κ_z. For both kinds of aquifers, the distribution curve of SDR versus $\bar{t}$ is independent of aquifer anisotropy κ_x.

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Appendix A: Finite integral transform

Latinopoulos (1985) provided the finite integral transform for a rectangular aquifer domain where each side can be under either the Dirichlet, no-flow, or Robin condition. The transform associated with the boundary conditions, Eqs. (12) − (15), is defined as

$$\tilde{h}(\alpha_m, \beta_n) = \Im\{\bar{h}(\bar{x}, \bar{y})\} = \int_0^{\bar{w}_x} \int_0^{\bar{w}_y} \bar{h}(\bar{x}, \bar{y}) \cos(\alpha_m \bar{x}) K(\bar{y})\, d\bar{y}\, d\bar{x} \quad \text{(A1)}$$

with

$$K(\bar{y}) = \sqrt{2}\, \frac{\beta_n \cos(\beta_n \bar{y}) + \kappa_1 \sin(\beta_n \bar{y})}{\sqrt{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2 /(\beta_n^2 + \kappa_2^2)] + \kappa_1}} \quad \text{(A2)}$$

where $\cos(\alpha_m \bar{x})\ K(\bar{y})$ is the kernel function. According to Latinopoulos (1985, Eq. (9)), the transform has the property of

$$\Im\left\{\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2}\right\} = -(\kappa_x \alpha_m^2 + \beta_n^2)\, \tilde{h}(\alpha_m, \beta_n) \quad \text{(A3)}$$

The formula for the inverse finite integral transform can be written as (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x}, \bar{y}) = \Im^{-1}\left\{\tilde{h}(\alpha_m, \beta_n)\right\} = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty} \tilde{h}(0, \beta_n) K(\bar{y}) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{h}(\alpha_m, \beta_n) \cos(\alpha_m \bar{x}) K(\bar{y})\right] \quad \text{(A4)}$$

Appendix B: Derivation of equation (31)

The function of p in Eq. (28) is defined as

$$F(p) = \frac{\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p\,\kappa_z\lambda(p\gamma\cosh\lambda+\kappa_z\lambda\sinh\lambda)} \quad \text{(B1)}$$

Notice that the term $\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$ in Eq. (28) is excluded because it is independent of p. $F(p)$ is a single-value function with respect to p. On the basis of the residue theorem, the inverse Laplace transform for $F(p)$ equals the summation of residues of poles in the complex plane. The residue of a simple pole can be derived according to the formula below:

$$\mathrm{Res}|_{p=p_i} = \lim_{p\to p_i} F(p)\exp(p\bar{t})\,(p-p_i) \quad \text{(B2)}$$

where p_i is the location of the pole in the complex plane.

The locations of poles are the roots of the equation obtained by letting the denominator in Eq. (B1) to be zero, denoted as

$$p\,\kappa_z\,\lambda(p\,\gamma\cosh\lambda+\kappa_z\,\lambda\sinh\lambda)=0 \quad \text{(B3)}$$

where λ is defined in Eq. (29). Notice that $p=-\kappa_x\alpha_m^2-\beta_n^2$ obtained by $\lambda=0$ is not a pole in spite of being a root. Apparently, one pole is at $p = 0$, and the residue based on Eq. (B2) with $p_i=0$ is expressed as

$$\mathrm{Res}|_{p=0} = \lim_{p\to 0}\frac{\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{\kappa_z\lambda(p\gamma\cosh\lambda+\kappa_z\lambda\sinh\lambda)}\exp(p\bar{t}) \quad \text{(B4)}$$

Eq. (B4) with $p=0$ and $\lambda=\lambda_s$ reduces to $\psi_{m,n}$ in Eq. (33).

Other poles are determined by the equation of

$$p\,\gamma\cosh\lambda+\kappa_z\,\lambda\sinh\lambda=0 \quad \text{(B5)}$$

which comes from Eq. (B3). One pole is at $p = p_0$ between $p = 0$ and $p=-\kappa_x\,\alpha_m^2-\beta_n^2$ in the negative part of the real axis. Newton's method can be used to obtain the value of p_0. In order to have proper initial guess for Newton's method, we let $\lambda=\lambda_0$ and then have $p=\kappa_z\,\lambda_0^2-\kappa_x\,\alpha_m^2-\beta_n^2$ based on Eq. (29). Substituting $\lambda=\lambda_0$, $p=\kappa_z\,\lambda_0^2-\kappa_x\,\alpha_m^2-\beta_n^2$, $\cosh\lambda_0=(e^{\lambda_0}+e^{-\lambda_0})/2$ and $\sinh\lambda_0=(e^{\lambda_0}-e^{-\lambda_0})/2$ into Eq. (B5) and rearranging the result leads to Eq. (40). Initial guess for finding root λ_0 of Eq. (40) is discussed in section 2.3. With known value of λ_0, one can obtain $p_0 = \kappa_z\, \lambda_0^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$. According to Eq. (B2), the residue of the simple pole at $p = p_0$ is written as

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0} \frac{\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p\kappa_z\lambda(p\gamma\cosh\lambda+\kappa_z\lambda\sinh\lambda)} \exp(p\bar{t})\,(p-p_0) \quad \text{(B6)}$$

where both the denominator and nominator equal zero when $p = p_0$. Applying L'Hospital's Rule to Eq. (B6) results in

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0} \frac{2\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p[(1+2\gamma)\kappa_z\lambda\cosh\lambda+(\gamma p+\kappa_z)\sinh\lambda]} \exp(p\bar{t}) \quad \text{(B7)}$$

Eq. (B7) with $p = p_0$ and $\lambda = \lambda_0$ reduces to $\psi_{m,n,0}$ in Eq. (34).

On the other hand, infinite poles are at $p = p_i$ behind $p = -\kappa_x\, \alpha_m^2 - \beta_n^2$. Similar to the derivation of Eq. (40), we let $\lambda = \sqrt{-1}\lambda_i$ and then have $p = -\kappa_z\, \lambda_i^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \sqrt{-1}\lambda_i$, $p = -\kappa_z\, \lambda_i^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B3) and rearranging the result yields Eq. (41). The determination of λ_i is discussed in section 2.3. With known value λ_i, one can have $p_i = -\kappa_z\, \lambda_i^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$. The residues of those simple poles at p=p_i can be expressed as $\psi_{m,n,i}$ in Eq. (35) by substituting $p_0 = p_i$, $p = p_i$, $\lambda = \sqrt{-1}\lambda_i$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B7). Eventually, the inverse Laplace transform for $F(p)$ equals the sum of those residues (i.e., $\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i}$). The time-domain result of $\Omega(a,b,c)$ in Eq. (28) is then obtained as $\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$. By substituting $\tilde{h}(\alpha_m,\beta_n) = \phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$ and $\tilde{h}(0,\beta_n) = \phi_n K(\bar{y}_0)$ into Eq. (A4) and letting $\bar{h}(\bar{x},\bar{y})$ to be $\Phi(a,b,c)$, the inverse finite integral transform for the result can be derived as

$$\Phi(a,b,c) = \frac{1}{\bar{w}_x}\Big[\sum_{n=1}^{\infty}\Big(\phi_n\, K(\bar{y}_0)K(\bar{y}) + 2\sum_{m=1}^{\infty}\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\cos(\alpha_m\,\bar{x})\,K(\bar{y})\Big)\Big] \quad \text{(B8)}$$

Moreover, Eq. (B8) reduces to Eq. (31) when letting the terms of $K(\bar{y}_0)K(\bar{y})$ and

$\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)\, K(\bar{y})$ to be $2X_n Y_n$ and $2X_{m,n} Y_n$, respectively.

~~Appendix C: Derivation of $\bar{\psi}$ in Eq. (65)~~

~~The dimensionless stream function $\bar{\psi}$ in Eq. (65) can be expressed as~~

$$\bar{\psi} = C - \sqrt{\kappa_z} \int \partial \bar{h}_w / \partial \bar{z}\, d\bar{x} \quad \text{at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \qquad \text{(C1)}$$

~~where C is a coefficient resulting from the integration, and $\bar{h}_w$ is defined in Eq. (44). Substituting Eq. (44) into Eq. (C1) leads to~~

$$\bar{\psi}(\bar{x}, \bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \int \partial \Phi(-\bar{z}_0, \bar{z}, 1)/\partial \bar{z}\, d\bar{x} & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \int \partial \Phi(\bar{z}, \bar{z}_0, -1)/\partial \bar{z}\, d\bar{x} & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \qquad \text{(C2)}$$

$$\Phi(a, b, c) = \frac{2}{\bar{w}_x} \left\{ \sum_{n=1}^{\infty} \left[\phi_n X_{n,k} + 2 \sum_{m=1}^{\infty} \phi_{m,n} X_{m,n,k} \cos(\alpha_m \bar{x}) \right] Y_n \right\} \qquad \text{(C3)}$$

~~where $\phi_{m,n}$, Y_n, $X_{n,k}$ and $X_{m,n,k}$ are defined in Eqs. (32), (38), (45) and (46), respectively, and ϕ_n equals $\phi_{m,n}$ with $\alpha_m = 0$. In Eq. (C3), variable $\bar{x}$ appears only in $\cos(\alpha_m \bar{x})$, and variable $\bar{z}$ appears only in ϕ_n and $\phi_{m,n}$ in Eq. (32). Eq. (C2) therefore becomes~~

$$\bar{\psi}(\bar{x}, \bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \Phi'(-\bar{z}_0, \bar{z}, 1) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \Phi'(-\bar{z}_0, \bar{z}, 1) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \qquad \text{(C4)}$$

$$\Phi'(a, b, c) = \frac{2}{\bar{w}_x} \left\{ \sum_{n=1}^{\infty} \left[\frac{\partial \phi_n}{\partial \bar{z}} X_{n,k} \int d\bar{x} + 2 \sum_{m=1}^{\infty} \frac{\partial \phi_{m,n}}{\partial \bar{z}} X_{m,n,k} \int \cos(\alpha_m \bar{x})\, d\bar{x} \right] Y_n \right\} \qquad \text{(C5)}$$

~~Consider $\bar{t} = 10^7$ for steady-state flow that the exponential terms of $\exp(p_0 \bar{t})$ and $\exp(p_i \bar{t})$ approach zero (i.e., $p_0 > 0$ and $p_i > 0$) for the default values of the parameters used to plot Figure 2. Then, we have $\phi_{m,n} = \psi_{m,n}$ defined in Eq. (33) because of $\psi_{m,n,0} \cong 0$, $\psi_{m,n,i} \cong 0$, $\mu_{m,n,0} \cong 0$ and $\nu_{m,n,i} \cong 0$. On the basis of $\phi_{m,n} = \psi_{m,n}$ and Eq. (33) with $a = -\bar{z}_0$ and $b = \bar{z}$ for $-\bar{z}_0 \le \bar{z} \le 0$ and $a = \bar{z}$ and $b = \bar{z}_0$ for $-1 \le \bar{z} \le -\bar{z}_0$, the result of differentiation, i.e., $\partial \phi_{m,n} / \partial \bar{z}$, in Eq. (C5) equals~~

$$\frac{\partial \phi_{m,n}}{\partial \bar{z}} = \begin{cases} -\lambda_s \cosh[(1 - \bar{z}_0)\lambda_s] \sinh(\bar{z}\, \lambda_s) / (\kappa_z \lambda_s \sinh \lambda_s) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ -\lambda_s \sinh[(1 + \bar{z})\lambda_s] \cosh(\bar{z}_0 \lambda_s) / (\kappa_z \lambda_s \sinh \lambda_s) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \qquad \text{(C6)}$$

~~Notice that $\partial \phi_n / \partial \bar{z}$ in Eq. (C5) equals Eq. (C6) with $\alpha_m = 0$. In addition, both integrations in Eq. (C5) can be done analytically as~~

$$\int \cos(\alpha_m \bar{x})\, d\bar{x} = \begin{cases} \sin(\alpha_m \bar{x}) / \alpha_m \text{ for } \alpha_m \neq 0 \\ \bar{x} \text{ for } \alpha_m = 0 \end{cases} \tag{C7}$$

~~On the other hand, coefficient C in Eq. (C4) is determined by the condition of $\bar{\psi} = 0$ at $\bar{x} = \bar{x}_0$ and results in~~

$$C = \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \begin{cases} \Phi^{t}(-\bar{z}_0, \bar{z}, 1) \text{ for } -\bar{z}_0 \leq \bar{z} \leq 0 \\ \Phi^{t}(-\bar{z}_0, \bar{z}, 1) \text{ for } -1 \leq \bar{z} \leq -\bar{z}_0 \end{cases} \tag{C8}$$

~~where Φ^{t} is defined in Eq. (C5) with Eqs. (C6) and (C7), $\bar{x} = \bar{x}_0$ and $\bar{y} = 1$.~~

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Table 1. Symbols used in the text and their definitions.

Symbol	Definition
a	Shortest horizontal distance between stream 1 and the far end of lateral
$\bar{a}$	a/y_0
b_1, b_2	Thicknesses of streambeds 1 and 2, respectively
d	Shortest horizontal distance between the far end of lateral and location of having only horizontal flow
$\bar{d}$	d/y_0
H	Aquifer thickness
h	Hydraulic head
$\bar{h}$	$(K_y\ H\ h)/Q$
K_x, K_y, K_z	Aquifer hydraulic conductivities in x, y and z directions, respectively
(K_1, K_2)	Hydraulic conductivities of streambeds 1 and 2, respectively
L_k	Length ~~from x axis to~~of k-th lateral where $k \in (1, 2, \dots N)$
$\bar{L}_k$	L_k/y_0
N	The number of laterals
Q	Pumping rate of point sink or radial collector well
p	Laplace parameter
p_i	$-\kappa_z\lambda_i^2 - \kappa_x\alpha_m^2 - \beta_n^2$
p_i'	$-\kappa_z\ \lambda_i^2 - \beta_n^2$
p_0	$\kappa_z\lambda_0^2 - \kappa_x\alpha_m^2 - \beta_n^2$
p_0'	$\kappa_z\ \lambda_0^2 - \beta_n^2$
R	Shortest horizontal distance between the far end of lateral and aquifer lateral boundary
S_s, S_y	Specific storage and specific yield, respectively
t	Time since pumping
$\bar{t}$	$(K_y\ t)/(S_s\ y_0^2)$
w_x, w_y	Aquifer widths in x and y directions, respectively
$\bar{w}_x$, $\bar{w}_y$	w_x/y_0, w_y/y_0
X_n	Equaling $X_{m,n}$ defined in Eq. (39) with $\alpha_m = 0$
$X_{n,k}$	Defined in Eq. (45)
x, y, z	Cartesian coordinate system
$\bar{x}$, $\bar{y}$, $\bar{z}$	x/y_0, y/y_0, z/H
$\bar{x}_k$	Coordinate $\bar{x}$ of the far end of the k-th lateral

x_0, y_0, z_0	Location of center of RCW
$\bar{x}_0,\ \bar{y}_0,\ \bar{z}_0$	x_0/y_0, 1, z_0/H
$x_0',\ y_0',\ z_0'$	Location of point sink
$\bar{x}_0',\ \bar{y}_0',\ \bar{z}_0'$	$x_0'/y_0,\ y_0'/y_0,\ z_0'/H$
α_m	$m\,\pi/\overline{w}_x$
β_n	Roots of Eq. (19)
ϕ_n	Equaling $\phi_{m,n}$ defined in Eq. (32) with $\alpha_m = 0$
γ	$S_y/(S_s\,H)$
$\kappa_x,\ \kappa_z$	$K_x/K_y,\ (K_z\,y_0^2)/(K_y\,H^2)$
$\kappa_1,\ \kappa_2$	$(K_1\,y_0)/(K_y\,b_1),\ (K_2\,y_0)/(K_y\,b_2)$
$\lambda_0,\ \lambda_i$	Roots of Eqs. (40) and (41), respectively
$\lambda_s,\ \lambda_s'$	$\sqrt{(\kappa_x\alpha_m^2+\beta_n^2)/\kappa_z},\ \beta_n/\sqrt{\kappa_z}$
$\mu_{n,0}$	Equaling $\mu_{m,n,0}$ defined in Eq. (36) with $\alpha_m = 0$
$\nu_{n,i}$	Equaling $\nu_{m,n,i}$ defined in Eq. (37) with $\alpha_m = 0$
θ_k	Counterclockwise angle from *x* axis to *k*-th lateral where $k \in (1, 2, \ldots N)$
$\max\bar{x}_k,\ \min\bar{x}_k$	Maximum and minimum of $\bar{x}_k$, respectively, where $k \in (1, 2, \ldots N)$

Figures

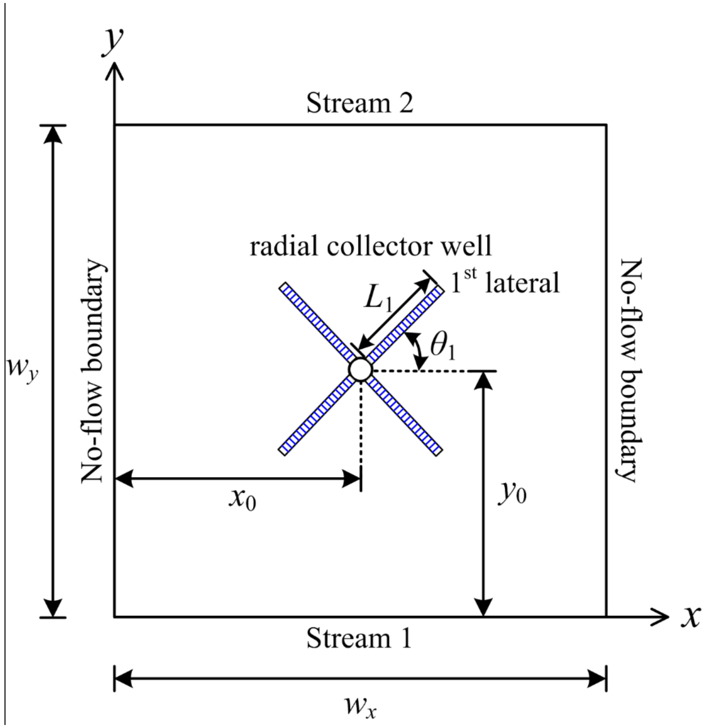
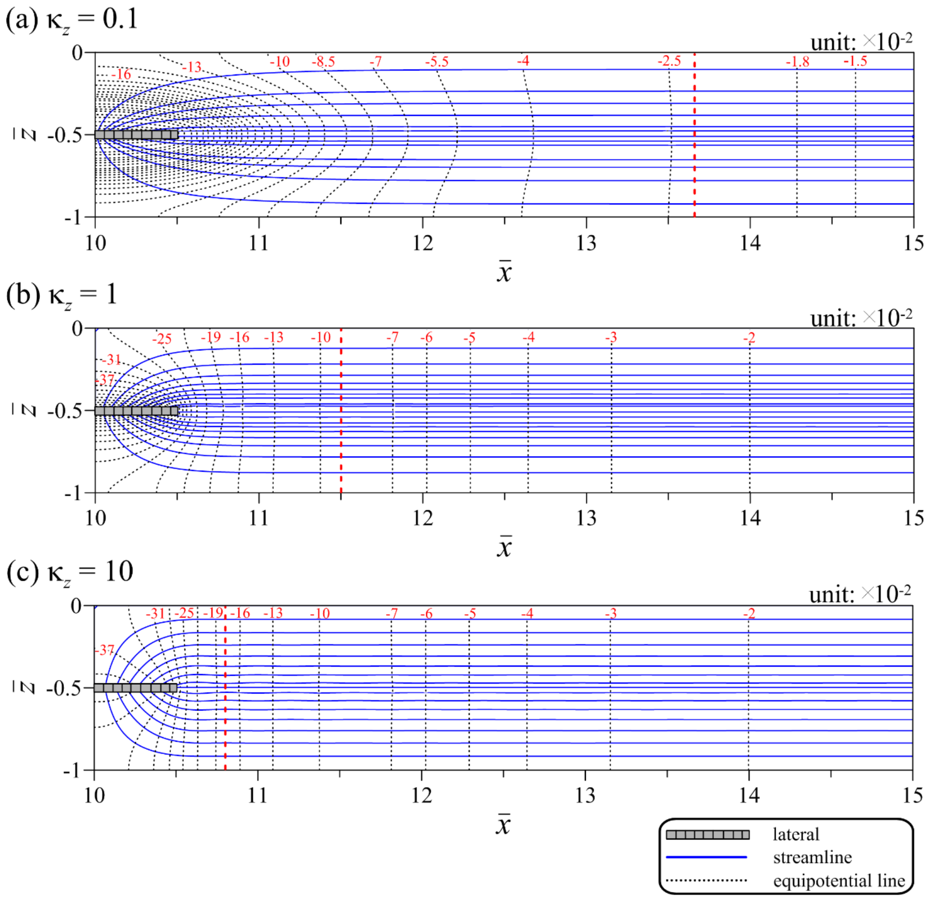


Figure 1. Schematic diagram of a radial collector well in a rectangular unconfined aquifer



(a) κz = 0.1
unit: ×10-2
-16
-13
-10
-8.5
-7
-5.5
-4
-2.5
-1.8
-1.5
(b) κz = 1
-25
-31
-37
-19
-16
-13
-10
-7
-6
-5
-4
-3
-2
(c) κz = 10
-37
-31
-25
-19
-16
-13
-10
-7
-6
-5
-4
-3
-2
lateral
streamline
equipotential line

818

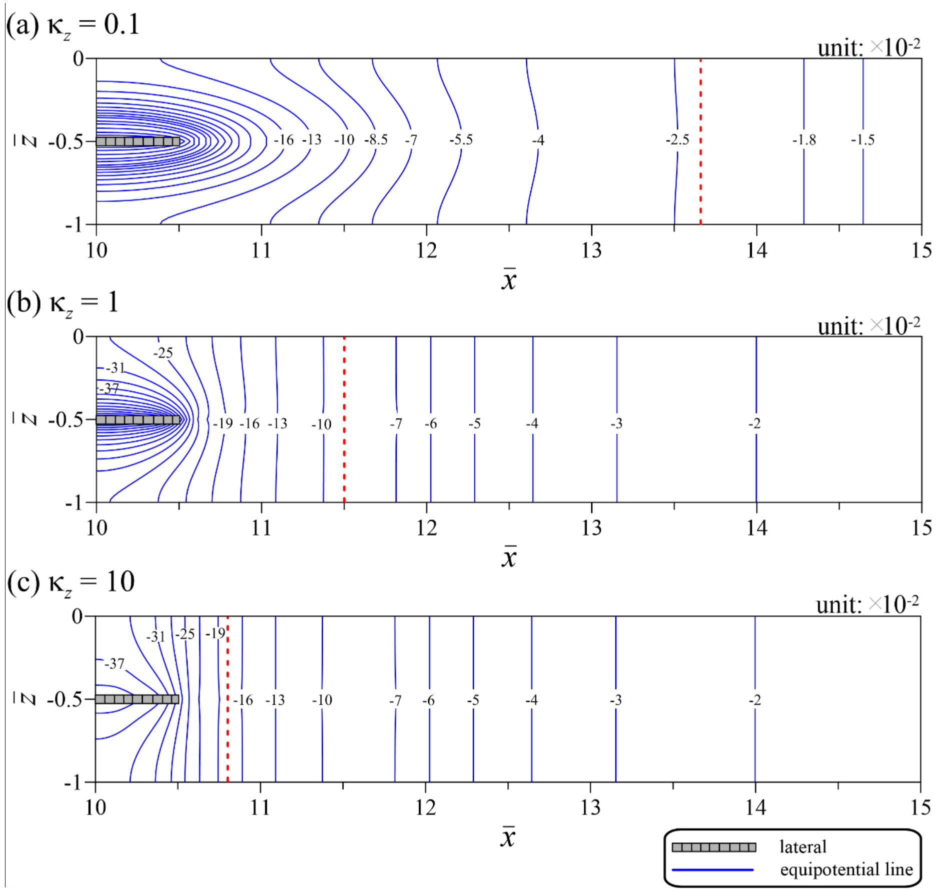


Figure 2. ~~Streamlines and e~~Equipotential lines predicted by the present solution for κ_z = (a) 0.1, (b) 1 and (c) 10.

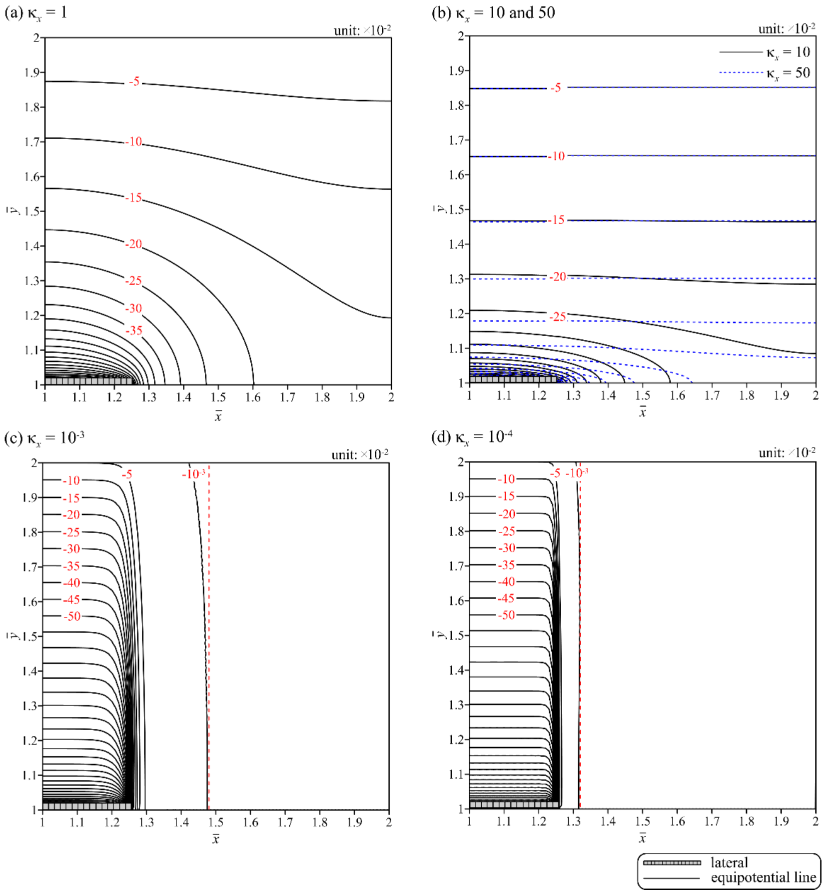


Figure 3. Spatial distributions of the dimensionless head predicted by the present head solution for κ_x = (a) 1, (b) 10 and 50, (c) 10^{-3} and (d) 10^{-4}.

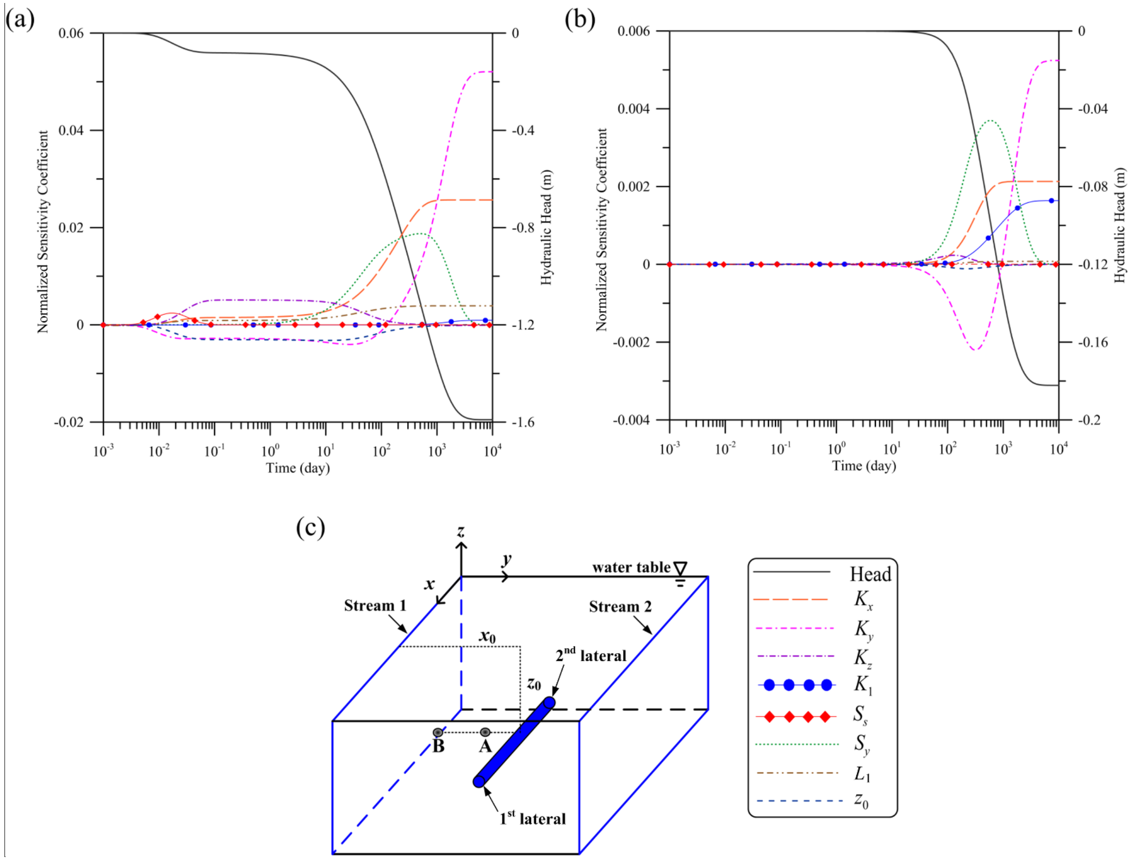


Figure 4. Temporal distribution curves of the normalized sensitivity coefficients for parameters K_x, K_y, K_z, S_s, S_y, K_1, L_1 and z_0 observed at piezometers (a) A of (400 m, 340 m, –10 m) and (b) B of (400 m, 80 m, –10 m).

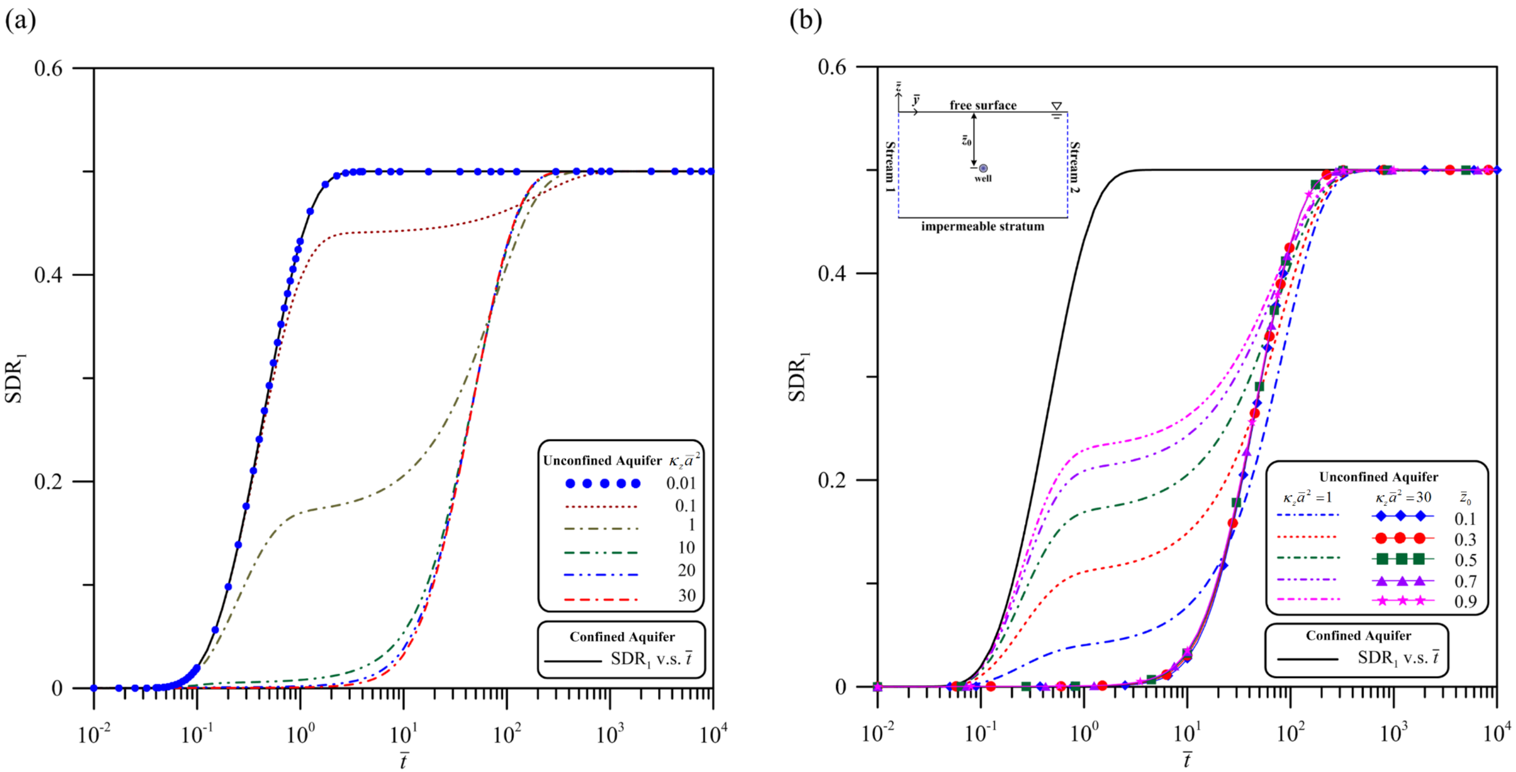


Figure 5. Temporal SDR_1 distributions predicted by Eq. (52) for stream 1 with various values of (a) $\kappa_z \bar{a}^2$ and (b) $\bar{z}_0$.

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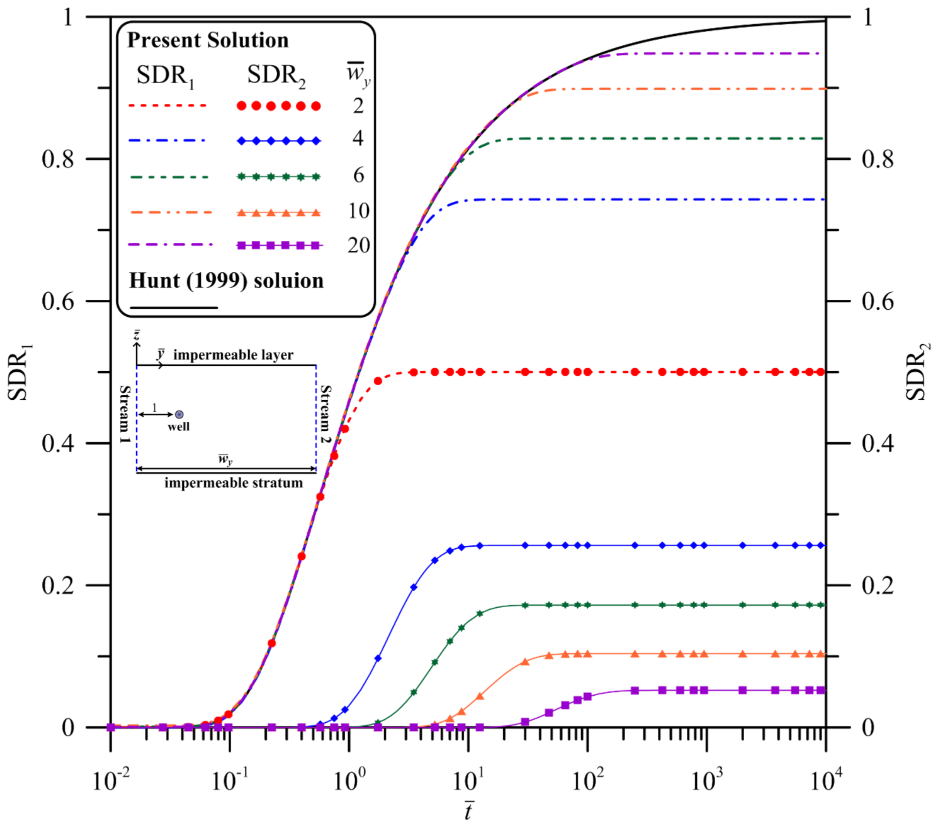

829 **Figure 6.** Temporal SDR distribution curves predicted by Eqs. (52) and (53) with $\gamma = 0$ for confined aquifers when $\overline{w}_y$ = 2, 4, 6, 10 and 20.