Referee #1: Comments and Responses

Interactive comment on "Analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer" by C.-S. Huang et al.

Anonymous Referee #1
Received and published: 31 August 2015

General Comments

This is my reivew of "Analysis of three-dimensional groudnwater flow toward a radial collector well in a finite-extent unconfined aquifer" submitted by Huang, Chen, and Yeh to HESSD. This manuscript presents a modification of one of the Latinopoulis (1985) solutions for a rectangular domain (combinations of type I, II and III boundary conditions on the lateral edges), by including the effects of a water table at the top of the aquifer. They then take the point source solution and integrate it to approximate line source solution to represent a horizontal well. I think the authors' derivation of the line-source approximation for a finite domain may be in error, as don't seem to have handled the boundary conditions in their transition from point source to line source rigorously (or it may not be presented clearly). The boundary condition used to represent the river (a fully penetrating type III boundary condition) is not realistic, and would not be widely useful. I have not seen any rectangular aquifers with trenches cut down to the bottom of the aquifer on one or two parallel sides.

Response: The solution based on the assumption of a fully penetrating stream is applicable to real-world stream depletion (or filtration) problems which satisfy the criterion $(K_z a^2)/(K_y H^2) \leq 0.01$ or $(K_z a^2)/(K_y H^2) \geq 30$ where K_y and K_z are aquifer horizontal and vertical hydraulic conductivities, respectively, H is aquifer thickness, and a is a shortest horizontal distance between the stream and a well. These two criteria mentioned above indicate that the effect of the vertical flow in aquifer systems on stream depletion/filtration rates (SDR) is ignorable as discussed in section 3.4 of the revised manuscript.

We consider a rectangular aquifer for two reasons. One is that the present model based on two parallel streams at two sides of the aquifer can be used to solve the problems involving water right distributions from the streams (Sun and Zhan, 2007). The other is that two no-flow boundaries at the other two sides of the aquifer significantly improve calculation efficiency in SDR by the fact that the triple series reduces to double series when deriving the present SDR solution (i.e., derivation from Eq. (50) to (52)). Conventional solutions derived based on aquifers of semi-infinite extent from a nearby stream can be considered as particular cases of the present solution if the three adjacent sides of the rectangular aquifer are far away from the pumping well. Regarding the derivation of the line-source approach, please refer to the response to Specific Comment 9.

We added following sentences in Introduction section:

"The streams fully penetrate the aquifer thickness and connect the aquifer with low-permeability streambeds. The model for the aquifer system with two parallel streams can be used to determine the fraction of water filtration from the streams and solve the associated water right problem (Sun and Zhan, 2007).” (lines 135 – 138 of the revised manuscript)

Specific Comments

1. The manuscript introduction and abstract should mention the river boundary conditions are "fully penetrating". The river is assumed to penetrate the entire thickness of the aquifer (treating river as a type III boundary condition), and the aquifer is not affected by anything occuring on the other side of th aquifer.

Response: Thanks for the comment. We insert following two sentences in Abstract and Introduction sections, respectively.

“The streams with low-permeability streambeds fully penetrate the aquifer thickness.” (lines 22 – 23 of the revised manuscript)

“The streams fully penetrate the aquifer thickness and connect the aquifer with low-permeability streambeds.” (lines 135 – 136 of the revised manuscript)

2. page 7505 line 9: your proposed solution also assumes flux along the well screen is uniform; please state this.

Response: Thanks for the suggestion. We add following sentence in the last paragraph of the Introduction section: “The flux across the well screen is assumed to be uniform along each of the laterals.” (lines 132 – 133 of the revised manuscript).

3. Figure 1 does not match the problem description in the text. The boundary conditions are rotated 90 degrees. Page 7509 indicates no-flow boundary conditions at $x = 0$ and $x = W_x$, but Fig 1 shows no-flow boundary conditions at $y = 0$ and $y = W_y$.

Response: The figure has been redrawn and also shown at the end of this reply.

4. Equation 7: The references associated with the water table boundary condition (Yeh et al 2010) should be Boulton (1954), Dagan (1967), and/or Neuman (1972).

- N. S. Boulton. The drawdown of the water-table under non-steady conditions near a pumped well in an unconfined formation. Proceedings Institution of Civil Engineers, 3(4):564–579, 1954.
- G. Dagan. A method of determining the permeability and effective porosity of unconfined anisotropic aquifers. Water Resources Research, 3(4):1059–1071, 1967.
- S. P. Neuman. Theory of flow in unconfined aquifers considering delayed response of the water table. Water Resources Research, 8(4):1031–1045, 1972.

Response: The citation “Yeh et al. (2010)” has changed to “Neuman (1972)”.

5. page 7510 lines 18-19: The boundary condition is linearized by uncoupling the water table location from the head and by fixing the water table position through time. "replacing $z = h$ with $z = 0$" is only partially true. This solution (and all analytical solutions) does not modify the position of the water table and boundary condition, even though the drawdown near the well increases with time.

Response: Thanks for the comment. The sentence is rewritten as "Eq. (7) is thus linearized by neglecting the quadratic terms, and the position of the water table is fixed at the initial condition (i.e., $z = 0$)." (lines 193 – 194 of the revised manuscript).

6. Equation 10: give some of the key values used to non-dimensionalize the solution in the text. Do not relegate all this to Table 1. Explicitly stating the characteristic length, time, and head would be useful here. Since this is a finite domain, there are multiple ways the characteristic length could be chosen.

Response: The characteristic length is y_0 defined as a distance from stream 1 at $y = 0$ to the center of a radial collector well. With definitions of dimensionless variables and parameters, the associated paragraph is rewritten as:

"Define dimensionless variables as $\bar{h} = (K_y\, H\, h)/Q$, $\bar{t} = (K_y\, t)/(S_s\, y_0^2)$, $\bar{x} = x/y_0$, $\bar{y} = y/y_0$, $\bar{z} = z/H$, $\bar{x}_0' = x_0'/y_0$, $\bar{y}_0' = y_0'/y_0$, $\bar{z}_0' = z_0'/H$, $\bar{w}_x = w_x/y_0$ and $\bar{w}_y = w_y/y_0$ where the overbar denotes a dimensionless symbol, and y_0, a distance between stream 1 and the center of the RCW, is chosen as a characteristic length. On the basis of the definitions, Eq. (1) can be written as

$$\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0') \quad (10)$$

where $\kappa_x = K_x/K_y$ and $\kappa_z = (K_z\, y_0^2)/(K_y\, H^2)$.

Similarly, the initial and boundary conditions are expressed as

$$\bar{h} = 0 \quad \text{at} \quad \bar{t} = 0 \quad (11)$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = 0 \quad (12)$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = \bar{w}_x \quad (13)$$

$$\partial \bar{h}/\partial \bar{y} - \kappa_1 \bar{h} = 0 \quad \text{at} \quad \bar{y} = 0 \quad (14)$$

$$\partial \bar{h}/\partial \bar{y} + \kappa_2 \bar{h} = 0 \quad \text{at} \quad \bar{y} = \bar{w}_y \quad (15)$$

$$\frac{\partial \bar{h}}{\partial \bar{z}} = -\frac{\gamma}{\kappa_z} \frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 0 \quad (16)$$

and

$$\partial \bar{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \quad (17)$$

where $\kappa_1 = (K_1\, y_0)/(K_y\, b_1)$, $\kappa_2 = (K_2\, y_0)/(K_y\, b_2)$ and $\gamma = S_y/(S_s\, H)$" (lines 212 – 228 of the revised manuscript)

7. Add "finite" before "integral transform" when referring to the Latinopoulus solution (e.g., p7511 l18, p7512 l3, p7513 l18)

Response: The phrase "double-integral transform" has changed to "finite integral transform".

8. page 7513 line 7: what exactly is meant by "$-z_0' -$" and "$-z_0' +$"? Either explain the notation, or use clearer notation.

Response: Thanks for pointing out the problem. They are revised as "$-\bar{z}_0'^{\,-}$" and "$-\bar{z}_0'^{\,+}$", respectively. In fact, the typo was made by the staff of HESSD in the proofread version.

9. Based on what is written on page 7515 (lines 18-21) and page 7516 (lines 1-2) (and the discussion about how the current approach is much faster than other approaches), it appears the line source solution is computed after the numerical inversions for the double finite x and y transforms are computed for a single point source. A single point source solution is computed, then this is shifted and added to a new solution. It is not totally clear exactly how it is being done (this should be more explicit). The finite domain requires a totally new solution for each point source, since the distance to each of the boundary conditions is part of the solution. If the solution is just shifted and summed up, the boundary conditions will not line up – the boundary conditions will extruded over the length of the well. The authors may be doing it the right way, but they are too vague in their specification of how they do it for me to tell one way or the other.

Response: The derivation of the solution from a point sink to a line sink (representing the water extraction over the lateral of radial collector well, RCW) is under the condition that the Cartesian coordinate system and aquifer boundaries are fixed. We integrate the point sink solution along the lateral by varying the locations of the point sinks without shifting the coordinate system. For more detailed derivation, following paragraph is provided:

"The lateral of RCW is approximately represented by a line sink composed of a series of adjoining point sinks. The locations of these point sinks are expressed in terms of $(\bar{x}_0 + \bar{l}\cos\theta,\ \bar{y}_0 + \bar{l}\sin\theta,\ \bar{z}_0)$ where $(\bar{x}_0,\ \bar{y}_0,\ \bar{z}_0) = (x_0/y_0,\ 1,\ z_0/H)$ is the central of the lateral, and $\bar{l}$ is a variable to define different locations of the point sink. The solution of head $\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t})$ for a lateral can therefore be derived by substituting $\bar{x}_0' = \bar{x}_0 + \bar{l}\cos\theta$, $\bar{y}_0' = 1 + \bar{l}\sin\theta$ and $\bar{z}_0' = \bar{z}_0$ into the point-sink solution, Eq. (30), then by integrating the resultant solution to $\bar{l}$, and finally by dividing the integration result into the sum of lateral lengths. The derivation can be denoted as

$$\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t}) = \left(\textstyle\sum_{k=1}^{N} \bar{L}_k\right)^{-1} \sum_{k=1}^{N} \int_0^{\bar{L}_k} \bar{h}(\bar{x},\bar{y},\bar{z},\bar{t})\, d\bar{l} \qquad (43)$$

where $\bar{L}_k = L_k/y_0$ is the k-th dimensionless lateral length. Note that the integration variable $\bar{l}$ (i.e., $\bar{x}_0'$ and $\bar{y}_0'$)

appears only in X_n and $X_{m,n}$ in Eq. (31). The integral in Eq. (43) can thus be done analytically by integrating X_n and $X_{m,n}$ with respect to $\bar{l}$. After the integration, Eq. (43) can be expressed as

$$\bar{h}_w(\bar{x},\bar{y},\bar{z},\bar{t}) = \left(\sum_{k=1}^{N}\bar{L}_k\right)^{-1}\sum_{k=1}^{N}\begin{cases}\Phi(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ \Phi(\bar{z},\bar{z}_0,-1) & \text{for } -1\le\bar{z}\le -\bar{z}_0\end{cases} \quad (44)$$

where Φ is defined by Eqs. (31) – (38), and X_n and $X_{m,n}$ in Eq. (31) are replaced, respectively, by

$$X_{n,k} = -G_k/(\beta_n \sin\theta_k) \quad (45)$$

and

$$X_{m,n,k} = \frac{\alpha_m F_k \cos\theta_k + \beta_n G_k \sin\theta_k}{\alpha_m^2 \cos^2\theta_k - \beta_n^2 \sin^2\theta_k} \quad (46)$$

with

$$F_k = \sin(X\alpha_m)[\beta_n \cos(Y\beta_n) + \kappa_1 \sin(Y\beta_n)] - \sin(\bar{x}_0\alpha_m)(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \quad (47)$$

$$G_k = \cos(X\alpha_m)[\kappa_1 \cos(Y\beta_n) - \beta_n \sin(Y\beta_n)] - \cos(\bar{x}_0\alpha_m)(\kappa_1 \cos\beta_n - \beta_n \sin\beta_n) \quad (48)$$

where $X = \bar{x}_0 + \bar{L}_k \cos\theta_k$ and $Y = 1 + \bar{L}_k \sin\theta_k$. Notice that Eq. (45) is obtained by substituting $\alpha_m = 0$ into Eq. (46). When $\theta_k = 0$ or π, Eq. (45) reduces to Eq. (49) by applying L'Hospital's rule.

$$X_{n,k} = \bar{L}_k(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \quad (49)"$$

(lines 310 – 335 of the revised manuscript)

10. page 7504 line 24: petroleum engineering does not use radial collector wells, and this solution would be of no use to a petroleum engineer (even though they have horizontal wells). Remove this statement.

Response: It has been removed as suggested.

Technical Corrections

1. page 7507 line 4: delete "depending on situations"
2. page 7507 line 15: change "One grouped the solutions involving" to "One group involved"
3. page 7507 line 17: change "organized the" to "group included"
4. page 7508 line 7: delete "The" before "Robin boundary conditions"
5. page 7509 line 6: the $\times$ in $0 \le \times \le W_x$ is a multiplication symbol, rather than the variable *x*

Responses: Thanks, we have done the corrections.

6. Figure 1: W_x is a capital *W* in the figure, and a lowercase w everywhere in the text and Table 1.

Response: The figure is redrawn with replacing W_x and W_y by w_x and w_y, respectively. The new figure is also shown at the end of this response.

7. page 7510 lines 3-5: these two sentences seem out of place, since they refer to equations on later pages. Move

this statement to the conclusions or summary section.

Response: One of the sentences is arranged in Concluding Remarks and rewritten as:

"The integration can be done analytically due to the aquifer of finite extent with Eqs. (3) − (6)." (lines 555 – 556 of the revised manuscript)

The other sentence of "The series term of $2\sum_{m=1}^{\infty} \phi_{m,n} X_{m,n} \cos(\alpha_m \bar{x})$ in Eq. (31) of the head solution disappears when deriving the SDR solution (i.e., Eqs. (50) and (51))" has been deleted.

8. page 7510 line 7: "permeability is usually less permeable" : remove "permeable"
9. page 7510 line 8: delete "the" before "Robin"

Responses: Thanks, They have been done as suggested.

10. page 7509 line 9: do not refer to a negative z coordinate as "depth". Depth is an always-positive scalar, which is the distance below the land surface.

Response: Thanks for the comment. The phrase "at depth z_0 measured from water table" has changed to "at $z = -z_0$".

11. page 7511 line 1: change "as the no-flow" to "a no-flow"
12. Equation 30: "for" should have spaces around it and should not be in italics (like equations 26 and 27)
13. page 7515 lines 2-3: "expended by" should be "expanded in"
14. Equation 42: the parentheses around κ_z and the square root should be large, to make association in the equation clearer.
15. page 7515 line 18: add commas between arguments of $\bar{h}_w$ like: $\bar{h}_w(\bar{x}, \bar{y}, \bar{z}, \bar{t})$

Responses: They have been revised according the suggestions.

16. Equation 44: remove the bar between the two options in the choice (it looks like a big fraction)

Response: The bar was inserted by the typesetter of this journal. We will remove it.

17. page 7518 line 18: in "under the confined condition" delete "under the" and "condition"

Response: Taken.

18. page 7520 line 1: add $y = 0$ before "and $y = w_y$"

Response: We would like to indicate the effect of boundaries at $\bar{x} = 0$, $\bar{x} = \bar{w}_x$ and $\bar{y} = \bar{w}_y$ on filtration from a stream at $\bar{y} = 0$. Therefore, $y = 0$ could not be added.

19. pages 7522,7523 & 7526: change "strap" to "strip" (lines 19 & 24 on 7522, line 6 on 7523, and line 22 on 7526)

Response: Done as suggested.

20. page 7527 line 20: change "no-flow" to "homogeneous Neumann" to be congruent with Dirichlet and Robin.

Response: The no-flow condition $\partial h/\partial n = 0$ is in fact a special case of the Neumann one $\partial h/\partial n = c$ with c = 0. The finite integral transform proposed by Latinopoulos (1985) is based on the former condition rather than the latter one.

21. Figure 2: what is the domain size assocaited with these figures? $W_x = W_y$ = 800? or 20?

Response: Thanks for the comment. We consider $\bar{w}_x = \bar{w}_y = 20$ for dimensionless aquifer domain ($\bar{w}_x = W_x/y_0$ and $\bar{w}_y = W_y/y_0$). We add the phrase "$\bar{w}_x = \bar{w}_y = 20$" in the associated text. (line 430 of the revised manuscript)

22. Figure 4: change "Nirmalized" to "Normalized" or "Scaled"

Response: We appreciate reviewer's eye for detail. The typo has been corrected as "Normalized".

Interactive comment on Hydrol. Earth Syst. Sci. Discuss., 12, 7503, 2015.

References

Latinopoulos, P.: Analytical solutions for periodic well recharge in rectangular aquifers with third-kind boundary conditions, J. Hydrol., 77(1), 293−306, 1985.

Sun, D. and Zhan, H.: Pumping induced depletion from two streams, Adv. Water Resour. 30(4), 1016−1026, 2007.

Figure

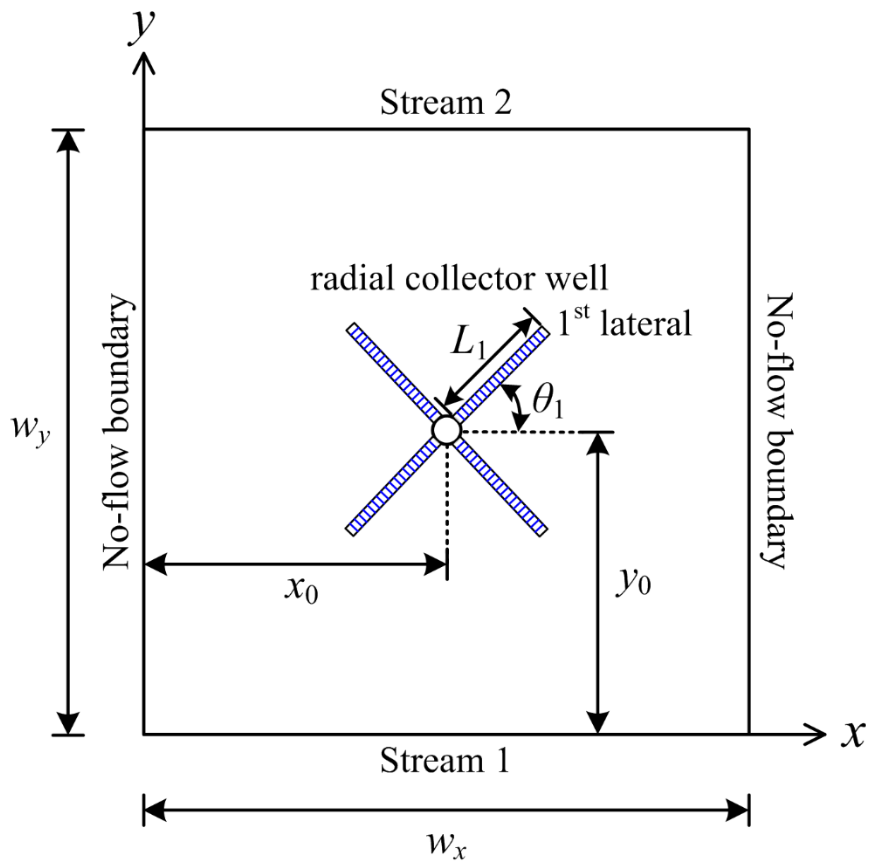

Figure 1. Schematic diagram of a radial collector well in a rectangular unconfined aquifer

Reply to 1st Comment of Referee #2

Analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer

The authors present a solution for transient flow toward a radial collector well. The title suggests that the solution covers transient flow in an unconfined aquifer, but the boundary conditions along the phreatic surface are simplified to such an extent that I doubt that the approximation is sufficiently close to the stated problem to be of much use. The phreatic surface is not only assumed to be a horizontal straight line, which in itself is a severe approximation, it is also assumed to remain in its original position at all times. The boundary along the moving phreatic surface, equation (7) in the paper, is simplified to equation (8), which implies that the vertical component of flow is equal to minus the specific yield multiplied by the rate of decrease in elevation of the phreatic surface, maintained at the original position ($z = 0$). Compressibility of the aquifer is included, but not in the sense of poro-elasticity, but using the Terzaghi approximation. I agree that this approximation is usually acceptable dealing with groundwater flow, but the authors should state their approximations carefully, including this one.

Response (1st): The simplification from Eq. (7) to Eq. (8) was first proposed by Boulton (1954) and later used to develop analytical solutions by, for example, Neuman (1972), Zhan and Zlotnik (2002), and Yeh et al. (2010). The simplification has been validated by agreement on drawdown measured by a field pumping test and predicted by Neuman (1972) solution based on Eq. (8) (e.g., Goldscheider and Drew, 2007, p. 88). We inserted the following sentence right below Eq. (8):

"Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test" (lines 202 – 204 of the revised manuscript)

We also inserted the following sentence to indicate the governing equation (i.e., Eq. (1)) is based on a concept proposed by Terzaghi: "The first term on the RHS of Eq. (1) depicts aquifer storage release based on the concept of effective stress proposed by Terzaghi (see, for example, Bear, 1979; Charbeneau, 2000)" (lines 172 – 174 of the revised manuscript)

The boundary conditions along the two streams are applied over the height of the aquifer (full penetration); this is not mentioned (referee 1 also mentions this point).

Response (2nd): We inserted following two sentences in Abstract and Introduction sections, respectively:

"The streams with low-permeability streambeds fully penetrate the aquifer thickness." (lines 22 – 23 of the revised manuscript) and "The streams fully penetrate the aquifer thickness and connect the aquifer with low-permeability streambeds." (lines 135 – 136 of the revised manuscript)

The authors integrate a point sink along the legs of the radial collector well, but fail to mention what boundary condition applies along the legs. The head should be maintained constant along the legs, whereas the condition applied by the authors is constant influx, as far as I have been able to gather from the description.

Response (3rd): Thanks for the suggestion. We add following sentence in the last paragraph of the Introduction section: "The flux across the well screen is assumed to be uniform along each of the laterals." (lines 132 – 133 of the revised manuscript).

The mathematical model resulting from the highly simplified boundary conditions and the application of the various transforms is not presented in sufficient detail for me to be able to verify the steps without re-deriving much of the work, which should not be necessary.

Response (4th): Please refer to the first response for the fact that the boundary condition is reasonably simplified. Regarding the application of those transforms, we added several intermediate equations and rewrote the associated text shown at the end of this reply.

The flow problem shown in Figure 2 is not clearly defined. The authors comment about existing models assuming 2-D flow with neglecting the vertical flow component; based on this comment, I assume that this figure applies to 3D flow, but this is not stated clearly. The sections shown in the figure do not mention whether these are horizontal or vertical; neither do they mention where the sections apply. If the flow considered is three-dimensional, then there does not exist a stream function, but the authors define one in equation (65). If the flow is transient ($\bar{t} = 10^7$), then the transient storage is yet another reason for the stream function not to exist; the divergence of the specific discharge vector is not zero. Perhaps the authors made the assumption that the time considered is so large that change in storage can be neglected, but this approximation must be stated. Furthermore, equation (65) is not obvious and, besides stating the approximation, the derivation should be presented.

Response (5th): Thanks for the comment. The derivation of the stream function is shown in Appendix C of the revised manuscript and also given at the end of this reply. In addition, we added the following sentence in section 3.1.

"$\bar{\psi} = K_y H \psi / Q$ is the dimensionless stream function describing 2-D streamlines at the vertical plane of $\bar{y} = 1$ based on $\bar{h}_w$ in Eq. (44) with $\bar{t} = 10^7$ for steady state." (lines 435 – 436 of the revised manuscript)

Summary

The authors present a very complex solution based on highly simplified boundary conditions and with insufficient detail. The authors do not present any comparison with existing solutions for simplified boundary conditions as a validation, both of their equations, and of their simplifying assumptions.

Response: Please refer to 1st response for the validation of the boundary condition.

The derivations are very difficult to follow and lack sufficient detail. The authors refer to equations further in the text, a procedure that violates standard approach in scientific work, and forces the reader to look ahead for equations that have not been digested yet.

Response: Please refer to 4th response for more detailed derivation.

I believe that the authors in their use of the stream function, violate basic principles; however, they may have made assumptions that are not stated clearly but if so, this needs to be rectified.

Response: Please refer to 5th response for the application of the stream function.

I suggest that the paper be shortened substantially and rewritten as follows:

- Remove the claim that the work applies to unconfined flow; it does not.
- Focus on one particular case, e.g., a radial collector well in a confined aquifer.

Response: Please refer to 1st response for the fact that the present solution is applicable to unconfined flow. In addition, we already demonstrated the application of the present solution to the well in confined aquifers in the second paragraph of section 3.4.

- State all boundary conditions clearly, including the ones along the legs of the radial collector well and the ones along the streams.

Response: Please refer to 2nd response for the statement of fully-penetrating streams and to 3rd response for the assumption of uniform flux on the laterals of the well.

- Make a comparison with an existing solution for at least one case.

Response: We already compared transient distributions of SDR predicted by the present solution and the Hunt (1999) solution in Fig. 6.

- Present the details of the analysis, taking into account that the reader should be able to follow the steps without the need to redo the analysis.

Response: Thanks for the comment. The text has been largely revised, and the new one is given at the end of this reply.

- If use is made of a stream function, make it clear that the flow is two-dimensional and steady. Otherwise, there does not exist a stream function at all.

Response: Please refer to 5th response for the statement of two-dimensional, steady-state flow.

References

Bear, J.: Hydraulics of Groundwater, McGraw-Hill, New York, 84, 1979.

Boulton, N. S.: The drawdown of the water table under non-steady conditions near a pumped well in an unconfined formation, Proc. Inst. Civil Eng., 3, 564–79, 1954.

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Kreyszig, E.: Advanced engineering mathematics, John Wiley & Sons, New York, 258, 1999.

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Neuman, S. P.: Theory of flow in unconfined aquifers considering delayed response of the water table, Water Resour. Res., 8(4), 1031–1045, 1972.

Yeh, H. D., Huang, C. S., Chang, Y. C., and Jeng, D. S.: An analytical solution for tidal fluctuations in unconfined aquifers with a vertical beach, Water Resour. Res., 46(10), W10535, doi:10.1029/2009WR008746, 2010.

Zhan, H., and Zlotnik, V. A.: Ground water flow to horizontal and slanted wells in unconfined aquifers, Water Resour. Res., 38 (7), 1108. Doi:10.1029/2001WR000401, 2002.

Text abstracted from lines 230 – 296 and 593 – 680 of the revised manuscript

Define dimensionless variables as $\bar{h} = (K_y\, H\, h)/Q$, $\bar{t} = (K_y\, t)/(S_s\, y_0^2)$, $\bar{x} = x/y_0$, $\bar{y} = y/y_0$, $\bar{z} = z/H$, $\bar{x}_0' = x_0'/y_0$, $\bar{y}_0' = y_0'/y_0$, $\bar{z}_0' = z_0'/H$, $\bar{w}_x = w_x/y_0$ and $\bar{w}_y = w_y/y_0$ where the overbar denotes a dimensionless symbol, and y_0, a distance between stream 1 and the center of the RCW, is chosen as a characteristic length. On the basis of the definitions, Eq. (1) can be written as

$$\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0') \qquad (10)$$

where $\kappa_x = K_x/K_y$ and $\kappa_z = (K_z\, y_0^2)/(K_y\, H^2)$.

Similarly, the initial and boundary conditions are expressed as

$$\bar{h} = 0 \quad \text{at} \quad \bar{t} = 0 \qquad (11)$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = 0 \qquad (12)$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = \bar{w}_x \qquad (13)$$

$$\partial \bar{h}/\partial \bar{y} - \kappa_1 \bar{h} = 0 \quad \text{at} \quad \bar{y} = 0 \qquad (14)$$

$$\partial \bar{h}/\partial \bar{y} + \kappa_2 \bar{h} = 0 \quad \text{at} \quad \bar{y} = \bar{w}_y \qquad (15)$$

$$\frac{\partial \bar{h}}{\partial \bar{z}} = -\frac{\gamma}{\kappa_z}\frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 0 \qquad (16)$$

and

$$\partial \bar{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \qquad (17)$$

where $\kappa_1 = (K_1\, y_0)/(K_y\, b_1)$, $\kappa_2 = (K_2\, y_0)/(K_y\, b_2)$ and $\gamma = S_y/(S_s\, H)$.

2.2 Head solution for point sink

The model, Eqs. (10) – (17), reduces to an ordinary differential equation (ODE) with two boundary conditions in terms of $\bar{z}$ after taking Laplace transform and finite integral transform. The former transform converts $\bar{h}(\bar{x},\bar{y},\bar{z},\bar{t})$ into $\hat{h}(\bar{x},\bar{y},\bar{z},p)$, $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')$ in Eq. (10) into $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')/p$, and $\partial \bar{h}/\partial \bar{t}$ in Eqs. (10) and (16) into $p\hat{h} - \bar{h}\big|_{\bar{t}=0}$ where p is the Laplace parameter, and the second term, initial condition in Eq. (11), equals zero (Kreyszig, 1999). The transformed model becomes a boundary value problem written as

$$\kappa_x \frac{\partial^2 \hat{h}}{\partial \bar{x}^2} + \frac{\partial^2 \hat{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} = p\hat{h} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0')/p \qquad (18)$$

with boundary conditions $\partial \hat{h}/\partial \bar{x} = 0$ at $\bar{x} = 0$ and $\bar{x} = \bar{w}_x$, $\partial \hat{h}/\partial \bar{y} - \kappa_1 \hat{h} = 0$ at $\bar{y} = 0$, $\partial \hat{h}/\partial \bar{y} + \kappa_2 \hat{h} = 0$ at $\bar{y} = \bar{w}_y$, $\partial \hat{h}/\partial \bar{z} = -p\gamma \hat{h}/\kappa_z$ at $\bar{z} = 0$ and $\partial \hat{h}/\partial \bar{z} = 0$ at $\bar{z} = -1$. We then apply finite integral transform to the problem. One can refer to Appendix A for its detailed definition. The transform converts $\hat{h}(\bar{x},\bar{y},\bar{z},p)$ in the problem into $\tilde{h}(\alpha_m, \beta_n, \bar{z}, p)$, and $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')$ in Eq. (18) into $\cos(\alpha_m \bar{x}_0')K(\bar{y}_0')$ and $\kappa_x\, \partial^2\hat{h}/\partial\bar{x}^2 + \partial^2\hat{h}/\partial\bar{y}^2$ in Eq. (18) into $-(\kappa_x\alpha_m^2 + \beta_n^2)\tilde{h}$ where $(m,\ n) \in 1,\ 2,\ 3,\ \ldots\ \infty$, $\alpha_m = m\,\pi/\bar{w}_x$, $K(\bar{y}_0')$ is defined in Eq. (A2) with $\bar{y} = \bar{y}_0'$, and β_n are eigenvalues equaling the roots of the following equation as (Latinopoulos, 1985)

$$\tan\left(\beta_n\, \bar{w}_y\right) = \frac{\beta_n(\kappa_1 + \kappa_2)}{\beta_n^2 - \kappa_1\, \kappa_2} \tag{19}$$

The method to determine the roots is discussed in section 2.3. In turn, Eq. (18) becomes a second-order ODE defined by

$$\kappa_z \frac{\partial^2 \tilde{h}}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h} = \cos(\alpha_m\, \bar{x}_0')\, K(\bar{y}_0')\, \delta(\bar{z} + \bar{z}_0')/\,p \tag{20}$$

with two boundary conditions denoted as

$$\frac{\partial \tilde{h}}{\partial \bar{z}} = -\frac{p\,\gamma}{\kappa_z}\tilde{h} \quad \text{at} \quad \bar{z} = 0 \tag{21}$$

and

$$\partial \tilde{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{22}$$

Eq. (20) can be separated into two homogeneous ODEs as

$$\kappa_z \frac{\partial^2 \tilde{h}_a}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_a = 0 \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \tag{23}$$

and

$$\kappa_z \frac{\partial^2 \tilde{h}_b}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_b = 0 \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{24}$$

where h_a and h_b, respectively, represent the heads above and below $\bar{z} = -\bar{z}_0'$ where the point sink is located. Two continuity requirements should be imposed at $\bar{z} = -\bar{z}_0'$. The first is the continuity of the hydraulic head denoted as

$$\tilde{h}_a = \tilde{h}_b \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{25}$$

The second describes the discontinuity of the flux due to point pumping represented by the Dirac delta function in Eq. (20). It can be derived by integrating Eq. (20) from $\bar{z} = -\bar{z}_0'^{\,-}$ to $\bar{z} = -\bar{z}_0'^{\,+}$ as

$$\frac{\partial \tilde{h}_a}{\partial \bar{z}} - \frac{\partial \tilde{h}_b}{\partial \bar{z}} = \frac{\cos(\alpha_m\, \bar{x}_0')\, K(\bar{y}_0')}{p\,\kappa_z} \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{26}$$

Solving Eqs. (23) and (24) simultaneously with Eqs. (21), (22), (25), and (26) yields the Laplace-domain head solution as

$$\tilde{h}_a(\alpha_m, \beta_n, \bar{z}, p) = \Omega(-\bar{z}_0', \bar{z}, 1) \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \tag{27a}$$

and

$$\tilde{h}_b(\alpha_m, \beta_n, \bar{z}, p) = \Omega(\bar{z}, \bar{z}_0', -1) \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{27b}$$

with

$$\Omega(a,b,c) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + c\, p\gamma \sinh(b\lambda)]\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)}{p\kappa_z \lambda (p\gamma \cosh\lambda + \kappa_z \lambda \sinh\lambda)} \tag{28}$$

$$\lambda = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2 + p)/\kappa_z} \tag{29}$$

where a, b, and c are arguments. Taking the inverse Laplace transform and finite integral transform to Eq. (28) results in Eq. (31). One is referred to Appendix B for the detailed derivation. A time-domain head solution for a point sink is therefore written as

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \begin{cases} \Phi(-\bar{z}_0', \bar{z}, 1) \text{ for } -\bar{z}_0' \le \bar{z} \le 0 \\ \Phi(\bar{z}, \bar{z}_0', -1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0' \end{cases} \tag{30}$$

with

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n X_n + 2\sum_{m=1}^{\infty}\phi_{m,n} X_{m,n} \cos(\alpha_m \bar{x})\right] Y_n\right\} \tag{31}$$

$$\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i} \tag{32}$$

$$\psi_{m,n} = -\cosh[(1+a)\lambda_s]\cosh(b\,\lambda_s)/(\kappa_z \lambda_s \sinh\lambda_s) \tag{33}$$

$$\psi_{m,n,0} = \mu_{m,n,0} \cosh[(1+a)\lambda_0]\,[-\kappa_z \lambda_0 \cosh(b\,\lambda_0) + c\, p_0\, \gamma \sinh(b\,\lambda_0)] \tag{34}$$

$$\psi_{m,n,i} = \nu_{m,n,i} \cos[(1+a)\lambda_i]\,[-\kappa_z \lambda_i \cos(b\,\lambda_i) + c\, p_i\, \gamma \sin(b\,\lambda_i)] \tag{35}$$

$$\mu_{m,n,0} = 2\exp(p_0\,\bar{t})/\{p_0[(1+2\gamma)\kappa_z \lambda_0 \cosh\lambda_0 + (p_0\gamma + \kappa_z)\sinh\lambda_0]\} \tag{36}$$

$$\nu_{m,n,i} = 2\exp(p_i\,\bar{t})/\{p_i[(1+2\gamma)\kappa_z \lambda_i \cos\lambda_i + (p_i\gamma + \kappa_z)\sin\lambda_i]\} \tag{37}$$

$$Y_n = \frac{\beta_n \cos(\beta_n \bar{y}) + \kappa_1 \sin(\beta_n \bar{y})}{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2/(\beta_n^2 + \kappa_2^2)] + \kappa_1} \tag{38}$$

and

$$X_{m,n} = \cos(\alpha_m \bar{x}_0')\,[\beta_n \cos(\beta_n \bar{y}_0') + \kappa_1 \sin(\beta_n \bar{y}_0')] \tag{39}$$

where $\lambda_s = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2)/\kappa_z}$, $p_0 = \kappa_z \lambda_0^2 - \kappa_x \alpha_m^2 - \beta_n^2$, $p_i = -\kappa_z \lambda_i^2 - \kappa_x \alpha_m^2 - \beta_n^2$, ϕ_n and X_n equal $\phi_{m,n}$ and $X_{m,n}$ with $\alpha_m = 0$, respectively, and the eigenvalues λ_0 and λ_i are, respectively, the roots of the following equations:

$$e^{2\lambda_0} = \frac{-\gamma \kappa_z \lambda_0^2 + \kappa_z \lambda_0 + \gamma(\kappa_x \alpha_m^2 + \beta_n^2)}{\gamma \kappa_z \lambda_0^2 + \kappa_z \lambda_0 - \gamma(\kappa_x \alpha_m^2 + \beta_n^2)} \tag{40}$$

$$\tan\lambda_i = \frac{-\gamma(\kappa_z \lambda_i^2 + \kappa_x \alpha_m^2 + \beta_n^2)}{\kappa_z \lambda_i} \tag{41}$$

The determination for those eigenvalues is introduced in the next section. Notice that the solution consists of simple series expanded in β_n, double series expanded in β_n and λ_i (or α_m and β_n), and triple series expanded in α_m, β_n and λ_i.

Appendix A: Finite integral transform

Latinopoulos (1985) provided the finite integral transform for a rectangular aquifer domain where each side can be under either the Dirichlet, no-flow, or Robin condition. The transform associated with the boundary conditions, Eqs. (12) – (15), is defined as

$$\tilde{h}(\alpha_m, \beta_n) = \Im\{\bar{h}(\bar{x}, \bar{y})\} = \int_0^{\bar{w}_x} \int_0^{\bar{w}_y} \bar{h}(\bar{x}, \bar{y}) \cos(\alpha_m \bar{x}) K(\bar{y}) \, d\bar{y} \, d\bar{x} \tag{A1}$$

with

$$K(\bar{y}) = \sqrt{2} \frac{\beta_n \cos(\beta_n \bar{y}) + \kappa_1 \sin(\beta_n \bar{y})}{\sqrt{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2 /(\beta_n^2 + \kappa_2^2)] + \kappa_1}} \tag{A2}$$

where $\cos(\alpha_m \bar{x}) K(\bar{y})$ is the kernel function. According to Latinopoulos (1985, Eq. (9)), the transform has the property of

$$\Im\left\{\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2}\right\} = -(\kappa_x \alpha_m^2 + \beta_n^2) \tilde{h}(\alpha_m, \beta_n) \tag{A3}$$

The formula for the inverse finite integral transform can be written as (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x}, \bar{y}) = \Im^{-1}\{\tilde{h}(\alpha_m, \beta_n)\} = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty} \tilde{h}(0, \beta_n) K(\bar{y}) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{h}(\alpha_m, \beta_n) \cos(\alpha_m \bar{x}) K(\bar{y})\right] \tag{A4}$$

Appendix B: Derivation of equation (31)

The function of p in Eq. (28) is defined as

$$F(p) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{p \, \kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \tag{B1}$$

Notice that the term $\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)$ in Eq. (28) is excluded because it is independent of p. $F(p)$ is a single-value function with respect to p. On the basis of the residue theorem, the inverse Laplace transform for $F(p)$ equals the summation of residues of poles in the complex plane. The residue of a simple pole can be derived according to the formula below:

$$\mathrm{Res}|_{p=p_i} = \lim_{p \to p_i} F(p) \exp(p\bar{t}) \, (p - p_i) \tag{B2}$$

where p_i is the location of the pole in the complex plane.

The locations of poles are the roots of the equation obtained by letting the denominator in Eq. (B1) to be zero, denoted as

$$p \, \kappa_z \, \lambda (p \, \gamma \cosh \lambda + \kappa_z \, \lambda \sinh \lambda) = 0 \tag{B3}$$

where λ is defined in Eq. (29). Notice that $p = -\kappa_x \alpha_m^2 - \beta_n^2$ obtained by $\lambda = 0$ is not a pole in spite of being a root. Apparently, one pole is at $p = 0$, and the residue based on Eq. (B2) with $p_i = 0$ is expressed as

$$\mathrm{Res}|_{p=0} = \lim_{p \to 0} \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{\kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \exp(p\bar{t}) \tag{B4}$$

Eq. (B4) with $p = 0$ and $\lambda = \lambda_s$ reduces to $\psi_{m,n}$ in Eq. (33).

Other poles are determined by the equation of

$$p\,\gamma \cosh\lambda + \kappa_z\,\lambda \sinh\lambda = 0 \tag{B5}$$

which comes from Eq. (B3). One pole is at $p = p_0$ between $p = 0$ and $p = -\kappa_x\,\alpha_m^2 - \beta_n^2$ in the negative part of the real axis. Newton's method can be used to obtain the value of p_0. In order to have proper initial guess for Newton's method, we let $\lambda = \lambda_0$ and then have $p = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \lambda_0$, $p = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$, $\cosh\lambda_0 = (e^{\lambda_0} + e^{-\lambda_0})/2$ and $\sinh\lambda_0 = (e^{\lambda_0} - e^{-\lambda_0})/2$ into Eq. (B5) and rearranging the result leads to Eq. (40). Initial guess for finding root λ_0 of Eq. (40) is discussed in section 2.3. With known value of λ_0, one can obtain $p_0 = \kappa_z\,\lambda_0^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$. According to Eq. (B2), the residue of the simple pole at $p = p_0$ is written as

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0} \frac{\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p\kappa_z\lambda(p\gamma\cosh\lambda+\kappa_z\lambda\sinh\lambda)} \exp(p\bar{t})\,(p-p_0) \tag{B6}$$

where both the denominator and nominator equal zero when $p = p_0$. Applying L'Hospital's Rule to Eq. (B6) results in

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0} \frac{2\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p[(1+2\gamma)\kappa_z\lambda\cosh\lambda+(\gamma p+\kappa_z)\sinh\lambda]} \exp(p\bar{t}) \tag{B7}$$

Eq. (B7) with $p = p_0$ and $\lambda = \lambda_0$ reduces to $\psi_{m,n,0}$ in Eq. (34).

On the other hand, infinite poles are at $p = p_i$ behind $p = -\kappa_x\,\alpha_m^2 - \beta_n^2$. Similar to the derivation of Eq. (40), we let $\lambda = \sqrt{-1}\lambda_i$ and then have $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \sqrt{-1}\lambda_i$, $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B3) and rearranging the result yields Eq. (41). The determination of λ_i is discussed in section 2.3. With known value λ_i, one can have $p_i = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$. The residues of those simple poles at $p=p_i$ can be expressed as $\psi_{m,n,i}$ in Eq. (35) by substituting $p_0 = p_i$, $p = p_i$, $\lambda = \sqrt{-1}\lambda_i$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B7). Eventually, the inverse Laplace transform for $F(p)$ equals the sum of those residues (i.e., $\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i}$). The time-domain result of $\Omega(a,b,c)$ in Eq. (28) is then obtained as $\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$. By substituting $\tilde{h}(\alpha_m,\beta_n) = \phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$ and $\tilde{h}(0,\beta_n) = \phi_n K(\bar{y}_0)$ into Eq. (A4) and letting $\bar{h}(\bar{x},\bar{y})$ to be $\Phi(a,b,c)$, the inverse finite integral transform for the result can be derived as

$$\Phi(a,b,c) = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty}\left(\phi_n\,K(\bar{y}_0)K(\bar{y}) + 2\sum_{m=1}^{\infty}\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\cos(\alpha_m\,\bar{x})\,K(\bar{y})\right)\right] \tag{B8}$$

Moreover, Eq. (B8) reduces to Eq. (31) when letting the terms of $K(\bar{y}_0)K(\bar{y})$ and $\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\,K(\bar{y})$ to be $2X_nY_n$ and $2X_{m,n}Y_n$, respectively.

Appendix C: Derivation of $\bar{\psi}$ in Eq. (65)

The dimensionless stream function $\bar{\psi}$ in Eq. (65) can be expressed as

$$\bar{\psi} = C - \sqrt{\kappa_z}\int \partial\bar{h}_w/\partial\bar{z}\,d\bar{x} \text{ at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \tag{C1}$$

where C is a coefficient resulting from the integration, and $\bar{h}_w$ is defined in Eq. (44). Substituting Eq. (44) into Eq. (C1) leads to

$$\bar{\psi}(\bar{x},\bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k} \sum_{k=1}^{N} \begin{cases} \int \partial\Phi(-\bar{z}_0,\bar{z},1)/\partial\bar{z}\, d\bar{x} & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \int \partial\Phi(\bar{z},\bar{z}_0,-1)/\partial\bar{z}\, d\bar{x} & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \text{ at } \bar{y}=1 \text{ and } \bar{t}=10^7 \quad \text{(C2)}$$

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n X_{n,k} + 2\sum_{m=1}^{\infty}\phi_{m,n} X_{m,n,k} \cos(\alpha_m \bar{x})\right] Y_n\right\} \quad \text{(C3)}$$

where $\phi_{m,n}, Y_n$, $X_{n,k}$ and $X_{m,n,k}$ are defined in Eqs. (32), (38), (45) and (46), respectively, and ϕ_n equals $\phi_{m,n}$ with $\alpha_m = 0$. In Eq. (C3), variable $\bar{x}$ appears only in $\cos(\alpha_m \bar{x})$, and variable $\bar{z}$ appears only in ϕ_n and $\phi_{m,n}$ in Eq. (32). Eq. (C2) therefore becomes

$$\bar{\psi}(\bar{x},\bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k} \sum_{k=1}^{N} \begin{cases} \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \text{ at } \bar{y}=1 \text{ and } \bar{t}=10^7 \quad \text{(C4)}$$

$$\Phi'(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\frac{\partial\phi_n}{\partial\bar{z}} X_{n,k}\int d\bar{x} + 2\sum_{m=1}^{\infty}\frac{\partial\phi_{m,n}}{\partial\bar{z}} X_{m,n,k} \int\cos(\alpha_m \bar{x})\, d\bar{x}\right] Y_n\right\} \quad \text{(C5)}$$

Consider $\bar{t} = 10^7$ for steady-state flow that the exponential terms of $\exp(p_0\, \bar{t})$ and $\exp(p_i\, \bar{t})$ approach zero (i.e., $p_0 > 0$ and $p_i > 0$) for the default values of the parameters used to plot Figure 2. Then, we have $\phi_{m,n} = \psi_{m,n}$ defined in Eq. (33) because of $\psi_{m,n,0} \cong 0$, $\psi_{m,n,i} \cong 0$, $\mu_{m,n,0} \cong 0$ and $\nu_{m,n,i} \cong 0$. On the basis of $\phi_{m,n} = \psi_{m,n}$ and Eq. (33) with $a = -\bar{z}_0$ and $b = \bar{z}$ for $-\bar{z}_0 \le \bar{z} \le 0$ and $a = \bar{z}$ and $b = \bar{z}_0$ for $-1 \le \bar{z} \le -\bar{z}_0$, the result of differentiation, i.e., $\partial\phi_{m,n}/\partial\bar{z}$, in Eq. (C5) equals

$$\frac{\partial\phi_{m,n}}{\partial\bar{z}} = \begin{cases} -\lambda_s \cosh[(1-\bar{z}_0)\lambda_s]\sinh(\bar{z}\,\lambda_s)/(\kappa_z\,\lambda_s\,\sinh\lambda_s) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ -\lambda_s \sinh[(1+\bar{z})\lambda_s]\cosh(\bar{z}_0\,\lambda_s)/(\kappa_z\,\lambda_s\,\sinh\lambda_s) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{(C6)}$$

Notice that $\partial\phi_n/\partial\bar{z}$ in Eq. (C5) equals Eq. (C6) with $\alpha_m = 0$. In addition, both integrations in Eq. (C5) can be done analytically as

$$\int\cos(\alpha_m \bar{x})\, d\bar{x} = \begin{cases} \sin(\alpha_m \bar{x})/\alpha_m & \text{for } \alpha_m \ne 0 \\ \bar{x} & \text{for } \alpha_m = 0 \end{cases} \quad \text{(C7)}$$

On the other hand, coefficient C in Eq. (C4) is determined by the condition of $\bar{\psi} = 0$ at $\bar{x} = \bar{x}_0$ and results in

$$C = \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k} \sum_{k=1}^{N} \begin{cases} \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad \text{(C8)}$$

where Φ' is defined in Eq. (C5) with Eqs. (C6) and (C7), $\bar{x} = \bar{x}_0$ and $\bar{y} = 1$.

Reply to 2nd Comment of Referee #2

We thank reviewer #2 for his/her valuable and constructive comments. Our point-by-point responses to the comments are shown below. Each point first shows our previous response, then reviewer's comment on the response, and finally our present response to reviewer's comment.

1. "Analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer"
   Comment: The word Approximate should be added as the first word in the title.
   Response: The title has been changed as suggested.

2. "The simplification from Eq. (7) to Eq. (8) was first proposed by Boulton (1954) and later used to develop analytical solutions by, for example, Neuman (1972), Zhan and Zlotnik (2002), and Yeh et al. (2010). The simplification has been validated by agreement on drawdown measured by a field pumping test and predicted by Neuman (1972) solution based on Eq. (8) (e.g., Goldscheider and Drew, 2007, p. 88). We inserted the following sentence right below Eq. (8):
   "Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test.""
   Comment: I am aware that this approximation is not uncommon, but radial collector wells are often used for pumping large quantities of water. The approximation breaks down when draw-downs become too large. The issue is that the authors fail to make this point clear, and to explain what the limitations are of their approach. In the case considered in the paper, the release from storage as a result of drawing down the water table will be larger than release from elastic storage and is therefore important. The approximation replacing (7) by (8) implies that the release from storage is entirely accounted for by the vertical component of flow, neglecting the horizontal components. This approximation breaks down when the water table slopes more than a certain amount. I suggest that the authors verify in the results section that the gradients of the water table are indeed within acceptable limits.
   Response: Thanks for the comment. The simplification from Eq. (7) to (8) is hold under two conditions. The first of the conditions is that the decline of the water table $|h|$ is smaller than 10% of initial aquifer thickness H (i.e., $|h|/H \leq 0.1$) (Nyholm et al., 2002; Yeh et al., 2010). The second is that the sum of the horizontal hydraulic gradients of $|\partial h/\partial x|$ and $|\partial h/\partial y|$ at the water table is smaller than 0.01 (Yeh et al., 2010). Nyholm et al. (2002) achieved agreement on drawdown measured in a field pumping test and predicted by MODFLOW which models flow in the study site as confined behavior because of $|h|/H \leq 0.1$ in the pumping well. In addition, Yeh et al. (2010) also achieved agreement on the hydraulic head predicted by their

analytical solution based on Eq. (8), their finite-difference solution based on Eq. (7) with $\partial h/\partial y = 0$ (referring to Eq. (7a)), and Teo et al. (2003) solution derived by applying the perturbation technique to deal with Eq. (7a) when $|h|/H = 0.1$ and $\partial h/\partial x = 0.01$ (i.e., $\alpha = 0.1$ and $\partial \phi/\partial x = 0.01$ at $x = 0$, respectively, in Fig. 5(a) in Yeh et al., 2010). We therefore conclude from the result that the second-order terms of the horizontal hydraulic gradient is negligible when $|\partial h/\partial x| + |\partial h/\partial y| \leq 0.01$. With the abovementioned results, the associated paragraph is rewritten as:

"The free surface equation describing water table decline is written as

$$K_x\left(\frac{\partial h}{\partial x}\right)^2 + K_y\left(\frac{\partial h}{\partial y}\right)^2 + K_z\left(\frac{\partial h}{\partial z}\right)^2 - K_z\frac{\partial h}{\partial z} = S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z = h \tag{7}$$

Neuman (1972) indicated that the effect of the second-order terms in Eq. (7) is generally ignorable to develop analytical solutions. Eq. (7) is thus linearized by neglecting the quadratic terms, and the position of the water table is fixed at the initial condition (i.e., $z = 0$). The result is written as

$$K_z\frac{\partial h}{\partial z} = -S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z = 0 \tag{8}$$

Notice that Eq. (8) is applicable when the conditions $|h|/H \leq 0.1$ and $|\partial h/\partial x| + |\partial h/\partial y| \leq 0.01$ are satisfied. These two conditions had been studied and verified by simulations in, for example, Nyholm et al. (2002), Goldscheider and Drew (2007) and Yeh et al. (2010). Nyholm et al. (2002) achieved agreement on drawdown measured in a field pumping test and predicted by MODFLOW which models flow in the study site as confined behavior because of $|h|/H \leq 0.1$ in the pumping well. Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test. In addition, Yeh et al. (2010) also achieved agreement on the hydraulic head predicted by their analytical solution based on Eq. (8), their finite difference solution based on Eq. (7) with $\partial h/\partial y = 0$ (referring to Eq. (7a)), and Teo et al. (2003) solution derived by applying the perturbation technique to deal with Eq. (7a) when $|h|/H = 0.1$ and $|\partial h/\partial x| = 0.01$ (i.e., $\alpha = 0.1$ and $|\partial \phi/\partial x| = 0.01$ at $x = 0$ in Yeh et al. (2010, Fig. 5(a))." (lines 190 - 209 of the revise manuscript)

3. "We also inserted the following sentence to indicate the governing equation (i.e., Eq. (1)) is based on a concept proposed by Terzaghi: "The first term on the RHS of Eq. (1) depicts aquifer storage release based on the concept of effective stress proposed by Terzaghi (see, for example, Bear, 1979, p.84; Charbeneau, 2000, p.57)."
Comment: The governing equation is not based upon the concept of effective stress, but rather on the approximation that the total vertical stress is not changing. This is a good assumption for most groundwater flow problems. The concept of effective stress is not, in itself, sufficient to obtain the equation used here from Biot's equations.

Response: Thanks for the comment. The phrase “which is valid under the assumption of constant total stress” is added after the sentence. (lines 174 - 175 of the revise manuscript)

4. “A stream of partial penetration can be considered as fully penetrating if the distance between the stream and well is larger than 1.5 times the aquifer thickness (Todd and Mays, 2005).”

   Comment: I disagree with this statement. The flow pattern at distances of about 1.5 times the aquifer thickness indeed reduces to flow that is uniform over the vertical (de Saint Venant's principle). A stream, or well, with given discharge is therefore indeed indistinguishable at such distances. However, the boundary condition along a partially penetrating stream certainly affects the discharge the stream captures and replacing a head boundary condition over limited depth by one over the full depth will have an impact, not on the flow pattern at distance, but on the discharges computed. If the authors apply the constant head boundary condition over the full vertical, they must state this clearly, and, if they do this, a resistance between stream and aquifer needs to be added to obtain the proper flow rates.

   Response: Thanks for the comment. The sentence has been removed. The present model uses the Robin boundary conditions (Eqs. (5) and (6) given below) to describe the fluxes across low-permeability streambeds of two parallel streams at $y = 0$ and w_y. The SDR solution of the present model can therefore predict stream filtration rate subject to the effect of streambed resistance.

$$K_y \frac{\partial h}{\partial y} - \frac{K_1}{b_1} h = 0 \quad \text{at} \quad y = 0 \tag{5}$$

   and

$$K_y \frac{\partial h}{\partial y} + \frac{K_2}{b_2} h = 0 \quad \text{at} \quad y = w_y \tag{6}$$

   where h is the hydraulic head at stream-aquifer interfaces, K_y is the aquifer hydraulic conductivity in the y direction normal to two parallel streams, K and b are the hydraulic conductivity and thickness of the streambeds, respectively, and subscripts 1 and 2 indicate streambeds at $y = 0$ and w_y, respectively.

5. “The flux across the well screen is assumed to be uniform along each of the laterals.”

   Comment: I am not sure how it is possible in practice to maintain the flow rate constant along the lengths of the radii of the well. If this is an approximation of an actual well of constant head radii, then it should be remembered, and stated, that the head varies along the legs in the solution. It would be of more practical value to break the legs up onto segments, and solve a system of equations to fix the heads at the centers of each segment to some prescribed value.

   Response: The assumption of uniform flux along the laterals of a radial collector well (RCW) is valid when the laterals are not long. We added following sentence to illustrate the validity of the assumption.

"The assumption is valid for a short lateral within 150 m verified by agreement on drawdown observed in field experiments and predicted by existing analytical solutions (Huang et al., 2011; 2012)." (lines 133 - 135 of the revise manuscript)

Regarding each lateral represented by several segments, it will be suitable for a long lateral of RCW to consider total head loss due to friction in the lateral but this is not our present concern. Two motives have combined to make us write this paper. One is to develop the present solution of the head and stream filtration/depletion rate (SDR) induced by RCW in finite-extent unconfined aquifers. The advantage of the present solution is that its calculation only relies on Newton's method and avoids laborious calculations involved in existing solutions. For example, Huang et al. (2012) solution should resort to numerical integration of a triple integral in predicting the hydraulic head and a quintuple integral in predicting SDR. The integrand is expressed in terms of an infinite series expanded by the roots of nonlinear equations. The integration variables are associated with those roots. The other is to provide insights or new findings accounting for flow and SDR induced by RCW, given below:

(1) We quantify a region where groundwater flow is three-dimensional. Beyond this region, the flow is horizontal and can be described by existing solutions neglecting the vertical flow (e.g., Mohamed and Rushton, 2006; Haitjema et al., 2010). Please refer to the 1st conclusion in the manuscript of Huang et al. (2015, p.7526).

(2) We provide a condition under which flow in aquifers is unidirectional and perpendicular to a horizontal well. Under the condition, existing models neglecting the flow component along the well give accurate head predictions (e.g., Anderson, 2000; Anderson, 2003; Kompani-Zare et al., 2005). Please refer to the 2nd conclusion in the manuscript of Huang et al. (2015, p.7526).

(3) We present a criterion describing that the effect of the vertical flow in aquifers on SDR can be ignored. A variety of existing solutions neglecting the vertical flow can predict accurate SDR only when the criterion is met (e.g., Glover and Balmer, 1954; Hantush, 1965; Hunt, 1999; Butler et al., 2001; Sun and Zhan, 2007; Zlotnik, 2014). Please refer to the 7th conclusion in the manuscript of Huang et al. (2015, p.7527).

(4) We find that SDR for unconfined aquifers depends on the depth of installing RCW, but for confined aquifers does not at all. Please refer to the 8th conclusion in the manuscript of Huang et al. (2015, p.7527).

6. "Please refer to 1st response for the fact that the present solution is applicable to unconfined flow."

Comment: I maintain that it should be made clear that this approximation is valid only under limited conditions, where drawdowns do not exceed some maximum.

Response: Please refer to 2nd response for the condition under which the approximation is valid.

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Approximate analysis of three-dimensional groundwater flow toward a radial collector well in a finite-extent unconfined aquifer

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Abstract

This study develops a three-dimensional mathematical model for describing transient hydraulic head distributions due to pumping at a radial collector well (RCW) in a rectangular confined or unconfined aquifer bounded by two parallel streams and no-flow boundaries. The streams with low-permeability streambeds fully penetrate the aquifer thickness. The governing equation with a point-sink term is employed. A first-order free surface equation delineating the water table decline induced by the well is considered. Robin boundary conditions are adopted to describe fluxes across the streambeds. The head solution for the point sink is derived by applying the methods of finite integral transform and Laplace transform. The head solution for a RCW is obtained by integrating the point-sink solution along the laterals of the RCW and then dividing the integration result by the sum of lateral lengths. On the basis of Darcy's law and head distributions along the streams, the solution for the stream depletion rate (SDR) can also be developed. With the aid of the head and SDR solutions, the sensitivity analysis can then be performed to explore the response of the hydraulic head to the change in a specific parameter such as the horizontal and vertical hydraulic conductivities, streambed permeability, specific storage, specific yield, lateral length and well depth. Spatial head distributions subject to the anisotropy of aquifer hydraulic conductivities are analyzed. A quantitative criterion is provided to identify whether groundwater flow at a specific region is 3-D or 2-D without the vertical component. In addition, another criterion is also given to allow the neglect of vertical flow effect on SDR. Conventional 2-D flow models can be used to provide accurate head and SDR predictions if satisfying these two criteria.

Keywords: Robin boundary condition, sensitivity analysis, stream depletion rate, first-order free surface equation, analytical solution

1. Introduction

The applications of a radial collector well (RCW) have received much attention in the aspects of water resource supply and groundwater remediation since rapid advances in drilling technology. An average yield for the well approximates 27,000 m^3/day (Todd and Mays, 2005). As compared to vertical wells, RCWs require less operating cost, produce smaller drawdown, and have better efficiency of withdrawing water from thin aquifers. In addition, RCWs can extract water from an aquifer underlying obstacles such as buildings, but vertical wells cannot. Recently, Huang et al. (2012) reviewed semi-analytical and analytical solutions associated with RCWs. Since then, Yeh and Chang (2013) provided a valuable overview of articles associated with RCWs.

A variety of analytical models involving a horizontal well, a specific case of a RCW with a single lateral, in aquifers were developed (e.g., Park and Zhan, 2003; Hunt, 2005; Anderson, 2013). The flux along the well screen is commonly assumed to be uniform. The equation describing three-dimensional (3-D) flow is used. Kawecki (2000) developed analytical solutions of the hydraulic heads for the early linear flow perpendicular to a horizontal well and late pseudo-radial flow toward the middle of the well in confined aquifers. They also developed an approximate solution for unconfined aquifers on the basis of the head solution and an unconfined flow modification. The applicability of the approximate solution was later evaluated in comparison with a finite difference solution developed by Kawecki and Al-Subaikhy (2005). Zhan et al. (2001) presented an analytical solution for drawdown induced by a horizontal well in confined aquifers and compared the difference in the type curves based on the well and a vertical well. Zhan and Zlotnik (2002) developed a semi-analytical solution of drawdown due to pumping from a nonvertical well in an unconfined aquifer accounting for the effect of instantaneous drainage or delayed yield when the free surface declines. They discussed the influences of the length, depth, and inclination of the well on temporal drawdown distributions. Park and Zhan (2002) developed a semi-analytical drawdown solution considering the effects of a finite diameter, the wellbore storage, and a skin zone around a horizontal well in anisotropic leaky aquifers. They found that those effects cause significant change in drawdown at an early pumping period. Zhan and Park (2003) provided a general semi-analytical solution for pumping-induced drawdown in a confined aquifer, an unconfined aquifer on a leaky bottom, or a leaky aquifer below a water reservoir. Temporal drawdown distributions subject to the aquitard storage effect were compared with those without that effect. Sun and Zhan (2006) derived a semi-analytical solution of drawdown due to pumping at a horizontal well in a leaky aquifer. A transient one-dimensional flow equation describing the vertical flow across the aquitard was considered. The derived solution was used to evaluate the Zhan and Park (2003) solution which assumed steady-state vertical flow in the aquitard.

Sophisticated numerical models involved in RCWs or horizontal wells were also reported. Steward (1999) applied the analytic element method to approximate 3-D steady-state flow induced by horizontal wells in contaminated aquifers. They discussed the relation between a pumping rate and the size of a polluted area. Chen et al. (2003) utilized the polygon finite difference method to deal with three kinds of seepage-pipe flows including laminar, turbulent, and transitional flows within a finite-diameter horizontal well. A sandbox experiment was also carried out to verify the prediction made by the method. Mohamad and Rushton (2006) used MODFLOW to predict flows inside an aquifer, from the aquifer to a horizontal well, and within the well. The predicted head distributions were compared with field data measured in Sarawak, Malaysia. Su et al. (2007) used software TOUGH2 based on the integrated finite difference method to handle irregular configurations of several laterals of two RCWs installed beside the Russian River, Forestville, California and analyzed pumping-induced unsaturated regions beneath the river. Lee et al. (2012) developed a finite element solution with triangle elements to assess whether the operation of a RCW near Nakdong River in South Korea can induce riverbank filtration. They concluded that the well can be used for sustainable water supply at the study site. In addition, Rushton and Brassington (2013a) extended Mohamad and Rushton (2006) study by enhancing the Darcy-Weisbach formula to describe frictional head lose inside a horizontal well. The spatial distributions of predicted flux along the well revealed that the flux at the pumping end is four times of the magnitude of that at the far end. Later, Rushton and Brassington (2013b) applied the same model to a field experiment at the Seton Coast, northwest England.

Well pumping in aquifers near streams may cause groundwater−surface water interactions (e.g., Rodriguez et al., 2013; Chen et al., 2013; Zhou et al., 2013; Exner-Kittridge et al., 2014; Flipo et al., 2014; Unland et al., 2014). The stream depletion rate (SDR), commonly used to quantify stream water filtration into the adjacent aquifer, is defined as the ratio of the filtration rate to a pumping rate. The SDR ranges from zero to a certain value which could be equal to or less than unity (Zlotnik, 2004). Tsou et al. (2010) developed an analytical solution of SDR for a slanted well in confined aquifers adjacent to a stream treated as a constant-head boundary. They indicated that a horizontal well parallel to the stream induces the steady-state SDR of unity more quickly than a slanted well. Huang et al. (2011) developed an analytical SDR solution for a horizontal well in unconfined aquifers near a stream regarded as a constant-head boundary. Huang et al. (2012) provided an analytical solution for SDR induced by a RCW in unconfined aquifers adjacent to a stream with a low-permeability streambed treated as the Robin condition. The influence of the configuration of the laterals on temporal SDR and spatial drawdown distributions was analyzed. Recently, Huang et al. (2014) gave an exhaustive review on analytical and semi-analytical SDR solutions and classified these solutions into two categories. One group involved two-dimensional (2-D) flow toward a fully-penetrating vertical well according to aquifer types and stream treatments. The other group included the solutions involving 3-D and quasi 3-D flows in the lights of aquifer types, well types, and stream treatments.

At present, existing analytical solutions associated with flow toward a RCW in unconfined aquifers have involved laborious calculation (Huang et al., 2012) and predicted approximate results (Hantush and Papadopoulos, 1962). The Huang et al. (2012) solution involves numerical integration of a triple integral in predicting the hydraulic head and a quintuple integral in predicting SDR. The integrand is expressed in terms of an infinite series expanded by roots of nonlinear equations. The integration variables are related to those roots. The application of their solution is therefore limited to those who are familiar with numerical methods. In addition, the accuracy of the Hantush and Papadopoulos (1962) solution is limited to some parts of a pumping period; that is, it gives accurate drawdown predictions at early and late times but divergent ones at middle time.

The objective of this study is to present new analytical solutions of the head and SDR, which overcome the above−mentioned limitations, for 3-D flow toward a RCW. A mathematical model is built to describe 3-D spatiotemporal hydraulic head distributions in a rectangular unconfined aquifer bounded by two parallel streams and by the no-flow stratums in the other two sides. The flux across the well screen is assumed to be uniform along each of the laterals. The assumption is valid for a short lateral within 150 m verified by agreement on drawdown observed in field experiments and predicted by existing analytical solutions (Huang et al., 2011; 2012). The streams fully penetrate the aquifer thickness and connect the aquifer with low-permeability streambeds. The model for the aquifer system with two parallel streams can be used to determine the fraction of water filtration from the streams and solve the associated water right problem (Sun and Zhan, 2007). The transient 3-D groundwater flow equation with a point-sink term is considered. The first-order free surface equation is used to describe water table decline due to pumping. Robin boundary conditions are adopted to describe fluxes across the streambeds. The head solution for a point sink is derived by the methods of Laplace transform and finite integral transform. The analytical head solution for a RCW is then obtained by integrating the point-sink solution along the well and dividing the integration result by the total lateral length. The RCW head solution is expressed in terms of a triple series expanded by eigenvalues which can be obtained by a numerical algorithm such as Newton's method. On the basis of Darcy's law and the RCW head solution, the SDR solution can then be obtained in terms of a double series with fast convergence. With the aid of both solutions, the sensitivity analysis is performed to investigate the response of the hydraulic head to the change in each of aquifer parameters. The spatial distributions of the head and streamline are discussed. Spatial head distributions subject to the anisotropy of aquifer hydraulic conductivities are analyzed. The influences of the vertical flow and well depth on temporal SDR distributions are investigated. Moreover, temporal SDR distributions induced by a RCW and a fully penetrating vertical well in confined aquifers are also compared. A quantitative criterion is provided to identify whether groundwater flow at a specific region is 3-D or 2-D without the vertical component. In addition, another criterion is also given to judge the suitability of neglecting the vertical flow effect on SDR.

2. Methodology

2.1. Mathematical model

Consider a RCW in a rectangular unconfined aquifer bounded by two parallel streams and no-flow stratums as illustrated in Figure 1. The symbols for variables and parameters are defined in Table 1. The origin of the Cartesian coordinate is located at the lower left corner. The aquifer domain falls in the range of $0 \leq x \leq w_x$, $0 \leq y \leq w_y$, and $-H \leq z \leq 0$. The RCW consists of a caisson and several laterals, each of which extends finitely with length L_k and counterclockwise with angle θ_k where $k \in 1, 2, \ldots N$ and N is the number of laterals. The caisson is located at (x_0, y_0), and the surrounding laterals are at $z = -z_0$.

First of all, a mathematical model describing 3-D flow toward a point sink in the aquifer is proposed. The equation describing 3-D hydraulic head distribution $h(x, y, z, t)$ is expressed as

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} + Q\,\delta(x - x_0')\delta(y - y_0')\delta(z + z_0') \quad (1)$$

where $\delta(\ )$ is the Dirac delta function, the second term on the right-hand side (RHS) indicates the point sink, and Q is positive for pumping and negative for injection. The first term on the RHS of Eq. (1) depicts aquifer storage release based on the concept of effective stress proposed by Terzaghi (see, for example, Bear, 1979; Charbeneau, 2000), which is valid under the assumption of constant total stress. By choosing water table as a reference datum where the elevation head is set to zero, the initial condition can therefore be denoted as

$$h = 0 \quad \text{at} \quad t = 0 \quad (2)$$

Note that equation (2) introduces negative hydraulic head for pumping, and the absolute value of the head equals drawdown.

The aquifer boundaries at $x = 0$ and $x = w_x$ are considered to be impermeable and thus expressed as

$$\partial h / \partial x = 0 \quad \text{at} \quad x = 0 \quad (3)$$

and

$$\partial h / \partial x = 0 \quad \text{at} \quad x = w_x \quad (4)$$

Streambed permeability is usually less than the adjacent aquifer formation. The fluxes across the streambeds can be described by Robin boundary conditions as

$$K_y \frac{\partial h}{\partial y} - \frac{K_1}{b_1} h = 0 \quad \text{at} \quad y = 0 \quad (5)$$

and

$$K_y \frac{\partial h}{\partial y} + \frac{K_2}{b_2} h = 0 \quad \text{at} \quad y = w_y \quad (6)$$

The free surface equation describing water table decline is written as

$$K_x\left(\frac{\partial h}{\partial x}\right)^2+K_y\left(\frac{\partial h}{\partial y}\right)^2+K_z\left(\frac{\partial h}{\partial z}\right)^2-K_z\frac{\partial h}{\partial z}=S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z=h \tag{7}$$

Neuman (1972) indicated that the effect of the second-order terms in Eq. (7) is generally ignorable to develop analytical solutions. Eq. (7) is thus linearized by neglecting the quadratic terms, and the position of the water table is fixed at the initial condition (i.e., $z = 0$). The result is written as

$$K_z\frac{\partial h}{\partial z}=-S_y\frac{\partial h}{\partial t} \quad \text{at} \quad z=0 \tag{8}$$

Notice that Eq. (8) is applicable when the conditions $|h|/H \le 0.1$ and $|\partial h/\partial x| + |\partial h/\partial y| \le 0.01$ are satisfied. These two conditions had been studied and verified by simulations in, for example, Nyholm et al. (2002), Goldscheider and Drew (2007) and Yeh et al. (2010). Nyholm et al. (2002) achieved agreement on drawdown measured in a field pumping test and predicted by MODFLOW which models flow in the study site as confined behavior because of $|h|/H \le 0.1$ in the pumping well. Goldscheider and Drew (2007) revealed that pumping drawdown predicted by Neuman (1972) analytical solution based on Eq. (8) agrees well with that obtained in a field pumping test. In addition, Yeh et al. (2010) also achieved agreement on the hydraulic head predicted by their analytical solution based on Eq. (8), their finite difference solution based on Eq. (7) with $\partial h/\partial y = 0$ (referring to Eq. (7a)), and Teo et al. (2003) solution derived by applying the perturbation technique to deal with Eq. (7a) when $|h|/H = 0.1$ and $|\partial h/\partial x| = 0.01$ (i.e., $\alpha = 0.1$ and $|\partial \phi/\partial x| = 0.01$ at $x = 0$ in Yeh et al. (2010, Fig. 5(a)). On the other hand, the bottom of the aquifer is considered as a no-flow boundary condition denoted as

$$\partial h/\partial z=0 \quad \text{at} \quad z=-H \tag{9}$$

Define dimensionless variables as $\bar{h} = (K_y\, H\, h)/Q$, $\bar{t} = (K_y\, t)/(S_s\, y_0^2)$, $\bar{x} = x/y_0$,

$\bar{y} = y/y_0$, $\bar{z} = z/H$, $\bar{x}_0' = x_0'/y_0$, $\bar{y}_0' = y_0'/y_0$, $\bar{z}_0' = z_0'/H$, $\bar{w}_x = w_x/y_0$ and $\bar{w}_y = w_y/y_0$ where the overbar denotes a dimensionless symbol, and y_0, a distance between stream 1 and the center of the RCW, is chosen as a characteristic length. On the basis of the definitions, Eq. (1) can be written as

$$\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = \frac{\partial \bar{h}}{\partial \bar{t}} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0') \tag{10}$$

where $\kappa_x = K_x/K_y$ and $\kappa_z = (K_z\, y_0^2)/(K_y\, H^2)$.

Similarly, the initial and boundary conditions are expressed as

$$\bar{h} = 0 \quad \text{at} \quad \bar{t} = 0 \tag{11}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = 0 \tag{12}$$

$$\partial \bar{h}/\partial \bar{x} = 0 \quad \text{at} \quad \bar{x} = \bar{w}_x \tag{13}$$

$$\partial \bar{h}/\partial \bar{y} - \kappa_1 \bar{h} = 0 \quad \text{at} \quad \bar{y} = 0 \tag{14}$$

$$\partial \bar{h}/\partial \bar{y} + \kappa_2 \bar{h} = 0 \quad \text{at} \quad \bar{y} = \bar{w}_y \tag{15}$$

$$\frac{\partial \bar{h}}{\partial \bar{z}} = -\frac{\gamma}{\kappa_z}\frac{\partial \bar{h}}{\partial \bar{t}} \quad \text{at} \quad \bar{z} = 0 \tag{16}$$

and

$$\partial \bar{h}/\partial \bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \tag{17}$$

where $\kappa_1 = (K_1\, y_0)/(K_y\, b_1)$, $\kappa_2 = (K_2\, y_0)/(K_y\, b_2)$ and $\gamma = S_y/(S_s\, H)$.

2.2 Head solution for point sink

The model, Eqs. (10) – (17), reduces to an ordinary differential equation (ODE) with two boundary conditions in terms of $\bar{z}$ after taking Laplace transform and finite integral transform. The former transform converts $\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ into $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$, $\delta(\bar{x} - \bar{x}_0')\, \delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')$ in Eq. (10) into $\delta(\bar{x} - \bar{x}_0')\, \delta(\bar{y} - \bar{y}_0')\delta(\bar{z} - \bar{z}_0')/p$, and $\partial \bar{h}/\partial \bar{t}$ in Eqs. (10) and (16) into $p\hat{h} - \bar{h}|_{\bar{t}=0}$ where p is the Laplace parameter, and the second term, initial condition in Eq. (11), equals zero (Kreyszig, 1999). The transformed model becomes a boundary value problem written as

$$\kappa_x \frac{\partial^2 \hat{h}}{\partial \bar{x}^2} + \frac{\partial^2 \hat{h}}{\partial \bar{y}^2} + \kappa_z \frac{\partial^2 \hat{h}}{\partial \bar{z}^2} = p\hat{h} + \delta(\bar{x} - \bar{x}_0')\delta(\bar{y}' - \bar{y}_0')\delta(\bar{z} + \bar{z}_0')/p \qquad (18)$$

with boundary conditions $\partial\hat{h}/\partial\bar{x} = 0$ at $\bar{x} = 0$ and $\bar{x} = \bar{w}_x$, $\partial\hat{h}/\partial\bar{y} - \kappa_1 \hat{h} = 0$ at $\bar{y} = 0$, $\partial\hat{h}/\partial\bar{y} + \kappa_2 \hat{h} = 0$ at $\bar{y} = \bar{w}_y$, $\partial\hat{h}/\partial\bar{z} = -p\gamma\hat{h}/\kappa_z$ at $\bar{z} = 0$ and $\partial\bar{h}/\partial\bar{z} = 0$ at $\bar{z} = -1$. We then apply finite integral transform to the problem. One can refer to Appendix A for its detailed definition. The transform converts $\hat{h}(\bar{x}, \bar{y}, \bar{z}, p)$ in the problem into $\tilde{h}(\alpha_m, \beta_n, \bar{z}, p)$, and $\delta(\bar{x} - \bar{x}_0')\,\delta(\bar{y} - \bar{y}_0')$ in Eq. (18) into $\cos(\alpha_m \bar{x}_0')K(\bar{y}_0')$ and $\kappa_x\, \partial^2\hat{h}/\partial\bar{x}^2 + \partial^2\hat{h}/\partial\bar{y}^2$ in Eq. (18) into $-(\kappa_x \alpha_m^2 + \beta_n^2)\tilde{h}$ where $(m,\ n) \in 1,\ 2,\ 3,\ \ldots\ \infty$, $\alpha_m = m\,\pi/\bar{w}_x$, $K(\bar{y}_0')$ is defined in Eq. (A2) with $\bar{y} = \bar{y}_0'$, and β_n are eigenvalues equaling the roots of the following equation as (Latinopoulos, 1985)

$$\tan\left(\beta_n\, \bar{w}_y\right) = \frac{\beta_n(\kappa_1 + \kappa_2)}{\beta_n^2 - \kappa_1\, \kappa_2} \qquad (19)$$

The method to determine the roots is discussed in section 2.3. In turn, Eq. (18) becomes a second-order ODE defined by

$$\kappa_z \frac{\partial^2 \tilde{h}}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h} = \cos(\alpha_m\, \bar{x}_0')\, K(\bar{y}_0')\, \delta(\bar{z} + \bar{z}_0')/p \qquad (20)$$

with two boundary conditions denoted as

$$\frac{\partial \tilde{h}}{\partial \bar{z}} = -\frac{p\,\gamma}{\kappa_z}\tilde{h} \quad \text{at} \quad \bar{z} = 0 \qquad (21)$$

and

$$\partial\tilde{h}/\partial\bar{z} = 0 \quad \text{at} \quad \bar{z} = -1 \qquad (22)$$

Eq. (20) can be separated into two homogeneous ODEs as

$$\kappa_z \frac{\partial^2 \tilde{h}_a}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_a = 0 \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \qquad (23)$$

and

$$\kappa_z \frac{\partial^2 \tilde{h}_b}{\partial \bar{z}^2} - (\kappa_x \alpha_m^2 + \beta_n^2 + p)\tilde{h}_b = 0 \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{24}$$

where h_a and h_b, respectively, represent the heads above and below $\bar{z} = -\bar{z}_0'$ where the point sink is located. Two continuity requirements should be imposed at $\bar{z} = -\bar{z}_0'$. The first is the continuity of the hydraulic head denoted as

$$\tilde{h}_a = \tilde{h}_b \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{25}$$

The second describes the discontinuity of the flux due to point pumping represented by the Dirac delta function in Eq. (20). It can be derived by integrating Eq. (20) from $\bar{z} = -\bar{z}_0'^{\,-}$ to $\bar{z} = -\bar{z}_0'^{\,+}$ as

$$\frac{\partial \tilde{h}_a}{\partial \bar{z}} - \frac{\partial \tilde{h}_b}{\partial \bar{z}} = \frac{\cos(\alpha_m \bar{x}_0') K(\bar{y}_0')}{p\kappa_z} \quad \text{at} \quad \bar{z} = -\bar{z}_0' \tag{26}$$

Solving Eqs. (23) and (24) simultaneously with Eqs. (21), (22), (25), and (26) yields the Laplace-domain head solution as

$$\tilde{h}_a(\alpha_m, \beta_n, \bar{z}, p) = \Omega(-\bar{z}_0', \bar{z}, 1) \quad \text{for} \quad -\bar{z}_0' \le \bar{z} \le 0 \tag{27a}$$

and

$$\tilde{h}_b(\alpha_m, \beta_n, \bar{z}, p) = \Omega(\bar{z}, \bar{z}_0', -1) \quad \text{for} \quad -1 \le \bar{z} \le -\bar{z}_0' \tag{27b}$$

with

$$\Omega(a,b,c) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + c\, p\gamma \sinh(b\lambda)]\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)}{p\kappa_z \lambda (p\gamma \cosh\lambda + \kappa_z \lambda \sinh\lambda)} \tag{28}$$

$$\lambda = \sqrt{(\kappa_x \alpha_m^2 + \beta_n^2 + p)/\kappa_z} \tag{29}$$

where a, b, and c are arguments. Taking the inverse Laplace transform and finite integral transform to Eq. (28) results in Eq. (31). One is referred to Appendix B for the detailed derivation. A time-domain head solution for a point sink is therefore written as

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \begin{cases} \Phi(-\bar{z}_0', \bar{z}, 1) \text{ for } -\bar{z}_0' \le \bar{z} \le 0 \\ \Phi(\bar{z}, \bar{z}_0', -1) \text{ for } -1 \le \bar{z} \le -\bar{z}_0' \end{cases} \tag{30}$$

with

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n\, X_n + 2\sum_{m=1}^{\infty}\phi_{m,n}\, X_{m,n}\, \cos(\alpha_m\, \bar{x})\right]\, Y_n\right\} \tag{31}$$

$$\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i} \tag{32}$$

$$\psi_{m,n} = -\cosh[(1+a)\lambda_s]\cosh(b\,\lambda_s)/(\kappa_z\,\lambda_s\,\sinh\lambda_s) \tag{33}$$

$$\psi_{m,n,0} = \mu_{m,n,0}\,\cosh[(1+a)\lambda_0]\,[-\kappa_z\,\lambda_0\,\cosh(b\,\lambda_0) + c\,p_0\,\gamma\,\sinh(b\,\lambda_0)] \tag{34}$$

$$\psi_{m,n,i} = \nu_{m,n,i}\,\cos[(1+a)\lambda_i]\,[-\kappa_z\,\lambda_i\,\cos(b\,\lambda_i) + c\,p_i\,\gamma\,\sin(b\,\lambda_i)] \tag{35}$$

$$\mu_{m,n,0} = 2\exp(p_0\,\bar{t})/\{p_0[(1+2\,\gamma)\,\kappa_z\,\lambda_0\,\cosh\lambda_0 + (p_0\,\gamma+\kappa_z)\sinh\lambda_0]\} \tag{36}$$

$$\nu_{m,n,i} = 2\exp(p_i\,\bar{t})/\{p_i[(1+2\,\gamma)\,\kappa_z\,\lambda_i\,\cos\lambda_i + (p_i\,\gamma+\kappa_z)\sin\lambda_i]\} \tag{37}$$

$$Y_n = \frac{\beta_n\cos(\beta_n\,\bar{y}) + \kappa_1\sin(\beta_n\,\bar{y})}{(\beta_n^2+\kappa_1^2)[\bar{w}_y + \kappa_2/(\beta_n^2+\kappa_2^2)] + \kappa_1} \tag{38}$$

and

$$X_{m,n} = \cos(\alpha_m\,\bar{x}_0')\,[\beta_n\,\cos(\beta_n\,\bar{y}_0') + \kappa_1\,\sin(\beta_n\bar{y}_0')] \tag{39}$$

where $\lambda_s = \sqrt{(\kappa_x\alpha_m^2+\beta_n^2)/\kappa_z}$, $p_0 = \kappa_z\lambda_0^2 - \kappa_x\alpha_m^2 - \beta_n^2$, $p_i = -\kappa_z\lambda_i^2 - \kappa_x\alpha_m^2 - \beta_n^2$, ϕ_n and X_n equal $\phi_{m,n}$ and $X_{m,n}$ with $\alpha_m = 0$, respectively, and the eigenvalues λ_0 and λ_i are, respectively, the roots of the following equations:

$$e^{2\lambda_0} = \frac{-\gamma\,\kappa_z\,\lambda_0^2 + \kappa_z\,\lambda_0 + \gamma\,(\kappa_x\,\alpha_m^2+\beta_n^2)}{\gamma\,\kappa_z\,\lambda_0^2 + \kappa_z\,\lambda_0 - \gamma\,(\kappa_x\,\alpha_m^2+\beta_n^2)} \tag{40}$$

$$\tan\lambda_i = \frac{-\gamma(\kappa_z\,\lambda_i^2 + \kappa_x\,\alpha_m^2 + \beta_n^2)}{\kappa_z\,\lambda_i} \tag{41}$$

The determination for those eigenvalues is introduced in the next section. Notice that the solution consists of simple series expanded in β_n, double series expanded in β_n and λ_i (or α_m and β_n), and triple series expanded in α_m, β_n and λ_i.

2.3 Evaluations for β_n, λ_0 and λ_i

Application of Newton's method with proper initial guesses to determine the eigenvalues β_n, λ_0 and λ_i has been proposed by Huang et al. (2014) and is briefly introduced herein. The eigenvalues are situated at the intersection points of the left-hand side (LHS) and RHS functions of Eq. (19) for β_n, Eq. (40) for λ_0, and Eq. (41) for λ_i. Hence, the initial guesses for β_n are considered as $\beta_v - \delta$ if $\beta_v > (\kappa_1\, \kappa_2)^{0.5}$ and as $\beta_v + \delta$ if $\beta_v < (\kappa_1\, \kappa_2)^{0.5}$ where $\beta_v = (2n-1)\pi/(2\,\bar{w}_y)$ and δ is a chosen small value such as 10^{-8} for avoiding being right at the vertical asymptote. In addition, the guess for λ_0 can be formulated as

$$\lambda_{0\,initial} = \delta + \left\{-\kappa_z - \sqrt{\kappa_z[\kappa_z + 4\,\gamma^2(\kappa_x \alpha_m^2 + \beta_n^2)]}\right\}/(2\gamma\kappa_z) \quad (42)$$

where the RHS second term represents the location of the vertical asymptote derived by letting the denominator of the RHS function in Eq. (40) to be zero and solving λ_0 in the resultant equation. Moreover, the guessed value for λ_i is $(2\,i-1)\pi/2 + \delta$.

2.4 Head solution for radial collector well

The lateral of RCW is approximately represented by a line sink composed of a series of adjoining point sinks. The locations of these point sinks are expressed in terms of $(\bar{x}_0 + \bar{l}\cos\theta,\ \bar{y}_0 + \bar{l}\sin\theta,\ \bar{z}_0)$ where $(\bar{x}_0,\ \bar{y}_0,\ \bar{z}_0) = (x_0/y_0,\ 1,\ z_0/H)$ is the central of the lateral, and $\bar{l}$ is a variable to define different locations of the point sink. The solution of head $\bar{h}_w(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ for a lateral can therefore be derived by substituting $\bar{x}_0' = \bar{x}_0 + \bar{l}\cos\theta$, $\bar{y}_0' = 1 + \bar{l}\sin\theta$ and $\bar{z}_0' = \bar{z}_0$ into the point-sink solution, Eq. (30), then by integrating the resultant solution to $\bar{l}$, and finally by dividing the integration result into the sum of lateral lengths. The derivation can be denoted as

$$\bar{h}_w(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \left(\sum_{k=1}^{N} \bar{L}_k\right)^{-1} \sum_{k=1}^{N} \int_0^{\bar{L}_k} \bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t})\, d\bar{l} \quad (43)$$

where $\bar{L}_k = L_k/y_0$ is the k-th dimensionless lateral length. Note that the integration variable $\bar{l}$ (i.e., $\bar{x}_0'$ and $\bar{y}_0'$) appears only in X_n and $X_{m,n}$ in Eq. (31). The integral in Eq. (43) can thus be done analytically by integrating X_n and $X_{m,n}$ with respect to $\bar{l}$. After the integration, Eq. (43) can be expressed as

$$\bar{h}_w(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \left(\sum_{k=1}^{N} \bar{L}_k\right)^{-1} \sum_{k=1}^{N} \begin{cases} \Phi(-\bar{z}_0, \bar{z}, 1) & \text{for } -\bar{z}_0 \le \bar{z} \le 0 \\ \Phi(\bar{z}, \bar{z}_0, -1) & \text{for } -1 \le \bar{z} \le -\bar{z}_0 \end{cases} \quad (44)$$

where Φ is defined by Eqs. (31) – (38), and X_n and $X_{m,n}$ in Eq. (31) are replaced, respectively, by

$$X_{n,k} = -G_k/(\beta_n \sin\theta_k) \tag{45}$$

and

$$X_{m,n,k} = \frac{\alpha_m F_k \cos\theta_k + \beta_n G_k \sin\theta_k}{\alpha_m^2 \cos^2\theta_k - \beta_n^2 \sin^2\theta_k} \tag{46}$$

with

$$F_k = \sin(X\alpha_m)[\beta_n \cos(Y\beta_n) + \kappa_1 \sin(Y\beta_n)] - \sin(\bar{x}_0\alpha_m)(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \tag{47}$$

$$G_k = \cos(X\alpha_m)[\kappa_1 \cos(Y\beta_n) - \beta_n \sin(Y\beta_n)] - \cos(\bar{x}_0\alpha_m)(\kappa_1 \cos\beta_n - \beta_n \sin\beta_n) \tag{48}$$

where $X = \bar{x}_0 + \bar{L}_k \cos\theta_k$ and $Y = 1 + \bar{L}_k \sin\theta_k$. Notice that Eq. (45) is obtained by substituting $\alpha_m = 0$ into Eq. (46). When $\theta_k = 0$ or π, Eq. (45) reduces to Eq. (49) by applying L'Hospital's rule.

$$X_{n,k} = \bar{L}_k(\beta_n \cos\beta_n + \kappa_1 \sin\beta_n) \tag{49}$$

2.5 SDR solution for radial collector well

On the basis of Darcy's law and the head solution for a RCW, the SDR from streams 1 and 2 can be defined, respectively, as

$$SDR_1(\bar{t}) = -\int_{\bar{x}=0}^{\bar{x}=\bar{w}_x} \left(\int_{\bar{z}=-\bar{z}_0}^{\bar{z}=0} \frac{\partial \bar{h}_w}{\partial \bar{y}} d\bar{z} + \int_{\bar{z}=-1}^{\bar{z}=-\bar{z}_0} \frac{\partial \bar{h}_w}{\partial \bar{y}} d\bar{z} \right) d\bar{x} \quad \text{at} \quad \bar{y} = 0 \tag{50}$$

and

$$SDR_2(\bar{t}) = \int_{\bar{x}=0}^{\bar{x}=\bar{w}_x} \left(\int_{\bar{z}=-\bar{z}_0}^{\bar{z}=0} \frac{\partial \bar{h}_w}{\partial \bar{y}} d\bar{z} + \int_{\bar{z}=-1}^{\bar{z}=-\bar{z}_0} \frac{\partial \bar{h}_w}{\partial \bar{y}} d\bar{z} \right) d\bar{x} \quad \text{at} \quad \bar{y} = \bar{w}_y \tag{51}$$

Again, the double integrals in both equations can be done analytically. Notice that the series term of $2\sum_{m=1}^{\infty} \phi_{m,n} X_{m,n} \cos(\alpha_m \bar{x})$ in Eq. (31) disappears due to the consideration of Eqs. (3) and (4) and the integration with respect to $\bar{x}$ in Eqs. (50) and (51) when deriving the SDR solution. The SDR_1 and SDR_2 are therefore expressed in terms of double series and given below:

$$SDR_1(\bar{t}) = -\frac{2}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \left(\psi_n' + \psi_{n,0}' + \sum_{i=1}^{\infty} \psi_{n,i}' \right) X_{n,k} Y_n'(0) \tag{52}$$

and

$$SDR_2(\bar{t}) = \frac{2}{\sum_{k=1}^{N} \bar{L}_k} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \left(\psi_n' + \psi_{n,0}' + \sum_{i=1}^{\infty} \psi_{n,i}'\right) X_{n,k}\, Y_n'(\bar{w}_y) \tag{53}$$

with

$$Y_n'(\bar{y}) = \frac{\kappa_1 \beta_n \cos(\beta_n \bar{y}) - \beta_n^2 \sin(\beta_n \bar{y})}{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2/(\beta_n^2 + \kappa_2^2)] + \kappa_1} \tag{54}$$

$$\psi_n' = -\{\sinh(\bar{z}_0 \lambda_s') \cosh[(1 - \bar{z}_0)\lambda_s'] + \sinh[(1 - \bar{z}_0)\lambda_s'] \cosh(\bar{z}_0 \lambda_s')\}/(\kappa_z \lambda_s'^2 \sinh \lambda_s') \tag{55}$$

$$\psi_{n,0}' = -\mu_{n,0}(\theta_{n,0} + \vartheta_{n,0})/\lambda_0 \tag{56}$$

$$\theta_{n,0} = \cosh[(1 - \bar{z}_0)\lambda_0]\{p_0' \gamma[-1 + \cosh(\bar{z}_0 \lambda_0) + \kappa_z \lambda_0 \sinh(\bar{z}_0 \lambda_0)]\} \tag{57}$$

$$\vartheta_{n,0} = \sinh[(1 - \bar{z}_0)\lambda_0][\kappa_z \lambda_0 \cosh(\bar{z}_0 \lambda_0) + p_0' \gamma \sinh(\bar{z}_0 \lambda_0)] \tag{58}$$

$$\psi_{n,i}' = \nu_{n,i}(\sigma_{n,i} - \eta_{n,i})/\lambda_i \tag{59}$$

$$\sigma_{n,i} = \cos[(1 - \bar{z}_0)\lambda_i]\{p_i' \gamma[-1 + \cos(\bar{z}_0 \lambda_i)] - \kappa_z \lambda_i \sin(\bar{z}_0 \lambda_i)\} \tag{60}$$

$$\eta_{n,i} = \sin[(1 - \bar{z}_0)\lambda_i][\kappa_z \lambda_i \cos(\bar{z}_0 \lambda_i) + p_i' \gamma \sin(\bar{z}_0 \lambda_i)] \tag{61}$$

where $\lambda_s' = \beta_n/\sqrt{\kappa_z}$; $p_0' = \kappa_z \lambda_0^2 - \beta_n^2$; $p_i' = -\kappa_z \lambda_i^2 - \beta_n^2$; $\mu_{n,0}$ equals $\mu_{m,n,0}$ in Eq. (36) with $\alpha_m = 0$; $\nu_{n,i}$ equals $\nu_{m,n,i}$ in Eq. (37) with $\alpha_m = 0$; $X_{n,k}$ is defined in Eq. (45) for $\theta_k \neq 0$ or π and Eq. (49) for $\theta_k = 0$ or π; and λ_0 and λ_i are the roots of Eqs. (40) and (41) with $\alpha_m = 0$, respectively.

2.6 Special cases of the present solution

2.6.1 Confined aquifer of finite extent

If $\gamma = 0$ (i.e., $S_y = 0$ in Eq. (8)), the top boundary is regarded as an impermeable stratum. The aquifer is then a confined system. Under this circumstance, Eq. (40) reduces to $e^{2\lambda_0} = 1$ having the root of $\lambda_0 = 0$, and Eq. (41) yields $\tan \lambda_i = 0$ having the roots of $\lambda_i = i\,\pi$ where $i \in 1, 2, 3, \ldots\ \infty$. With $\gamma = 0$, $\lambda_0 = 0$ and $\lambda_i = i\,\pi$, the head solution for a confined aquifer can be expressed as Eq. (44) with Eqs. (31) – (38) and (45) – (49) where $\psi_{m,n,0}$ in Eq. (32)

is replaced by

$$\psi_{m,n,0} = -\exp(p_0\,\bar{t})/p_0 \quad (62)$$

Similarly, the SDR solution for a confined aquifer can be written as Eqs. (52) and (53) where the RHS function in Eq. (56) reduces to that in Eq. (62) by applying L'Hospital's rule with $\gamma = 0$ and $\lambda_0 = 0$.

2.6.2 Confined aquifer of infinite extent

The head solution introduced in section 2.6.1 is applicable to spatiotemporal head distributions in confined aquifers of infinite extent before the lateral boundary effect comes. Wang and Yeh (2008) indicated that the time can be quantified, in our notation, as $t = R^2S_s/(16K_y)$ (i.e., $\bar{t} = R^2/(16y_0^2)$ for dimensionless time) where R is the shortest distance between a RCW and aquifer lateral boundary. Prior to the time, the present head solution with N = 1 for a horizontal well in a confined aquifer gives very close results given in Zhan et al. (2001).

2.6.3 Unconfined aquifer of infinite extent

Prior to the beginning time mentioned in section 2.6.2, the absolute value calculated by the present head solution, Eqs. (44) with N = 1, represents drawdown induced by a horizontal well in unconfined aquifers of infinite extent. The calculated drawdown should be close to that from Zhan and Zlotnik (2002) solution for the case of the instantaneous drainage from water table decline.

2.6.4 Unconfined aquifer of semi-infinite extent

When $\kappa_1 \rightarrow \infty$ (i.e., $b_1 = 0$), Eq. (14) reduces to the Dirichlet condition of $\bar{h} = 0$ for stream 1 in the absence from a low-permeability streambed, and Eq. (19) becomes $\tan(\beta_n \bar{w}_y) = -\beta_n/\kappa_2$. In addition, the boundary effect occurring at the other three sides of the aquifer can be neglected prior to the beginning time. Moreover, when N = 1 and $\theta_1 = 0$, a RCW can be regarded as a horizontal well parallel to stream 1. Under these three conditions, the present head and SDR predictions are close to those in Huang et al. (2011), the head solution of which agrees well with measured data from a field experiment executed by Mohamed and Rushton (2006). On the other hand, before the time when the boundary effect occurs at $\bar{x} = 0$, $\bar{x} = \bar{w}_x$ and $\bar{y} = \bar{w}_y$, the present head and SDR solutions for a RCW give close predictions to those in Huang et al. (2012), the head and SDR solutions of which agree well with observation data taken from two field experiments carried out by Schafer (2006) and Jasperse (2009), respectively.

2.7 Sensitivity analysis

The hydraulic parameters determined from field observed data are inevitably subject to measurement errors. Consequently, head predictions from the analytical model have uncertainty due to the propagation of measurement errors. Sensitivity analysis can be considered as a tool of exploring the response of the head to the change in a specific parameter (Zheng and Bennett, 2002). One may define the normalized sensitivity coefficient as

$$S_{i,t} = \frac{P_i}{H}\frac{\partial h}{\partial P_i} \tag{63}$$

where $S_{i,t}$ is the normalized sensitivity coefficient for the ith parameter at time t, and P_i represents the magnitude of the ith parameter. Eq. (63) can be approximated as

$$S_{i,t} = \frac{h(P_i + \Delta P_i) - h(P_i)}{\Delta P_i} \times \frac{P_i}{H} \tag{64}$$

where ΔP_i is an increment chosen as $10^{-3} P_i$ (Yeh et al., 2008).

3. Results and discussion

This section demonstrates head and SDR predictions and explores some physical insights regarding flow behavior. In section 3.1, groundwater flow and equipotential lines induced by pumping are discussed. In section 3.2, the influence of anisotropy on spatial head and temporal SDR distributions is studied. In section 3.3, the sensitivity analysis is performed to investigate the response of the head to the change in each hydraulic parameter. In section 3.4, the effects of the vertical flow and well depth on temporal SDR distributions for confined and unconfined aquifers are investigated. For conciseness, we consider a RCW with two laterals with N = 2, $\bar{L}_1 = \bar{L}_2 = 0.5$, $\theta_1 = 0$ and $\theta_2 = \pi$. The well can be viewed as a horizontal well parallel to streams 1 and 2. The default values for the other dimensionless parameters are $\bar{w}_x = \bar{w}_y = 2$, $\gamma = 100$, $\bar{x}_0 = 1$, $\bar{y}_0 = 1$, $\bar{z}_0 = 0.5$, $\kappa_x = \kappa_z = 1$, and $\kappa_1 = \kappa_2 = 20$.

3.1 Groundwater flow and hydraulic head

Most existing models assume 2-D flow with neglecting the vertical flow for pumping at a horizontal well (e.g., Mohamed and Rushton, 2006; Haitjema et al., 2010). The head distributions predicted by those models are inaccurate if an observation well is close to the region where the vertical flow prevails. Figure 2 demonstrates the streamlines and equipotential lines predicted by the present solution for a horizontal well in an unconfined aquifer for $\bar{x}_0 = 10$, $\bar{w}_x = \bar{w}_y = 20$ and $\kappa_z = 0.1$, 1, and 10. The well is located at $9.5 \leq \bar{x} \leq 10.5$, $\bar{y} = 1$ and $\bar{z} = 0.5$ as illustrated in the figure. The equipotential lines are based on steady-state head distributions plotted by Eq. (44) with $\bar{y} = 1$ and $\bar{t} = 10^7$. The stream function ψ can be derived via the Cauchy-Riemann equation, in our notation, as

$$\frac{\partial \bar{\psi}}{\partial \bar{x}} = -\sqrt{\kappa_z}\frac{\partial \bar{h}_w}{\partial \bar{z}} \tag{65}$$

where $\bar{\psi} = K_y H \psi / Q$ is the dimensionless stream function describing 2-D streamlines at the vertical plane of $\bar{y} = 1$ based on $\bar{h}_w$ in Eq. (44) with $\bar{t} = 10^7$ for steady state. The function $\bar{\psi}$ is obtained firstly by substituting Eq. (44) into Eq. (65), then by differentiating the result with respect to $\bar{z}$, and eventually by integrating the differentiation result to $\bar{x}$. The coefficient arising from the integration is determined by the condition of $\bar{\psi} = 0$ at $\bar{x} = \bar{x}_0$. The detailed derivation of the stream function is shown in Appendix C. When $\kappa_z = 0.1$, in the range of $10 \leq \bar{x} \leq 13.66$, the contours of the hydraulic head are in a curved path, and the flow toward the well is slanted. Moreover, the range decreases to $10 \leq \bar{x} \leq 11.5$ when $\kappa_z = 1$ and to $10 \leq \bar{x} \leq 10.82$ when $\kappa_z = 10$. Beyond these ranges, the head contours are nearly vertical, and the flow is essentially horizontal. Define $\bar{d}$ = d/y_0 as a shortest dimensionless horizontal distance between the well and a nearest location of only horizontal flow. The $\bar{d}$ is therefore chosen as 3.16, 1 and 0.32 for the cases of κ_z = 0.1, 1 and 10, respectively. Substituting (κ_z, $\bar{d}$) = (0.1, 3.16), (1, 1) and (10, 0.32) into $\kappa_z\,\bar{d}^2$ leads to about unity. We may therefore conclude that the vertical flow at an observation location is negligible if a shortest dimensionless horizontal distance between the location and a RCW is less than $\bar{d} = \sqrt{1/\kappa_z}$ (i.e., $d = H\sqrt{K_y/K_z}$) for thin aquifers, observation locations far from the well, and/or a small ratio of K_y/K_z.

3.2 Anisotropy analysis of hydraulic head and stream depletion rate

Previous articles have seldom analyzed flow behavior for anisotropic aquifers, i.e., κ_x (K_x/K_y) ≠ 1. Head predictions based on the models, developed for isotropic aquifers, will be inaccurate if $\kappa_x \neq 1$. Consider $\bar{w}_x = \bar{w}_y = 2$, $\bar{t} = 10^7$ for steady-state head distributions, and a RCW with $\bar{L}_1 = \bar{L}_2 = 0.25$, $\theta_1 = 0$, $\theta_2 = \pi$, and $(\bar{x}_0, \bar{y}_0, \bar{z}_0)$ = (1, 1, -0.5) for symmetry. The contours of the dimensionless head at $\bar{z} = -0.5$ are shown in Figures 3(a) – 3(d) for κ_x = 1, 10 and 50, 10^{-3}, and 10^{-4}, respectively. The figure indicates that the anisotropy causes a significant effect on the head distributions in comparison with the case of κ_x = 1. In Figure 3(b), the contours exhibit smooth curves in the strip regions of $1 \leq \bar{y} \leq 1.45$ for the case of κ_x = 10 and $1 \leq \bar{y} \leq 1.2$ for the case of κ_x = 50. For the region of $\bar{y} \geq 1.45$, the predicted heads for both cases agree well, and all the contour lines are parallel, indicating that the flow is essentially unidirectional. Substituting (κ_x, $\bar{y}$) = (10, 1.45) and (50, 1.2) into $\kappa_x\,(\bar{y}-1)^2$ results in a value about 2. Accordingly, we may draw the conclusion that plots from the inequality of $\kappa_x\,(\bar{y}-1)^2 \leq 2$ indicate the strip region for κ_x being greater than 10. Some existing models assuming 2-D flow in a vertical plane with neglecting the flow component along a horizontal well give accurate head predictions beyond the region (e.g., Anderson, 2000; Anderson, 2003; Kompani-Zare et al., 2005).

Aquifers with $K_yH \geq 10^3$ m^2/day can efficiently produce plenty of water from a well. RCWs usually operate with $Q \leq 10^5$ m^3/day for field experiments (e.g., Schafer, 2006; Jasperse, 2009). We therefore define significant dimensionless head drop as $|\bar{h}| > 10^{-5}$ (i.e., $|h| > 1$ mm). The anisotropy of κ_x <1 produces the drop in the strip areas of $1 \leq \bar{x} \leq 1.48$ for the case of $\kappa_x = 10^{-3}$ in Figure 3(c) and $1 \leq \bar{y} \leq 1.32$ for the case of $\kappa_x = 10^{-4}$ in Figure 3(d). Substituting $(\kappa_x, \bar{x}) = (10^{-3}, 1.48)$ and $(10^{-4}, 1.32)$ into $(\bar{x} - \bar{x}_0 - \bar{L}_1)^2/\kappa_x$ approximates 52.9. This result leads to the conclusion that the area can be determined by the inequalities of $(\bar{x} - \bar{x}_0 - \bar{L}_1)^2 \leq 52.9\kappa_x$ and $(\bar{x} - \bar{x}_0 + \bar{L}_2)^2 \leq 52.9\kappa_x$ for any value of κ_x in the range κ_x <1. For a RCW with irregular lateral configurations, the inequalities become $(\bar{x} - \max \bar{x}_k)^2 \leq 52.9\ \kappa_x$ and $(\bar{x} - \min \bar{x}_k)^2 \leq 52.9\kappa_x$ where $\bar{x}_k$ is coordinate $\bar{x}$ of the far end of the k-th lateral. The conclusion applies in principle to reduction in grid points for numerical solutions based on finite difference methods or finite element methods. On the other hand, we have found that Eq. (52) or (53) with various κ_x predicts the same temporal SDR distribution (not shown), indicating that the SDR is independent of κ_x.

3.3 Sensitivity analysis of hydraulic head

Consider an unconfined aquifer of H = 20 m and $w_x = w_y$ = 800 m with a RCW having two laterals of $L_1 = L_2$ = 50 m, $\theta_1 = 0$ and $\theta_2 = \pi$ and two piezometers installed at point A of (400 m, 340 m, −10 m) and point B of (400 m, 80 m, −10 m) illustrated in Figure 4. As discussed in section 3.1, the temporal head distribution at point A exhibits the unconfined behavior in Figure 4(a) because of $\kappa_z\,\bar{d}^2$<1 while at point B displays the confined one in Figure 4(b) due to $\kappa_z\,\bar{d}^2$>1. The sensitivity analysis is conducted with the aid of equation (64) to observe head responses at these two piezometers to the change in each of K_x, K_y, K_z, S_s, S_y, K_1, L_1 and z_0. The temporal distribution curves of the normalized sensitivity coefficients for those eight parameters are shown in Figures 4(a) for point A and 4(b) for point B when $K_x = K_y = 1$

m/day, K_z = 0.1 m/day, $S_s = 10^{-5}$ m^{-1}, S_y = 0.2, $K_1 = K_2$ = 0.1 m/day, $b_1 = b_2$ = 1 m, Q = 100 m^3/day, $x_0 = y_0$ = 400 m, and z_0 = 10 m. The figure demonstrates that the hydraulic heads at both piezometers are most sensitive to the change in K_y, second sensitive to the change in K_x and thirdly sensitive to the change in S_y, indicating that K_y, K_x and S_y are the most crucial factors in designing a pumping system. This figure also shows that the heads at point A is sensitive to the change in S_s at the early period of 4×10^{-3} day < t < 10^{-1} day but at point B is insensitive to the change over the entire period. In addition, the head at point A is sensitive to the changes in K_z and z_0 due to 3-D flow (i.e., $\kappa_z \bar{d}^2 < 1$) as discussed in section 3.1. In contrast, the head at point B is insensitive to the changes in K_z and z_0 because the vertical flow diminishes (i.e., $\kappa_z \bar{d}^2 > 1$). Moreover, the head at point A is sensitive to the change in L_1 but the head at point B is not because its location is far away from the well. Furthermore, the normalized sensitivity coefficient of K_1 for point A away from stream 1 approaches zero but for point B in the vicinity of stream 1 increases with time and finally maintains a certain value at the steady state. Regarding the sensitivity analysis of SDR, Huang et al. (2014) has performed the sensitivity analysis of normalized coefficients of SDR_1 to the changes in K_y, K_1 and S_s for a confined aquifer and in K_y, K_z, K_1, S_s and S_y for an unconfined aquifer.

3.4 Effects of vertical flow and well depth on stream depletion rate

Huang et al. (2014) reveals that the effect of the vertical flow on SDR induced by a vertical well is dominated by the magnitude of the key factor κ_z (i.e., $K_z y_0^2/(K_y H^2)$) where y_0 herein is a distance between stream 1 and the vertical well. They concluded that the effect is negligible when $\kappa_z \geq 10$ for a leaky aquifer. The factor should be replaced by $\kappa_z \bar{a}^2$ (i.e., $K_z a^2/(K_y H^2)$) where a is a shortest distance measured from stream 1 to the end of a lateral of a RCW, and $\bar{a} = a/y_0 = 1$ in this study due to $N = 2$, $\theta_1 = 0$ and $\theta_2 = \pi$. We investigate SDR in response to various $\bar{z}_0$ and $\kappa_z \bar{a}$ for unconfined and confined aquifers. The temporal SDR_1 distributions predicted by Eq. (52) for stream 1 adjacent to an unconfined aquifer are shown in Fig. 5(a) for

$\bar{z}_0$ = 0.5 and $\kappa_z\bar{a}^2$ = 0.01, 0.1, 1, 10, 20 and 30 and Fig. 5(b) for $\kappa_z\bar{a}^2$ = 1 and 30 when $\bar{z}_0$ = 0.1, 0.3, 0.5, 0.7 and 0.9. The curves of SDR_1 versus $\bar{t}$ is plotted in both panels by the present SDR solution for a confined aquifer. In Fig. 5(a), the present solution for an unconfined aquifer predicts a close SDR_1 to that for the confined aquifer when $\kappa_z\bar{a}^2$ = 0.01, indicating that the vertical flow in the unconfined aquifer is ignorable. The SDR_1 for the unconfined aquifer with $\kappa_z\bar{a}^2$ = 30 behaves like that for a confined one, indicating the vertical flow is also ignorable. The SDR_1 is therefore independent of well depths $\bar{z}_0$ when $\kappa_z\bar{a}^2$ = 30 as shown in Fig. 5(b). We may therefore conclude that, under the condition of $\kappa_z\bar{a}^2 \leq 0.01$ or $\kappa_z\bar{a}^2 \geq 30$, a 2-D horizontal flow model can give good predictions in SDR_1 for unconfined aquifers. In contrast, SDR_1 increases with decreasing $\kappa_z\bar{a}^2$ when $0.01 < \kappa_z\bar{a}^2 < 30$ in Fig. 5(a), indicating that the vertical flow component induced by pumping in unconfined aquifers significantly affects SDR_1. The effect of well depth $\bar{z}_0$ on SDR_1 is also significant as shown in Fig. 5(b) when $\kappa_z\bar{a}^2$ = 1. Obviously, the vertical flow effect should be considered in a model when $0.01 < \kappa_z\bar{a}^2 < 30$ for unconfined aquifers.

It is interesting to note that the SDR_1 or SDR_2 induced by two laterals (i.e., $\theta_1 = 0$ and $\theta_2 = \pi$) parallel to the streams adjacent to a confined aquifer is independent of $\kappa_z\bar{a}^2$ and $\bar{z}_0$ but depends on aquifer width of $\bar{w}_y$. The temporal SDR distribution curves based on Eqs. (52) and (53) with $\gamma = 0$ for a confined aquifer with $\bar{w}_y$ = 2, 4, 6, 10 and 20 are plotted in Fig. 6. The dimensionless distance between the well and stream 1 is set to unity (i.e., $\bar{y}_0 = 1$) for each case. The SDR_1 predicted by Hunt (1999) solution based on a vertical well in a confined aquifer extending infinitely is considered. The present solution for each $\bar{w}_y$ gives the same SDR_1 as the Hunt solution before the time when stream 2 contributes filtration water to the aquifer and influences the supply of SDR_1. It is interesting to note that the sum of steady-state SDR_1 and SDR_2 is always unity for a fixed $\bar{w}_y$. The former and latter can be estimated by $(\bar{w}_y - 1)/\bar{w}_y$

and $1/\overline{w}_y$, respectively. Such a result corresponds with that in Sun and Zhan (2007) which investigates the distribution of steady-state SDR_1 and SDR_2 induced by a vertical well.

4. Concluding remarks

This study develops a new analytical model describing 3-D flow induced by a RCW in a rectangular confined or unconfined aquifer bounded by two parallel streams and no-flow stratums in the other two sides. The flow equation in terms of the hydraulic head with a point sink term is employed. Both streams fully penetrate the aquifer and are under the Robin condition in the presence of low-permeability streambeds. A first-order free surface equation (8) describing the water table decline gives good predictions when the conditions $|h|/H \leq 0.1$ and $|\partial h/\partial x| + |\partial h/\partial y| \leq 0.01$ are satisfied. The flux across the well screen might be uniform on a lateral within 150 m. The head solution for the point sink is expressed in terms of a triple series derived by the methods of Laplace transform and finite integral transform. The head solution for a RCW is then obtained by integrating the point-sink solution along the laterals and dividing the integration result by the sum of lateral lengths. The integration can be done analytically due to the aquifer of finite extent with Eqs. (3) – (6). On the basis of Darcy's law and the head solution, the SDR solution for two streams can also be acquired. The double integrals of defining the SDR in Eqs. (50) and (51) can also be done analytically due to considerations of Eqs. (3) – (6). The sensitivity analysis is performed to explore the response of the head to the change in each of the hydraulic parameters and variables. New findings regarding the responses of flow and SDR to pumping at a RCW are summarized below:

1. Groundwater flow in a region based on $\bar{d} = \sqrt{1/\kappa_z}$ is 3-D, and temporal head distributions exhibit the unconfined behavior. A mathematical model should consider 3-D flow when predicting the hydraulic head in the region. Beyond this region, groundwater flow is horizontal, and temporal head distributions display the confined behavior. A 2-D flow model can predict accurate hydraulic head.

2. The aquifer anisotropy of κ_x >10 causes unidirectional flow in the strip region determined based on $\kappa_x\,(\bar{y}-1)^2 > 2$ for a horizontal well. Existing models assuming 2-D flow in a vertical plane with neglecting the flow component along the well give accurate head predictions in the region.
3. The aquifer anisotropy of κ_x <1 produces significant change in the head (i.e., $|\bar{h}| > 10^{-5}$ or $|h| > 1$ mm) in the strip area determined by $(\bar{x} - \max \bar{x}_k)^2 \leq 52.9\,\kappa_x$ and $(\bar{x} - \min \bar{x}_k)^2 \leq 52.9\,\kappa_x$ for a RCW with irregular lateral configurations.
4. The hydraulic head in the whole domain is most sensitive to the change in K_y, second sensitive to the change in K_x, and thirdly sensitive to the change in S_y. They are thus the most crucial factors in designing a pumping system.
5. The hydraulic head is sensitive to changes in K_z, S_s, z_0 and L_k in the region of $\bar{d} < \sqrt{1/\kappa_z}$ and is insensitive to the changes of them beyond the region.
6. The hydraulic head at observation locations near stream 1 is sensitive to the change in K_1 but away from the stream isn't.
7. The effect of the vertical flow on SDR is ignorable when $\kappa_z \bar{a}^2 \leq 0.01$ or $\kappa_z \bar{a}^2 \geq 30$ for unconfined aquifers. In contrast, neglecting the effect will underestimate SDR when $0.01 < \kappa_z \bar{a}^2 < 30$.
8. For unconfined aquifers, SDR increases with dimensionless well depth $\bar{z}_0$ when $0.01 < \kappa_z < 30$ and is independent of $\bar{z}_0$ when $\kappa_z \leq 0.01$ or $\kappa_z \geq 30$. For confined aquifers, SDR is independent of $\bar{z}_0$ and κ_z. For both kinds of aquifers, the distribution curve of SDR versus $\bar{t}$ is independent of aquifer anisotropy κ_x.

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Appendix A: Finite integral transform

Latinopoulos (1985) provided the finite integral transform for a rectangular aquifer domain where each side can be under either the Dirichlet, no-flow, or Robin condition. The transform associated with the boundary conditions, Eqs. (12) – (15), is defined as

$$\tilde{h}(\alpha_m, \beta_n) = \Im\{\bar{h}(\bar{x}, \bar{y})\} = \int_0^{\bar{w}_x} \int_0^{\bar{w}_y} \bar{h}(\bar{x}, \bar{y})\ \cos(\alpha_m\, \bar{x})\, K(\bar{y})\, d\bar{y}\, d\bar{x} \tag{A1}$$

with

$$K(\bar{y}) = \sqrt{2}\frac{\beta_n \cos(\beta_n\, \bar{y}) + \kappa_1 \sin(\beta_n\, \bar{y})}{\sqrt{(\beta_n^2 + \kappa_1^2)[\bar{w}_y + \kappa_2 /(\beta_n^2 + \kappa_2^2)] + \kappa_1}} \tag{A2}$$

where $\cos(\alpha_m\, \bar{x})\ K(\bar{y})$ is the kernel function. According to Latinopoulos (1985, Eq. (9)), the transform has the property of

$$\Im\left\{\kappa_x \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2}\right\} = -(\kappa_x\, \alpha_m^2 + \beta_n^2)\, \tilde{h}(\alpha_m, \beta_n) \tag{A3}$$

The formula for the inverse finite integral transform can be written as (Latinopoulos, 1985, Eq. (14))

$$\bar{h}(\bar{x}, \bar{y}) = \Im^{-1}\{\tilde{h}(\alpha_m, \beta_n)\} = \frac{1}{\bar{w}_x}\left[\sum_{n=1}^{\infty} \tilde{h}(0, \beta_n) K(\bar{y}) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{h}(\alpha_m, \beta_n)\cos(\alpha_m\, \bar{x}) K(\bar{y})\right] \tag{A4}$$

Appendix B: Derivation of equation (31)

The function of p in Eq. (28) is defined as

$$F(p) = \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{p\, \kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \tag{B1}$$

Notice that the term $\cos(\alpha_m \bar{x}_0) K(\bar{y}_0)$ in Eq. (28) is excluded because it is independent of p. $F(p)$ is a single-value function with respect to p. On the basis of the residue theorem, the inverse Laplace transform for $F(p)$ equals the summation of residues of poles in the complex plane. The residue of a simple pole can be derived according to the formula below:

$$\mathrm{Res}|_{p=p_i} = \lim_{p \to p_i} F(p) \exp(p\bar{t})\,(p - p_i) \quad \text{(B2)}$$

where p_i is the location of the pole in the complex plane.

The locations of poles are the roots of the equation obtained by letting the denominator in Eq. (B1) to be zero, denoted as

$$p\, \kappa_z\, \lambda (p\, \gamma \cosh \lambda + \kappa_z\, \lambda \sinh \lambda) = 0 \quad \text{(B3)}$$

where λ is defined in Eq. (29). Notice that $p = -\kappa_x \alpha_m^2 - \beta_n^2$ obtained by $\lambda = 0$ is not a pole in spite of being a root. Apparently, one pole is at $p = 0$, and the residue based on Eq. (B2) with $p_i = 0$ is expressed as

$$\mathrm{Res}|_{p=0} = \lim_{p \to 0} \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{\kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \exp(p\bar{t}) \quad \text{(B4)}$$

Eq. (B4) with $p = 0$ and $\lambda = \lambda_s$ reduces to $\psi_{m,n}$ in Eq. (33).

Other poles are determined by the equation of

$$p\, \gamma \cosh \lambda + \kappa_z\, \lambda \sinh \lambda = 0 \quad \text{(B5)}$$

which comes from Eq. (B3). One pole is at $p = p_0$ between $p = 0$ and $p = -\kappa_x\, \alpha_m^2 - \beta_n^2$ in the negative part of the real axis. Newton's method can be used to obtain the value of p_0. In order to have proper initial guess for Newton's method, we let $\lambda = \lambda_0$ and then have $p = \kappa_z\, \lambda_0^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \lambda_0$, $p = \kappa_z\, \lambda_0^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$, $\cosh \lambda_0 = (e^{\lambda_0} + e^{-\lambda_0})/2$ and $\sinh \lambda_0 = (e^{\lambda_0} - e^{-\lambda_0})/2$ into Eq. (B5) and rearranging the result leads to Eq. (40). Initial guess for finding root λ_0 of Eq. (40) is discussed in section 2.3. With known value of λ_0, one can obtain $p_0 = \kappa_z\, \lambda_0^2 - \kappa_x\, \alpha_m^2 - \beta_n^2$. According to Eq. (B2), the residue of the simple pole at $p = p_0$ is written as

$$\mathrm{Res}|_{p=p_0} = \lim_{p \to p_0} \frac{\cosh[(1+a)\lambda][-\kappa_z \lambda \cosh(b\lambda) + cp\gamma \sinh(b\lambda)]}{p \kappa_z \lambda (p\gamma \cosh \lambda + \kappa_z \lambda \sinh \lambda)} \exp(p\bar{t})\,(p - p_0) \quad \text{(B6)}$$

where both the denominator and nominator equal zero when $p = p_0$. Applying L'Hospital's

Rule to Eq. (B6) results in

$$\mathrm{Res}|_{p=p_0} = \lim_{p\to p_0} \frac{2\cosh[(1+a)\lambda][-\kappa_z\lambda\cosh(b\lambda)+cp\gamma\sinh(b\lambda)]}{p[(1+2\gamma)\kappa_z\lambda\cosh\lambda+(\gamma p+\kappa_z)\sinh\lambda]}\exp(p\bar{t}) \quad \text{(B7)}$$

Eq. (B7) with $p = p_0$ and $\lambda = \lambda_0$ reduces to $\psi_{m,n,0}$ in Eq. (34).

On the other hand, infinite poles are at $p = p_i$ behind $p = -\kappa_x\,\alpha_m^2 - \beta_n^2$. Similar to the derivation of Eq. (40), we let $\lambda = \sqrt{-1}\lambda_i$ and then have $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$ based on Eq. (29). Substituting $\lambda = \sqrt{-1}\lambda_i$, $p = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B3) and rearranging the result yields Eq. (41). The determination of λ_i is discussed in section 2.3. With known value λ_i, one can have $p_i = -\kappa_z\,\lambda_i^2 - \kappa_x\,\alpha_m^2 - \beta_n^2$. The residues of those simple poles at $p=p_i$ can be expressed as $\psi_{m,n,i}$ in Eq. (35) by substituting $p_0 = p_i$, $p = p_i$, $\lambda = \sqrt{-1}\lambda_i$, $\cosh\lambda = \cos\lambda_i$ and $\sinh\lambda = \sqrt{-1}\sin\lambda_i$ into Eq. (B7). Eventually, the inverse Laplace transform for $F(p)$ equals the sum of those residues (i.e., $\phi_{m,n} = \psi_{m,n} + \psi_{m,n,0} + \sum_{i=1}^{\infty}\psi_{m,n,i}$). The time-domain result of $\Omega(a,b,c)$ in Eq. (28) is then obtained as $\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$. By substituting $\tilde{h}(\alpha_m,\beta_n) = \phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)$ and $\tilde{h}(0,\beta_n) = \phi_n K(\bar{y}_0)$ into Eq. (A4) and letting $\bar{h}(\bar{x},\bar{y})$ to be $\Phi(a,b,c)$, the inverse finite integral transform for the result can be derived as

$$\Phi(a,b,c) = \frac{1}{\bar{w}_x}\Big[\sum_{n=1}^{\infty}\Big(\phi_n\,K(\bar{y}_0)K(\bar{y}) +$$

$$2\sum_{m=1}^{\infty}\phi_{m,n}\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\cos(\alpha_m\,\bar{x})\,K(\bar{y})\Big)\Big] \quad \text{(B8)}$$

Moreover, Eq. (B8) reduces to Eq. (31) when letting the terms of $K(\bar{y}_0)K(\bar{y})$ and $\cos(\alpha_m\bar{x}_0)K(\bar{y}_0)\,K(\bar{y})$ to be $2X_nY_n$ and $2X_{m,n}Y_n$, respectively.

Appendix C: Derivation of $\bar{\psi}$ in Eq. (65)

The dimensionless stream function $\bar{\psi}$ in Eq. (65) can be expressed as

$$\bar{\psi} = C - \sqrt{\kappa_z}\int \partial\bar{h}_w/\partial\bar{z}\,d\bar{x} \quad \text{at } \bar{y} = 1 \text{ and } \bar{t} = 10^7 \quad \text{(C1)}$$

where C is a coefficient resulting from the integration, and $\bar{h}_w$ is defined in Eq. (44).

Substituting Eq. (44) into Eq. (C1) leads to

$$\bar{\psi}(\bar{x},\bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k}\sum_{k=1}^{N}\begin{cases}\int \partial\Phi(-\bar{z}_0,\bar{z},1)/\partial\bar{z}\;d\bar{x} & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ \int \partial\Phi(\bar{z},\bar{z}_0,-1)/\partial\bar{z}\;d\bar{x} & \text{for } -1\le\bar{z}\le-\bar{z}_0\end{cases} \quad \text{at } \bar{y}=1 \text{ and } \bar{t}=10^7 \qquad \text{(C2)}$$

$$\Phi(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\phi_n\, X_{n,k} + 2\sum_{m=1}^{\infty}\phi_{m,n}\, X_{m,n,k}\,\cos(\alpha_m\,\bar{x})\right]\, Y_n\right\} \qquad \text{(C3)}$$

where $\phi_{m,n}$, Y_n, $X_{n,k}$ and $X_{m,n,k}$ are defined in Eqs. (32), (38), (45) and (46), respectively, and ϕ_n equals $\phi_{m,n}$ with $\alpha_m = 0$. In Eq. (C3), variable $\bar{x}$ appears only in $\cos(\alpha_m\,\bar{x})$, and variable $\bar{z}$ appears only in ϕ_n and $\phi_{m,n}$ in Eq. (32). Eq. (C2) therefore becomes

$$\bar{\psi}(\bar{x},\bar{z}) = C - \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k}\sum_{k=1}^{N}\begin{cases}\Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -1\le\bar{z}\le-\bar{z}_0\end{cases} \quad \text{at } \bar{y}=1 \text{ and } \bar{t}=10^7 \qquad \text{(C4)}$$

$$\Phi'(a,b,c) = \frac{2}{\bar{w}_x}\left\{\sum_{n=1}^{\infty}\left[\frac{\partial\phi_n}{\partial\bar{z}}\, X_{n,k}\int d\bar{x} + 2\sum_{m=1}^{\infty}\frac{\partial\phi_{m,n}}{\partial\bar{z}}\, X_{m,n,k}\int\cos(\alpha_m\,\bar{x})\,d\bar{x}\right]\, Y_n\right\} \qquad \text{(C5)}$$

Consider $\bar{t} = 10^7$ for steady-state flow that the exponential terms of $\exp(p_0\,\bar{t})$ and $\exp(p_i\,\bar{t})$ approach zero (i.e., $p_0 > 0$ and $p_i > 0$) for the default values of the parameters used to plot Figure 2. Then, we have $\phi_{m,n} = \psi_{m,n}$ defined in Eq. (33) because of $\psi_{m,n,0} \cong 0$, $\psi_{m,n,i} \cong 0$, $\mu_{m,n,0} \cong 0$ and $\nu_{m,n,i} \cong 0$. On the basis of $\phi_{m,n} = \psi_{m,n}$ and Eq. (33) with $a = -\bar{z}_0$ and $b = \bar{z}$ for $-\bar{z}_0 \le \bar{z} \le 0$ and $a = \bar{z}$ and $b = \bar{z}_0$ for $-1 \le \bar{z} \le -\bar{z}_0$, the result of differentiation, i.e., $\partial\phi_{m,n}/\partial\bar{z}$, in Eq. (C5) equals

$$\frac{\partial\phi_{m,n}}{\partial\bar{z}} = \begin{cases}-\lambda_s\cosh[(1-\bar{z}_0)\lambda_s]\sinh(\bar{z}\,\lambda_s)/(\kappa_z\,\lambda_s\,\sinh\lambda_s) & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ -\lambda_s\sinh[(1+\bar{z})\lambda_s]\cosh(\bar{z}_0\,\lambda_s)/(\kappa_z\,\lambda_s\,\sinh\lambda_s) & \text{for } -1\le\bar{z}\le-\bar{z}_0\end{cases} \qquad \text{(C6)}$$

Notice that $\partial\phi_n/\partial\bar{z}$ in Eq. (C5) equals Eq. (C6) with $\alpha_m = 0$. In addition, both integrations in Eq. (C5) can be done analytically as

$$\int\cos(\alpha_m\,\bar{x})\,d\bar{x} = \begin{cases}\sin(\alpha_m\,\bar{x})/\alpha_m & \text{for } \alpha_m\ne 0\\ \bar{x} & \text{for } \alpha_m = 0\end{cases} \qquad \text{(C7)}$$

On the other hand, coefficient C in Eq. (C4) is determined by the condition of $\bar{\psi} = 0$ at $\bar{x} = \bar{x}_0$ and results in

$$C = \frac{\sqrt{\kappa_z}}{\sum_{k=1}^{N}\bar{L}_k}\sum_{k=1}^{N}\begin{cases}\Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -\bar{z}_0\le\bar{z}\le 0\\ \Phi'(-\bar{z}_0,\bar{z},1) & \text{for } -1\le\bar{z}\le-\bar{z}_0\end{cases} \qquad \text{(C8)}$$

where Φ' is defined in Eq. (C5) with Eqs. (C6) and (C7), $\bar{x} = \bar{x}_0$ and $\bar{y} = 1$.

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Table 1. Symbols used in the text and their definitions.

Symbol	Definition
a	Shortest horizontal distance between stream 1 and the far end of lateral
$\bar{a}$	a/y_0
b_1, b_2	Thicknesses of streambeds 1 and 2, respectively
d	Shortest horizontal distance between the far end of lateral and location of having only horizontal flow
$\bar{d}$	d/y_0
H	Aquifer thickness
h	Hydraulic head
$\bar{h}$	$(K_y\ H\ h)/Q$
K_x, K_y, K_z	Aquifer hydraulic conductivities in x, y and z directions, respectively
(K_1, K_2)	Hydraulic conductivities of streambeds 1 and 2, respectively
L_k	Length from x axis to k-th lateral where $k \in (1, 2, \ldots N)$
$\bar{L}_k$	L_k/y_0
N	The number of laterals
Q	Pumping rate of point sink or radial collector well
p	Laplace parameter
p_i	$-\kappa_z\lambda_i^2 - \kappa_x\alpha_m^2 - \beta_n^2$
p_i'	$-\kappa_z\ \lambda_i^2 - \beta_n^2$
p_0	$\kappa_z\lambda_0^2 - \kappa_x\alpha_m^2 - \beta_n^2$
p_0'	$\kappa_z\ \lambda_0^2 - \beta_n^2$
R	Shortest horizontal distance between the far end of lateral and aquifer lateral boundary
S_s, S_y	Specific storage and specific yield, respectively
t	Time since pumping
$\bar{t}$	$(K_y\ t)/(S_s\ y_0^2)$
w_x, w_y	Aquifer widths in x and y directions, respectively
$\bar{w}_x$, $\bar{w}_y$	w_x/y_0, w_y/y_0
X_n	Equaling $X_{m,n}$ defined in Eq. (39) with $\alpha_m = 0$
$X_{n,k}$	Defined in Eq. (45)
x, y, z	Cartesian coordinate system
$\bar{x}$, $\bar{y}$, $\bar{z}$	x/y_0, y/y_0, z/H
$\bar{x}_k$	Coordinate $\bar{x}$ of the far end of the k-th lateral

x_0, y_0, z_0	Location of center of RCW
$\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$	x_0/y_0, 1, z_0/H
x_0', y_0', z_0'	Location of point sink
$\bar{x}_0'$, $\bar{y}_0'$, $\bar{z}_0'$	x_0'/y_0, y_0'/y_0, z_0'/H
α_m	$m\,\pi/\bar{w}_x$
β_n	Roots of Eq. (19)
ϕ_n	Equaling $\phi_{m,n}$ defined in Eq. (32) with $\alpha_m = 0$
γ	$S_y/(S_s\,H)$
κ_x, κ_z	K_x/K_y, $(K_z\,y_0^2)/(K_y\,H^2)$
κ_1, κ_2	$(K_1\,y_0)/(K_y\,b_1)$, $(K_2\,y_0)/(K_y\,b_2)$
λ_0, λ_i	Roots of Eqs. (40) and (41), respectively
λ_s, λ_s'	$\sqrt{(\kappa_x\alpha_m^2 + \beta_n^2)/\kappa_z}$, $\beta_n/\sqrt{\kappa_z}$
$\mu_{n,0}$	Equaling $\mu_{m,n,0}$ defined in Eq. (36) with $\alpha_m = 0$
$\nu_{n,i}$	Equaling $\nu_{m,n,i}$ defined in Eq. (37) with $\alpha_m = 0$
θ_k	Counterclockwise angle from x axis to k-th lateral where $k \in (1, 2, \dots N)$
$\max \bar{x}_k$, $\min \bar{x}_k$	Maximum and minimum of $\bar{x}_k$, respectively, where $k \in (1, 2, \dots N)$

Figures

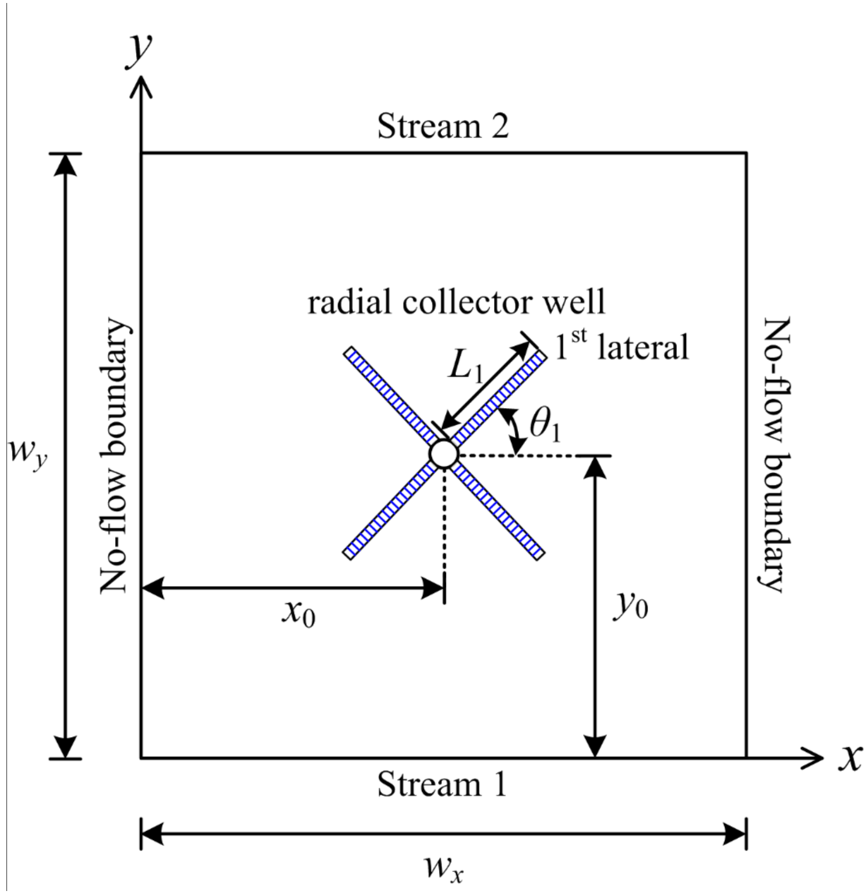
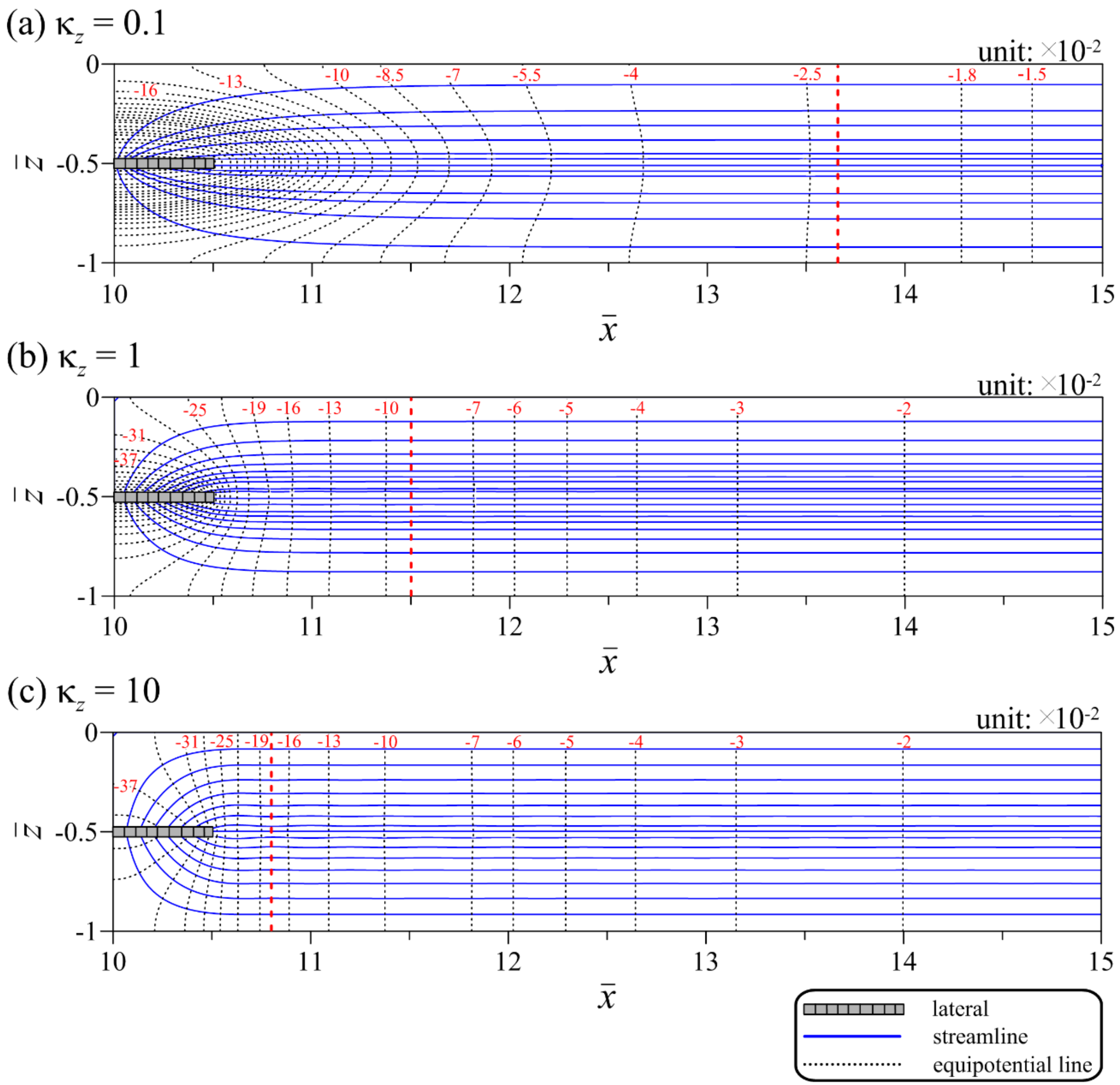


Figure 1. Schematic diagram of a radial collector well in a rectangular unconfined aquifer

813



814 **Figure 2.** Streamlines and equipotential lines predicted by the present solution for κ_z = (a) 0.1, (b) 1 and (c) 10.

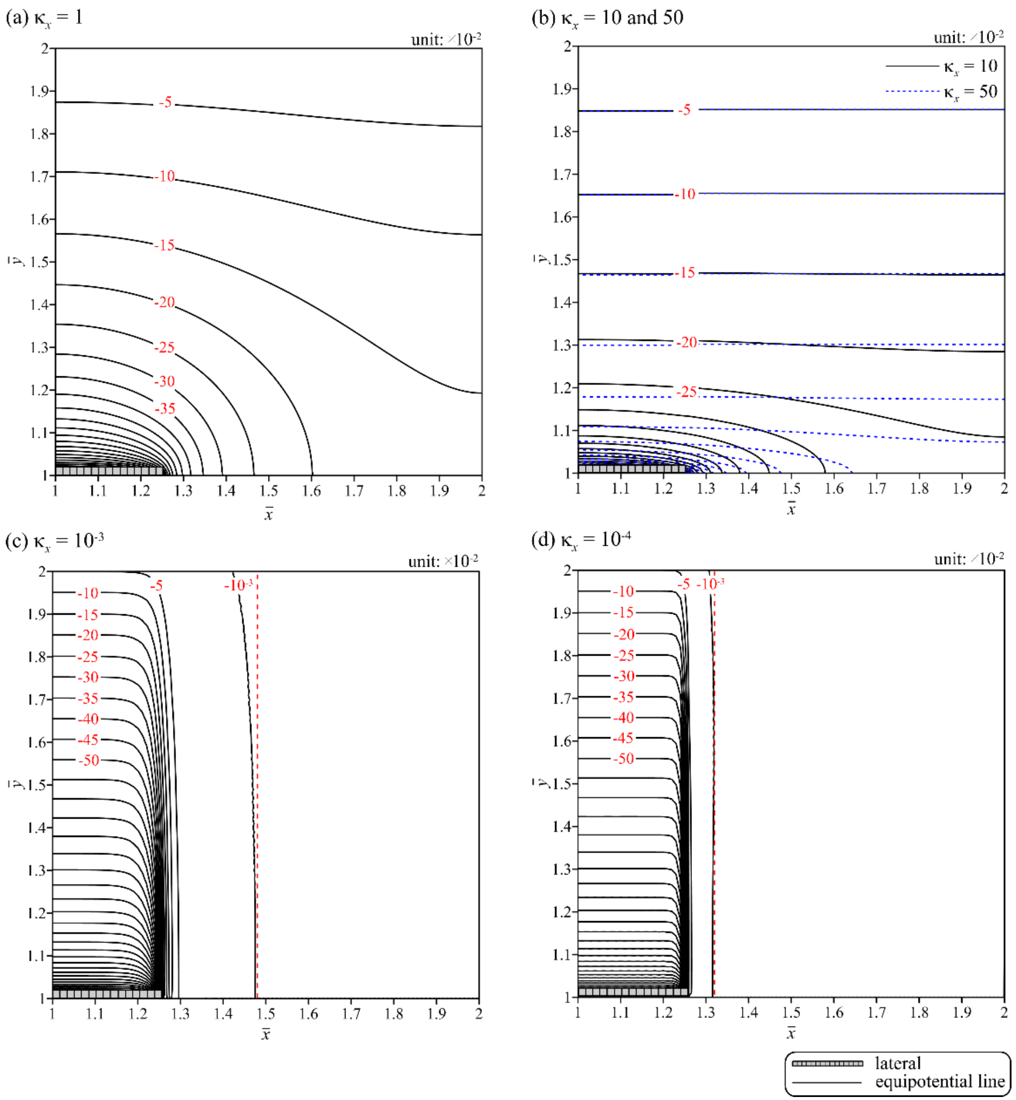


Figure 3. Spatial distributions of the dimensionless head predicted by the present head solution for κ_x = (a) 1, (b) 10 and 50, (c) 10^{-3} and (d) 10^{-4}.

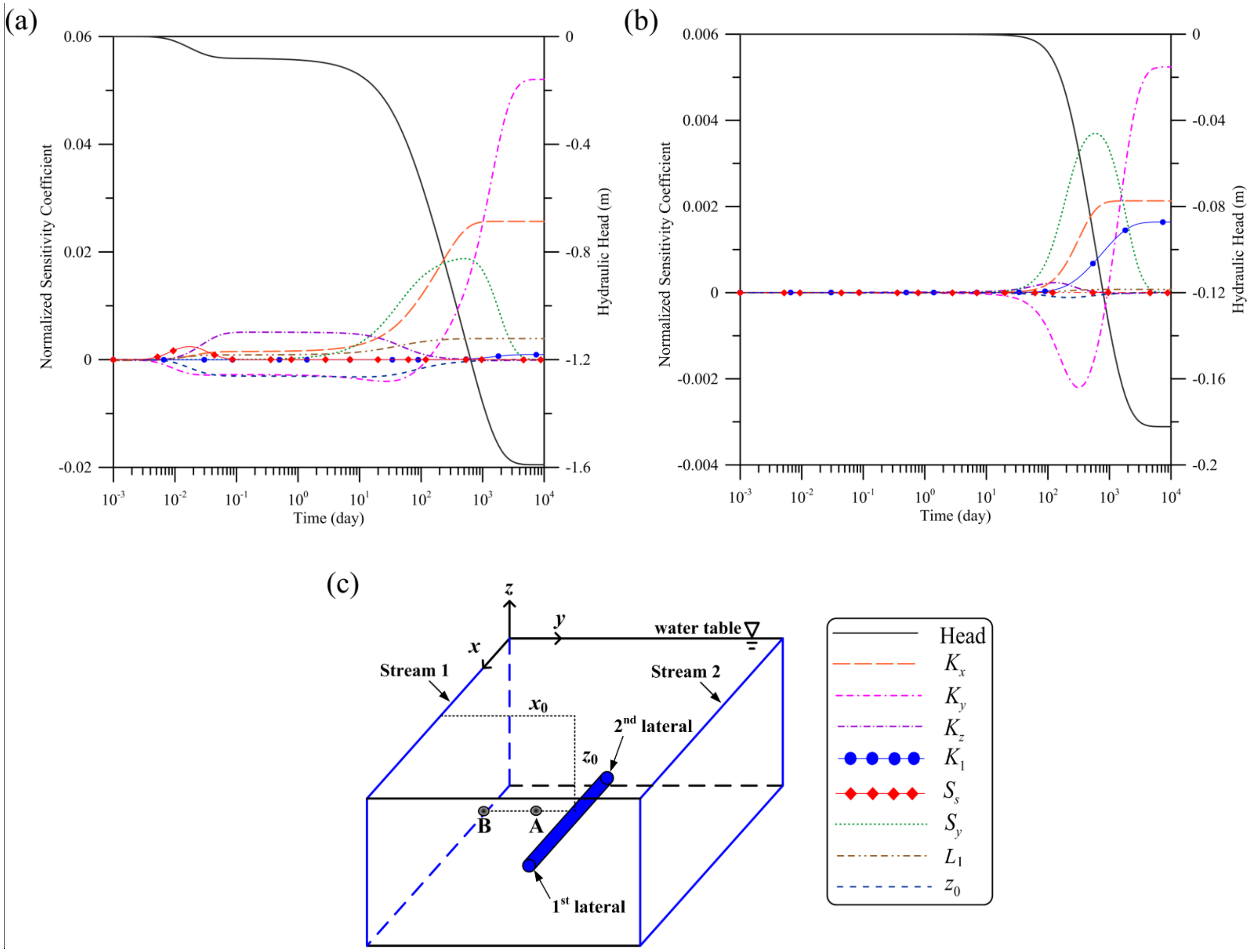


Figure 4. Temporal distribution curves of the normalized sensitivity coefficients for parameters K_x, K_y, K_z, S_s, S_y, K_1, L_1 and z_0 observed at piezometers (a) A of (400 m, 340 m, −10 m) and (b) B of (400 m, 80 m, −10 m).

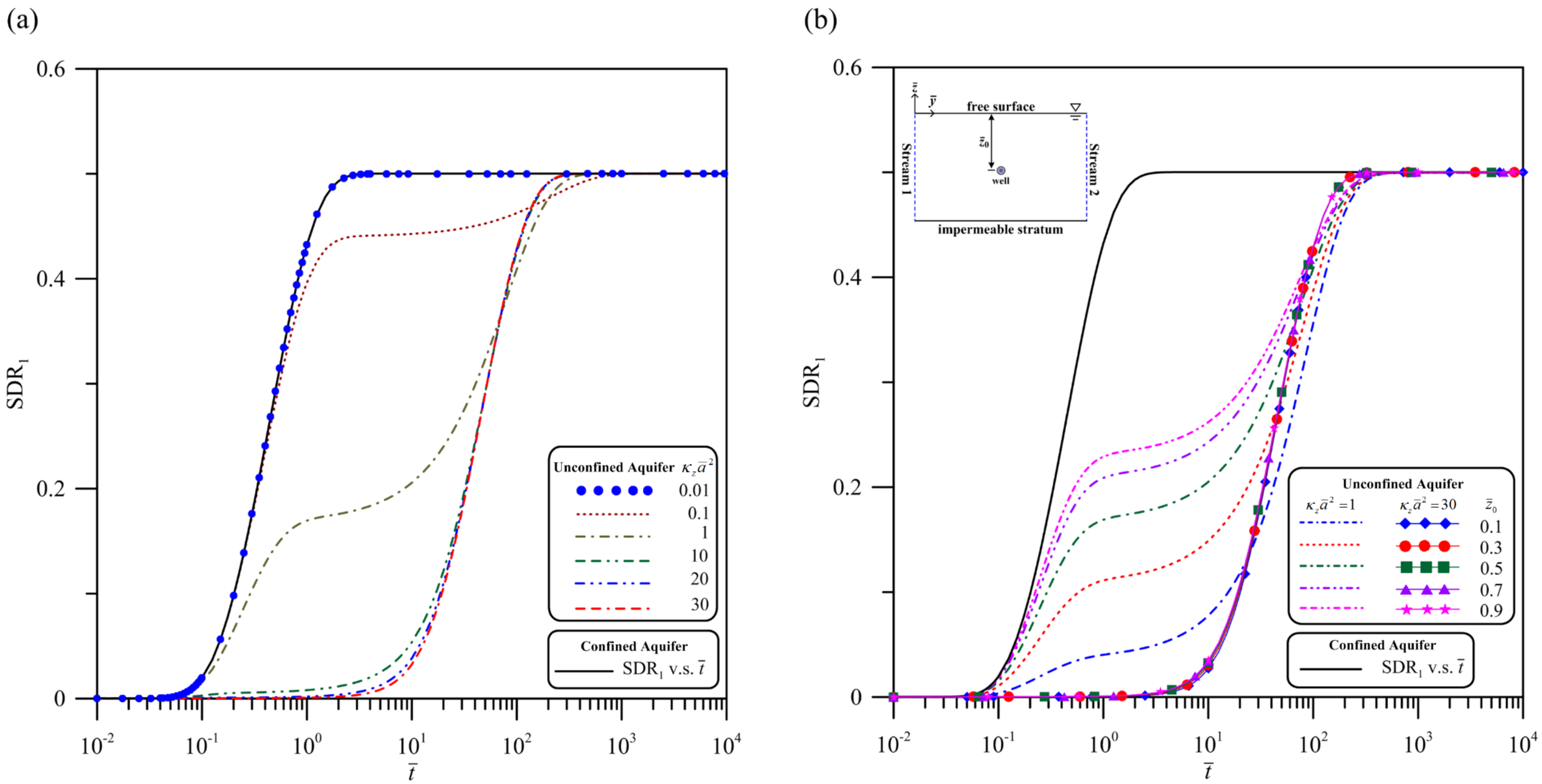


Figure 5. Temporal SDR_1 distributions predicted by Eq. (52) for stream 1 with various values of (a) $\kappa_z \bar{a}^2$ and (b) $\bar{z}_0$.

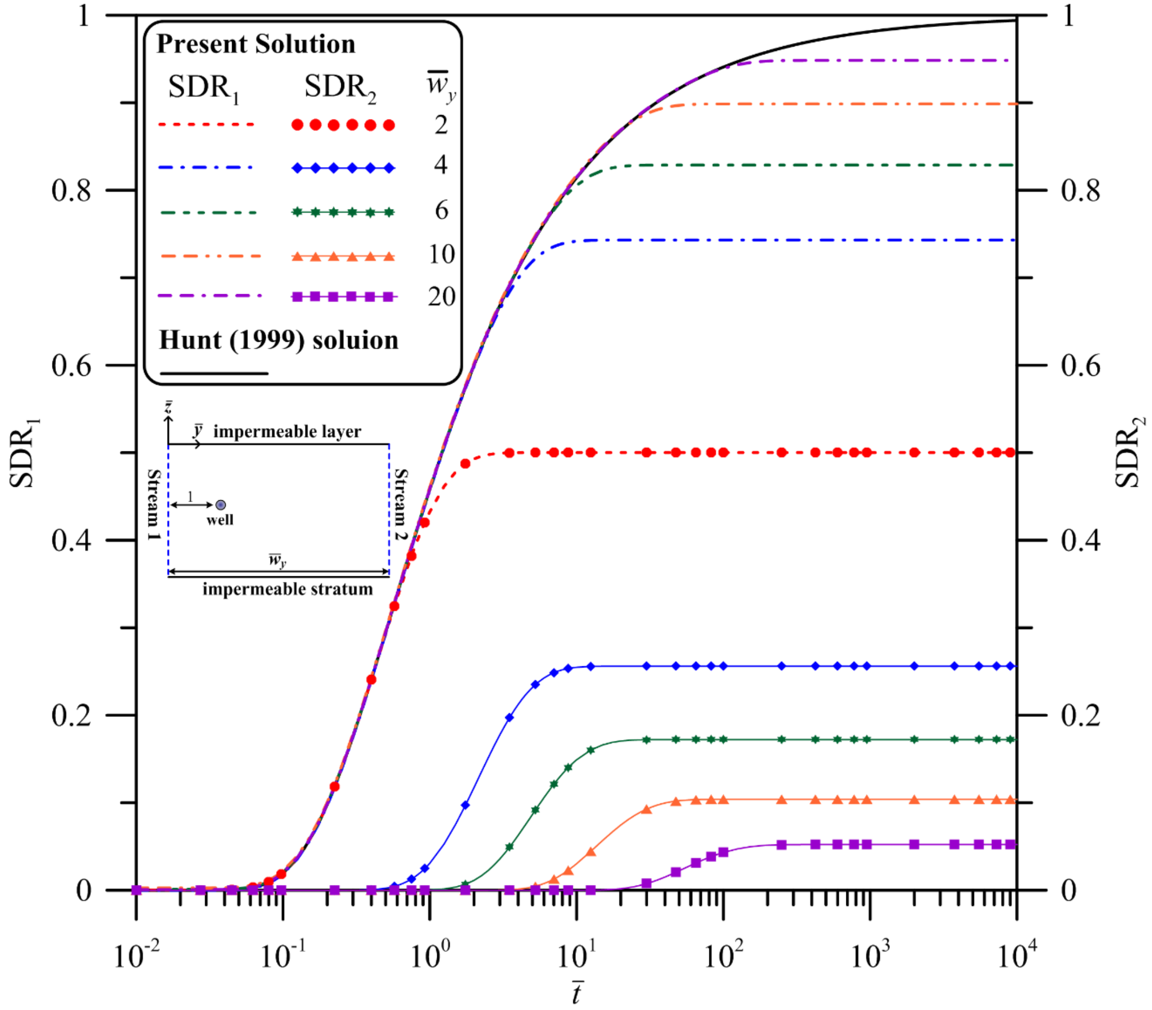


Figure 6. Temporal SDR distribution curves predicted by Eqs. (52) and (53) with $\gamma = 0$ for confined aquifers when $\overline{w}_y$ = 2, 4, 6, 10 and 20.