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Technical note: Analytical solution for the mean drawdown of steady state pumping tests in two-dimensional isotropic heterogeneous aquifers

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Abstract

A new method is presented which allows to interpret steady state pumping test in heterogeneous isotropic transmissivity fields. In contrast to mean uniform flow, the pumping test drawdowns in heterogeneous media cannot be described by a single effective or equivalent value of hydraulic transmissivity. A radially depending description of transmissivity is required, including the parameters of aquifer heterogeneity: mean, variance and correlation length. Such a model is provided by the upscaling procedure Radial Coarse Graining, which describes the transition of near well to far field transmissivity effectively. Based on the Radial Coarse Graining Transmissivity, an analytical solution for a steady state pumping test drawdown is derived. The so-called effective well flow solution is derived for two cases: the ensemble mean of pumping tests and the drawdown at an individual heterogeneous transmissivity field. The analytical form of the solution allows to inversely estimate the parameters of aquifer heterogeneity from pumping test data. This is shown making use of virtual pumping test data, for both cases the ensemble mean drawdown and pumping tests at individual transmissivity fields. The effective well flow solution reproduces the drawdown for two-dimensional pumping tests in heterogeneous media and is a promising tool to estimate parameters of aquifer heterogeneity, in particular for the horizontal correlation length.

1 Introduction

Pumping tests are a widely used tool to identify horizontal hydraulic conductivity, which is the parameter determining the groundwater flow velocity. Analytical solutions of the radial flow equation are used in practice to analyze measured drawdowns. In general, these solutions assume a constant homogeneous hydraulic conductivity like Thiem's solution for steady state (Thiem, 1906):

$$h_{\mathrm{Thiem}}(r) = -\frac{Q_{\mathrm{w}}}{2\pi D K_{\mathrm{h}}} \ln \frac{r}{R} + h(R)\,. \tag{1}$$

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Thiem's solution (1) gives the hydraulic head $h_{\mathrm{Thiem}}(r)$ depending on the radial distance r from the well for homogeneous horizontal hydraulic conductivity K_{h}. It is valid in a confined aquifer of thickness D with fully penetrating well and a constant discharge Q_{w}, $h(R)$ is a known reference head at an arbitrary distance R from the well.

In large scale pumping tests the vertical extension of the aquifer is negligible compared to horizontal aquifer extend. Thus, flow is assumed to be horizontal and modelled as two-dimensional. Hydraulic conductivity is then replaced by transmissivity, which is defined as the product of conductivity and aquifer thickness $T = K_{\mathrm{h}}D$. In the following, transmissivity will be used instead of horizontal hydraulic conductivity, since the focus of the work will be on two-dimensional well flow.

Most natural aquifers exhibit geological heterogeneity in the sedimentary composition, which evolved from the complex geomorphological processes through which they were formed. In particular, transmissivity shows a strong spatial variability. Values measured in the field vary over orders of magnitude (Gelhar, 1993). Geostatistical distributions are generally used to capture the effects of aquifer heterogeneities. Transmissivity $T(\boldsymbol{x})$ is modelled as log-normally distributed spatial random function: $\log T(\boldsymbol{x}) = Y(\boldsymbol{x})$ is normally distributed with a Gaussian probability density function $\mathrm{pdf}_Y(x) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ in uni-variate form with μ and σ^2 being the mean and the variance of Y, respectively. The correlation structure of transmissivity in space is captured by a covariance model $\mathrm{Cov}[T(\boldsymbol{x}+\boldsymbol{s}), T(\boldsymbol{x})] = \exp\left(2\mu + \sigma^2 + \mathrm{CV}_Y(\boldsymbol{s})\right)$.

In the stochastic framework, the solution of the radial flow equation with a log-normal distributed transmissivity is also a random spatial function. Since the solution of the stochastic differential equation is out of scope, the focus of investigation was on homogeneous substitute values for describing well flow effectively. As a first approach, Thiem's solution (1) was applied to pumping tests in heterogeneous media. However, this requires a representative transmissivity value T for the whole range of the depression cone (Matheron, 1967), which does not exist. Effective or equivalent descriptions of transmissivity in pumping tests were investigated, e.g. by Desbarats (1992);

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Sánchez-Vila et al. (1999); Neuman et al. (2007); Dagan and Lessoff (2007); Schneider and Attinger (2008) and many more. For a detailed review see Sánchez-Vila et al. (2006).

In contrast to mean uniform flow, the pumping test drawdowns in heterogeneous media cannot be described by a single constant value of transmissivity (Matheron, 1967). Different transmissivities characterize the behavior near and far from the well: the representative value close to the well is the harmonic mean of the log-normal transmissivity $T_\mathrm{H} = \exp\left(\mu - \frac{1}{2}\sigma^2\right) = T_\mathrm{G}\exp\left(-\frac{1}{2}\sigma^2\right)$. With increasing distance from the well, the drawdown behaviour is characterized by the effective transmissivity for uniform flow, which is the geometric mean $T_\mathrm{G} = \exp(\mu)$ for flow in two dimensional isotropic porous media.

It seems obvious, that a representative description of transmissivity for well flow needs to be a radially depending function, which interpolates between the harmonic and the geometric mean. The equivalent transmissivity T_eq is a well established approach of a radially depending description, visualized in Fig. 1a. $T_\mathrm{eq}(r) = -\frac{Q_\mathrm{w}}{2\pi(h(r)-h(r_\mathrm{w}))}\ln\frac{r}{r_\mathrm{w}}$ was derived from Thiem's solution (1) (Matheron, 1967; Indelman et al., 1996; Dagan and Lessoff, 2007). In this sense, the equivalent transmissivity T_eq is defined as the value for a homogeneous medium, which reproduces locally the same total outflow as observed in the heterogeneous domain of radius r. T_eq is strongly impacted by the reference point r_w and the corresponding head $h(r_\mathrm{w})$, which is generally chosen to be the drawdown at the well $h(r_\mathrm{w})$. Therefore, the equivalent conductivity stays close to the harmonic mean T_H, which is representative for the drawdown behaviour at the well (Fig. 1a). It takes more than 20 correlation length for T_eq to reach the far field representative value of T_G.

It is important to mention, that T_eq is not constructed to reproduce the drawdown which was used for calculating T_eq. Strictly speaking, replacing the heterogeneous transmissivity field with the equivalent transmissivity in a single forward model does not give the drawdown, with which T_eq was constructed as visualized in Fig. 1b. Instead $h_\mathrm{eq}(r)$ stays close to Thiem's solution with T_H as homogeneous substitute value.

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Schneider and Attinger (2008) introduced a novel approach to describe well flow effectively. They derived a radial adapted transmissivity $T_{\mathrm{RCG}}(r)$ by applying the upscaling technique Coarse Graining to well flow. $T_{\mathrm{RCG}}(r)$ does not only depend on the radial distance r but also on the statistical parameters of aquifer heterogeneity T_{G}, σ^2 and ℓ. $T_{\mathrm{RCG}}(r)$ captures the transition from near well to far field representative transmissivities, based on the radial distance to the well and the parameters of aquifer heterogeneity, as visualized in Fig. 1a.

In this study, an analytical solution for the hydraulic head $h_{\mathrm{efw}}(r)$ is presented, which is based on the Radial Coarse Graining transmissivity $T_{\mathrm{RCG}}(r)$ as an extension to the work of Schneider and Attinger (2008). Similar work has been done by (Zech et al., 2012) for pumping test in three dimensional porous media. The effective well flow solution $h_{\mathrm{efw}}(r)$ describes the mean depression cone of a pumping test in two dimensional heterogeneous media effectively. It can be interpreted as an extension of Thiem's formula (1) to log-normal distributed heterogeneous media. It accounts for the statistical parameters T_{G}, σ^2 and ℓ and thus allows to inversely estimate them from measured drawdown data. In contrast to existing head solutions for well flow, $h_{\mathrm{efw}}(r)$ is not limited to low variances, but is applicable to highly heterogeneous media with variances $\sigma^2 \gg 1$.

The work is organized the following: Sect. 2 is dedicated to the method of Radial Coarse Graining and the derivation of the effective well flow head solution. Thereby, it is distinguished between the solution for an ensemble mean and single realizations of heterogeneous transmissivity fields. Section 3 contains the application of the effective well flow solution to simulated pumping tests. It is shown, that $h_{\mathrm{efw}}(r)$ reproduces the drawdown in heterogeneous media and can be used to inversely estimate the statistical parameters of aquifer heterogeneity for both, ensemble mean and single realizations. Concluding remarks are given in Sect. 4.

2 Radial coarse graining transmissivity and effective well flow head

2.1 Steady state well flow with radially depending transmissivity

The drawdown of a steady state pumping test with a radially depending transmissivity $T(r)$ is given as the solution of the differential equation:

$$0 = T(r)\left(\frac{1}{r}\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2h}{\mathrm{d}r^2}\right) + \frac{T(r)}{\mathrm{d}r}\frac{\mathrm{d}h}{\mathrm{d}r} = \left(\frac{1}{r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}r}\right)\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2h}{\mathrm{d}r^2}. \tag{2}$$

The equation can be solved in $\frac{\mathrm{d}h}{\mathrm{d}r}$ by separation of variables, resulting in $\frac{\mathrm{d}h}{\mathrm{d}r} = C_1\frac{1}{rT(r)}$. The hydraulic head $h(r)$ is then given as the solution of the integral

$$h(r_2) - h(r_1) = C_1\int_{r_1}^{r_2}\frac{1}{rT(r)}\,\mathrm{d}r. \tag{3}$$

The integration constant C_1 is determined by the boundary condition. Supposing a constant flux boundary condition at the well, gives $Q_\mathrm{w} = -2\pi r_\mathrm{w}T(r_\mathrm{w})\frac{\mathrm{d}h}{\mathrm{d}r}(r_\mathrm{w}) = -2\pi r_\mathrm{w}T(r_\mathrm{w})\frac{C_1}{r_\mathrm{w}T(r_\mathrm{w})}$ and thus, $C_1 = -\frac{Q_\mathrm{w}}{2\pi}$.

Equation (3) is the general solution of the radial flow equation (2) for radially depending transmissivity, independent of the functional form of $T(r)$.

When comparing Eq. (3) with the definition of the equivalent transmissivity, it becomes obvious that T_eq is not constructed to solve the equation. The combination of both formulas results in $\int_{r_1}^{r_2}\frac{1}{rT(r)}\,\mathrm{d}r = \frac{1}{T_\mathrm{eq}(r)}\ln\frac{r_1}{r_2}$, which is only fulfilled, when $T(r)$ is constant in r.

2.2 Radial coarse graining transmissivity

A radially depending transmissivity for log-normal distributed media with Gaussian correlation structure was derived by Schneider and Attinger (2008), denoted as the Radial

Coarse Graining Transmissivity $T_{\mathrm{RCG}}(r)$. It was derived making use of the upscaling procedure Coarse Graining, introduced for uniform flow by Attinger (2003). The basic idea of Radial Coarse Graining is to perform a spatial filtering on the flow equation which is appropriate to the non-uniform flow character of a pumping test. The filter was chosen proportional to the radial distance from the well. Hence, the filter length is very small close to the well, so nearly no filtering is applied and the heterogeneity of the local transmissivities is resolved. Far away from the well, the filter volumes are very large and the local heterogeneous transmissivity values are replaced by the effective value for uniform flow. Detailed discussions on Radial Coarse Graining can be found in Schneider and Attinger (2008) and Zech et al. (2012).

Schneider and Attinger (2008) presented two forms of $T_{\mathrm{RCG}}(r)$, one for an ensemble of pumping tests and an adapted version for drawdowns of individual pumping tests, which are different from the ensemble behaviour due to a lack of ergodicity at the well. The ensemble version of $T_{\mathrm{RCG}}(r)$ is given as

$$T_{\mathrm{RCG}}(r) = T_{\mathrm{G}} \exp\left(-\frac{1}{2}\frac{\sigma^2}{(1+\zeta^2 r^2/\ell^2)}\right), \tag{4}$$

where r is the radial distance from the well, T_{G} is the geometric mean, σ^2 is the variance and ℓ is the correlation length of the log-normally distributed transmissivity $T(\boldsymbol{x})$. ζ is a factor of proportionality, which was determined to be $\zeta = 1.6$, as discussed in detail by Zech et al. (2012). $T_{\mathrm{RCG}}(r)$ allows a transition from near field transmissivity $T_{\mathrm{H}} = T_{\mathrm{G}} \exp\left(-\frac{1}{2}\sigma^2\right)$ to far field value T_{G} depending on the radial distance r, controlled by the correlation length ℓ (Fig. 1a).

The adapted radial coarse graining transmissivity accounts for local effects of individual pumping tests and is given by

$$T_{\mathrm{RCG}}^{\mathrm{local}}(r) = T_{\mathrm{G}} \exp\left(\frac{\ln T_{\mathrm{well}} - \ln T_{\mathrm{G}}}{1+\zeta^2 r^2/\ell^2}\right) = T_{\mathrm{well}}^{\frac{1}{1+\zeta^2 r^2/\ell^2}} T_{\mathrm{G}}^{\frac{\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}}, \tag{5}$$

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where T_{well} is the local transmissivity at the well of an individual transmissivity field. $T_{\text{RCG}}^{\text{local}}(r)$ was derived from $T_{\text{RCG}}(r)$ by substituting the variance by $\sigma^2 = -2\ln\frac{T_{\text{well}}}{T_{\text{G}}}$. The substitution is derived from the relation of the variance to the harmonic mean, as asymptotic representative value at the well, $T_{\text{H}} = T_{\text{G}}\exp\left(\frac{1}{2}\sigma^2\right)$, thus $\sigma^2 = -2\ln\frac{T_{\text{H}}}{T_{\text{G}}}$. The theoretical value of T_{H} is than replaced by the local transmissivity at the well T_{well}. Due to the lack of ergodicity at the well for a single realization, T_{well} generally does not equal the harmonic mean transmissivity T_{H} of the entire field $T(\boldsymbol{x})$.

2.3 Effective well flow head

Explicit results for the hydraulic head drawdown in steady state pumping test with a radially depending transmissivity can be achieved by solving the integral in Eq. (3) making use of $T_{\text{RCG}}(r)$ (Eq. 4). Details on the derivation of $h_{\text{efw}}(r)$ can be found in the Appendix. A fully analytical solution, called the effective well flow head $h_{\text{efw}}(r)$ is given by

$$\begin{aligned} h_{\text{efw}}(r) = &-\frac{Q_{\text{w}}}{4\pi T_{\text{G}}}\exp\left(\frac{\sigma^2}{2}\right)\left(\Gamma\left(\frac{\sigma^2}{2}\frac{-\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2}\frac{-\zeta^2 R^2/\ell^2}{1+\zeta^2 R^2/\ell^2}\right)\right) \\ &+\frac{Q_{\text{w}}}{4\pi T_{\text{G}}}\left(\Gamma\left(\frac{\sigma^2}{2}\frac{1}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2}\frac{1}{1+\zeta^2 R^2/\ell^2}\right)\right) + h_R\,, \end{aligned} \tag{6}$$

where r is the radial distance from the well, Q_{w} is the pumping rate, T_{G} is the geometric mean, σ^2 is the log-transmissivity variance, and ℓ is the correlation length. Again, ζ is the factor of proportionality determined to be 1.6 and R is an arbitrary distance from the well, where the hydraulic head $h(R) = h_R$ is known. Γ is the exponential integral function with

$$\Gamma(x) = \int_{\infty}^{x}\frac{\exp(z)}{z}\,\mathrm{d}z\,.$$

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An approximate solution $h_{\mathrm{efw}}^{\mathrm{approx}}(r)$ can be derived from Eq. (6) by making use of an approximation of the exponential integral function Γ. Details are given in the Appendix.

$$h_{\mathrm{efw}}^{\mathrm{approx}}(r) = -\frac{Q_{\mathrm{w}}}{2\pi T_{\mathrm{H}}} \ln\frac{r}{R} - \frac{Q_{\mathrm{w}}}{4\pi T_{\mathrm{G}}}\left(e^{\frac{\sigma^2}{2}} - 1\right) \cdot \left(\ln\frac{1+\zeta^2 R^2/\ell^2}{1+\zeta^2 r^2/\ell^2} + \frac{\sigma^2}{2}\frac{1}{(1+\zeta^2 r^2/\ell^2)} - \frac{\sigma^2}{2}\frac{1}{(1+\zeta^2 R^2/\ell^2)}\right) + h_R\,, \tag{7}$$

$h_{\mathrm{efw}}(r)$ is constructed to describe the mean drawdown of a pumping test in two dimensional heterogeneous media effectively. The drawdown curve of $h_{\mathrm{efw}}(r)$ for a specific choice of parameters (Ensemble A of Table 1) is given in Fig. 1b in comparison to the equivalent drawdown $h_{\mathrm{eq}}(r)$, as the solution of the radial flow equation using the equivalent transmissivity T_{eq}, based on the same statistical parameters.

The effective well flow solution $h_{\mathrm{efw}}(r)$ can be adapted to analyze individual pumping tests by using $T_{\mathrm{RCG}}^{\mathrm{local}}(r)$ (Eq. 5) instead of $T_{\mathrm{RCG}}(r)$ (Eq. 4). The appropriate local effective well flow solutions $h_{\mathrm{efw}}^{\mathrm{local}}(r)$ are than given similarly to Eqs. (6) and (7) with $\frac{\sigma^2}{2}$ substituted by $-\ln\frac{T_{\mathrm{well}}}{T_{\mathrm{G}}}$ and T_{H} substituted by T_{well}.

The local effective well flow solution $h_{\mathrm{efw}}^{\mathrm{local}}(r)$ can be used to analyze drawdowns of single pumping tests in heterogeneous media as encountered in practice. The solution is adapted to the lack of ergodicity at the well, by using transformed parameters T_{well}, T_{G} and ℓ. The geometric mean T_{G} and the correlation length ℓ for a single realization should also be interpreted as local values, not necessarily representing the mean values of the entire field, but those of the pumping well vicinity. Owing to the nature of the pumping test, the drawdown signal does not sample the heterogeneity in transmissivity in a symmetric way, but the shape of the drawdown is mainly determined by the local heterogeneity close to well.

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2.4 Impact of parameters

The analytical form of $h_{\mathrm{efw}}(r)$ allows to analyze the impact of the statistical parameters T_G, σ^2 and ℓ on the drawdown. The drawdown behaviour for different choices of parameters can be seen in Fig. 2, which is discussed in detail later on.

Every parameter impacts on the drawdown in a different region. The geometric mean T_G as representative value for mean uniform flow determines the far field behaviour. The variance σ^2 determines the drawdown at the well due to the dependence of the near-well asymptotic value $T_\mathrm{H} = T_\mathrm{G} \exp\left(-\frac{1}{2}\sigma^2\right)$. The larger the variance the larger are the differences between T_G and T_H and the steeper is the drawdown at the well. Whereas, the correlation length ℓ determines the transition from near to far field behavior.

The asymptotic behaviour of $h_{\mathrm{efw}}(r)$ can easily be analyzed using approximate functional description in Eq. (7): for distances close to the well, thus $r \ll \ell$, $h_{\mathrm{efw}}(r)$ converges to Thiem's solution with T_H as homogeneous substitute value. All terms, except the first one in Eq. (7), tend to zero or become constant. Thus, they are negligible compared to logarithmic first term for very small r,

$$\begin{aligned} h_{\mathrm{efw}}^{\mathrm{approx}}(r \ll \ell) &\approx -\frac{Q_\mathrm{w}}{2\pi T_\mathrm{H}} \ln\frac{r}{R} - \frac{Q_\mathrm{w}}{4\pi T_\mathrm{G}}\left(\mathrm{e}^{\frac{\sigma^2}{2}} - 1\right) \cdot \left(\ln\left(1 + \zeta^2 R^2/\ell^2\right) + \frac{\sigma^2}{2}\right) + h_R \\ &\approx -\frac{Q_\mathrm{w}}{2\pi T_\mathrm{H}} \ln\frac{r}{R} + h_R. \end{aligned}$$

For large distances from the well, i.e. $r \gg \ell$, the solution converges to Thiem's solution with T_G as homogeneous substitute value. The third and fourth term in Eq. (7) tend to zero and cancel out. The ones in the second term can be neglected, thus

$$h_{\mathrm{efw}}^{\mathrm{approx}}(r \gg \ell) \approx -\frac{Q_\mathrm{w}}{2\pi T_\mathrm{H}} \ln\frac{r}{R} - \left(\frac{Q_\mathrm{w}}{4\pi T_\mathrm{H}} - \frac{Q_\mathrm{w}}{4\pi T_\mathrm{G}}\right) \cdot \ln\left(\frac{\zeta^2 R^2/\ell^2}{\zeta^2 r^2/\ell^2}\right) + h_R \approx -\frac{Q_\mathrm{w}}{2\pi T_\mathrm{G}} \ln\frac{r}{R} + h_R.$$

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The larger the correlation length ℓ the longer takes the transition of the drawdown from near well to far field behaviour. The influence of ℓ on $h_{\mathrm{efw}}(r)$ vanishes quickly with increasing distance to the well. The drawdown reaches the far field behavior after approximately two correlation lengths $h^{\mathrm{efw}}(r > 2\ell) = h_{\mathrm{Thiem}}(r > 2\ell)$ with T_G as homogeneous substitute value (Fig. 1).

3 Robust estimation of statistical parameters

3.1 Numerical pumping tests

Numerical pumping tests in heterogeneous porous media were generated as artificial measurements. They serve to test the capability of $h_{\mathrm{efw}}(r)$ in reproducing the mean drawdown and in estimating the underlying parameters of heterogeneity. Pumping tests were simulated using the finite element software OpenGeoSys. The software was successfully tested against a wide range of benchmarks (Kolditz et al., 2012b). Results of a steady state simulation with homogeneous transmissivity are in perfect agreement with Thiem's analytical solution Eq. (1).

The numerical grid is a square of 256 × 256 elements with a uniform grid cell size of 1 m except for cells in the vicinity of the pumping well. The mesh was refined in the range of 4 m around the well, which ensures a fine resolution of the steep head gradients at the well. The well in the center of the mesh is included as a hollow cylinder with radius $r_w = 0.01$ m. The constant pumping rate of $Q_w = -10^{-4}\,\mathrm{m}^3\,\mathrm{s}^{-1}$ is distributed equally to all elements at the well. At the radial distance $R = 128$ m a constant head of $h(R) = 0$ m is applied giving a circular outer head boundary condition.

Log-normally distributed, Gaussian correlated transmissivity fields were generated using a statistical field generator based on the randomization method (Hesse et al., 2014). Multiple ensembles with different statistical parameter values were generated, including high variances up to $\sigma^2 = 4$ (Table 1). Ensemble A with $T_G = 10^{-4}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\sigma^2 = 1$ and $\ell = 10$ m will serve as reference ensemble for specific cases. Every ensem-

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ble consist of $N = 5000$ realizations, which was tested as sufficiently large to ensure ensemble convergence.

Pumping test simulations are post-processed by performing an angular and an ensemble average. For every realization i, the simulated drawdown $\langle h_i(r,\phi)\rangle$ at the radial and angular location (r,ϕ) in polar coordinates is averaged over the four axial directions: $\langle h_i(r)\rangle = \sum_{\phi_j}\langle h_i(r,\phi_j)\rangle$. The ensemble mean is the sum over the angular mean of all individual realizations: $\overline{\langle h(r)\rangle} = \sum_{i=1}^{N}\langle h_i(r)\rangle$.

Non-linear regression is used to find the best fit values for the statistical parameters, denoted by $\hat{T}_\mathrm{G}$, $\hat{\sigma}^2$, and $\hat{\ell}$. The best fit estimates are determined by minimizing the mean square error of the difference between the analytical solution $h_\mathrm{efw}(r)$ and the measured drawdown samples $h(r)$: $\min_{T_\mathrm{G},\sigma^2,\ell}\sum_r (h(r) - h_\mathrm{efw}(r))^2$ making use of the Levenberg–Marquardt algorithm. The reliability of the estimated parameters is evaluated using 95 %-confidence intervals.

The estimation procedure is applied to the head measurements at every meter distance starting at the well up to a distance of 80 m. The range beyond 80 m is not taken into account to avoid boundary effects. The range of 80 m includes at least 4 correlation lengths for all tested ensembles, which is sufficient to ensure convergence to the far field behaviour. The question of the applicability of $h_\mathrm{efw}(r)$ on limited head data is of quite complex nature. For a detailed discussion on that issue the reader is referred to Zech et al. (2015).

3.2 Ensemble pumping test interpretation

We first analyze the simulated ensemble means making use of the ensemble version of $T_\mathrm{RCG}(r)$ and $h_\mathrm{efw}(r)$ (Eqs. 4 and 6). Simulated ensemble means $\overline{\langle h(r)\rangle}$ for multiple choices of statistical parameters T_G, σ^2 and ℓ are visualized in Fig. 2 in combination with $h_\mathrm{efw}(r)$ for the best fit parameter estimates $\hat{T}_\mathrm{G}$, $\hat{\sigma}^2$, and $\hat{\ell}$. Input parameters as well as inverse estimation results for all tested ensembles are listed in Table 1.

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The best fit estimates show, that all three parameters can be inferred from the ensemble mean with a high degree of accuracy. The deviation of the geometric mean from the input value is in general less than 10 %, only for high variances the deviations are up to 30 %. Variances deviate in a range of 20 % and estimated correlation lengths are accurate within 10 % of the initial input parameter.

The confidence intervals of the estimates $\hat{T}_G$ and $\hat{\sigma}^2$ are very small, showing a high sensitivity of the effective well flow solution $h_{efw}(r)$ towards geometric mean and variance. The confidence intervals of the correlation length are larger due to the dependence of the estimate of $\hat{\ell}$ on the estimates $\hat{T}_G$ and $\hat{\sigma}^2$. This is due to the fact, that the correlation length determines the transition from $\hat{T}_{well} = \hat{T}_G \exp\left(-\frac{1}{2}\hat{\sigma}^2\right)$ to $\hat{T}_G$, which results in larger uncertainties in the estimates of $\hat{\ell}$.

3.3 Individual pumping test interpretation

In the following, pumping test drawdowns of individual transmissivity fields are interpreted based on the adaption version $h_{efw}^{local}(r)$ as discussed in Sect. 2.3. The drawdowns along the four axial directions as well as the radial mean for two realizations from Ensemble A ($T_G = 10^{-4}\,m^2\,s^{-1}$, $\sigma^2 = 1$, $\ell = 10\,m$) are visualized in Fig. 3a and b.

Both realizations from Fig. 3a and b differ significantly in the value of the local transmissivity at the well. The analysis of the transmissivity fields at the well gave "measured" values of $T_{well}^{(a)} = 0.204 \times 10^{-4}\,m^2\,s^{-1}$ and $T_{well}^{(b)} = 1.11 \times 10^{-4}\,m^2\,s^{-1}$, which is in both cases far from the theoretical harmonic mean value $T_H = 0.61 \times 10^{-4}\,m^2\,s^{-1}$ as being the representative value for the near well behaviour.

Inverse estimation results for realization in Fig. 3a differ for the drawdowns along the four axial directions $\langle h(r,\phi_j)\rangle$ and the radial mean $\langle h(r)\rangle$: the estimated geometric mean ranges between 1.03×10^{-4} and $1.45 \times 10^{-4}\,m^2\,s^{-1}$ for the four axial directions, with an average value of $\hat{T}_G = 1.17 \times 10^{-4}\,m^2\,s^{-1}$. The estimates for the local transmissivity at the well are between 0.195×10^{-4} and $0.212 \times 10^{-4}\,m^2\,s^{-1}$, with an average value of

$\hat{T}_{\text{well}} = 0.204 \times 10^{-4}\,\text{m}^2\,\text{s}^{-1}$, which is exactly the "measured" local transmissivity $T^{(a)}_{\text{well}}$. The value of $\hat{T}_{\text{well}} = 0.204 \times 10^{-4}\,\text{m}^2\,\text{s}^{-1}$ is equivalent to a local variance of $\hat{\sigma}^2 = 3.49$. The estimated correlation length ranges between 7.95 and 18.15 m, with an average of $\hat{\ell} = 12.77$ m. The differences in the estimates for the drawdowns in different direction for the same realization of transmissivity shows that the parameter estimates reflect local heterogeneity in the vicinity of the well rather than the global statistical parameters of the transmissivity field. This was studied and discussed in detail for pumping tests in three dimensional heterogeneous media by Zech et al. (2015).

Realizations in Fig. 3b does not allow to infer the parameters of variance and correlation length, due to the similarity of T_{well} and T_{G}. Near and far field representative transmissivities are nearly identical, thus the pumping test appears to behave like in a homogeneous medium (Fig. 3b). However, the behaviour is not representative but a result of the coincidental choice of the location of the pumping well.

A statistical analysis of the estimation results is presented in Fig. 4 for 100 realizations of Ensemble A. Estimation results in normalized form are shown as standard box plots, where normalization of results means that they are divided by the input parameters. It can be inferred that the estimate of the geometric mean $\hat{T}_{\text{G}}$ is in general close to the input value T_{G}. A few exceptionally small and large values of $\hat{T}_{\text{G}}$ show that the mean transmissivity observed by a pumping test in the vicinity of the well is not necessary close to the mean value of the entire transmissivity field. The estimate of the local transmissivity at the well $\hat{T}_{\text{well}}$ from the drawdown data is very close to the "measured" values T_{well} for nearly all realizations. Thus, the method reproduces very well the local transmissivity. However, the local value T_{well} of every realization can be far from the theoretical value of T_{H}, where both realizations in Fig. 3 gave example. The estimates of the correlation length show a very large scatter. Exceptionally large and small value for $\hat{\ell}$ refer to realizations, where it is nearly impossible to infer it due to the similarity of T_{well} and T_{G}, as for the realization of Fig. 3b. The large range of estimated correlation lengths also point towards the fact that $\hat{\ell}$ of a single drawdown needs to be interpreted as a highly local value, which is determined by the transmissivity distribution in the

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vicinity of the well rather than the distribution of the entire field. However, the median of the normalized estimated correlation lengths is close to one, pointing to the fact that representative values can be inferred by taking the mean from multiple pumping tests.

Representative values of T_G, σ^2 and ℓ for individual transmissivity fields can be inferred by performing pumping tests at multiple location of the field. Thus, the sampled area increases and the effect of local heterogeneity at the well is reduced. Parameters should be estimated for every test separately and statistically analyzed. Mean values and the range of estimation results can than be interpreted as representative for the underlying transmissivity field.

4 Conclusions

The analytical effective well flow solution $h_{\text{efw}}(r)$ is presented, which can be interpreted as extension of Thiem's equation to heterogeneous media. $h_{\text{efw}}(r)$ depends on the statistical parameters of log-normal distributed transmissivity: geometric mean T_G, variance σ^2 and correlation length ℓ. $h_{\text{efw}}(r)$ was derived based on the Radial Coarse Graining transmissivity $T_{\text{RCG}}(r)$ introduced by Schneider and Attinger (2008), which interpolates between the near well and far field representative transmissivities for well flow. Simulation of pumping tests were performed in log-normally distributed transmissivity fields and compared with $h_{\text{efw}}(r)$. Based on the results, the following conclusions can be drawn:

1. $h_{\text{efw}}(r)$ describes the mean drawdown of a pumping test in two dimensional heterogeneous isotropic media effectively. It is not limited to small variance, but is tested to reproduce ensemble means for highly heterogeneous media with variances up to $\sigma^2 = 4$.

2. The analytical character of $h_{\text{efw}}(r)$ allows to perform inverse estimation of the statistical parameters of the transmissivity fields from measured drawdowns. Geometric mean T_G, variances σ^2 and correlation length ℓ can be estimated for

a wide range of parameters with a high accuracy and certainty from ensemble mean drawdowns.

3. Parameter estimates from individual drawdowns reflect local heterogeneity at the well rather than the global statistical parameters of the transmissivity field.
4. Representative values of geometric mean, variance and correlation length for an individual field of transmissivity can be determined by performing pumping test at multiple locations of that field, estimating the parameters for every test separately and than performing a statistical analysis of the results.

$h_{\mathrm{efw}}(r)$ is a promising tool to interpret steady state pumping tests in order to infer the statistical parameters of the underlying transmissivity field without time- and cost-intensive laboratory investigations. Future steps will include the expansion of the method to interpret transient pumping test data.

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Appendix

The effective well flow head $h_{\mathrm{efw}}(r)$ as solution of the well flow equation (2) is derived by solving the integral (Eq. 3) with the analytical expression of $T_{\mathrm{RCG}}(r)$ from Eq. (4).

$$h(r_2) - h(r_1) = \frac{C_1}{T_G} \int_{r_1}^{r_2} \frac{1}{r} \exp\left(\frac{\sigma^2}{2} \frac{1}{(1+\zeta^2 r^2/\ell^2)} \right) \mathrm{d}r \,. \tag{A1}$$

The integral is evaluated analytically by making use of the exponential integral function

$$\Gamma(x) - \Gamma(X) = \int_X^x \frac{\exp(z)}{z} \mathrm{d}z = \ln \frac{x}{X} + \sum_{k=1}^{\infty} \frac{x^k - X^k}{k!k} \,. \tag{A2}$$

The argument in the exponent in Eq. (A1) is substituted by $z(r) = \frac{\sigma^2}{2}\frac{1}{(1+\zeta^2 r^2/\ell^2)}$ with integrator $\mathrm{d}r = -\frac{\ell\sigma^2}{4\zeta z^2}\left(\frac{\sigma^2}{2z} - 1\right)^{-\frac{1}{2}}\mathrm{d}z$, furthermore partial fraction decomposition is used, resulting in

$$
\begin{aligned}
h(r_2) - h(r_1) &= \frac{C_1}{T_\mathrm{G}}\frac{\sigma^2}{4}\int\limits_{z(r_1)}^{z(r_2)}\frac{\exp(z)}{z(z-\frac{\sigma^2}{2})}\mathrm{d}z \\
&= \frac{C_1}{2T_\mathrm{G}}\int\limits_{z(r_1)-\frac{\sigma^2}{2}}^{z(r_2)-\frac{\sigma^2}{2}}\frac{\exp\left(z+\frac{\sigma^2}{2}\right)}{z}\mathrm{d}z - \frac{C_1}{2T_\mathrm{G}}\int\limits_{z(r_1)}^{z(r_2)}\frac{\exp(z)}{z}\mathrm{d}z \qquad \text{(A3)} \\
&= \frac{C_1}{2T_\mathrm{G}}\mathrm{e}^{\frac{\sigma^2}{2}}\left(\Gamma\left(z(r_2)-\frac{\sigma^2}{2}\right) - \Gamma\left(z(r_1)-\frac{\sigma^2}{2}\right)\right) \\
&\quad - \frac{C_1}{2T_\mathrm{G}}\left(\Gamma(z(r_2)) - \Gamma(z(r_1))\right).
\end{aligned}
$$

The final solution for the effective well flow head as given in Eq. (6) results by re-substituting the abbreviation $z(r) = \frac{\sigma^2}{2}\frac{1}{(1+\zeta^2 r^2/\ell^2)}$ with $r_2 = r$ and $r_1 = R$ and inserting $C_1 = -\frac{Q_\mathrm{w}}{2\pi}$ as derived from the constant flux boundary condition (Sect. 2.1).

An approximate formulation of Eq. (A3) can be derived by using the definition of the exponential integral function as infinite sum, given in Eq. (A2) in combination with the relationship $z(r) - \frac{\sigma^2}{2} = \frac{\sigma^2}{2}\frac{-\zeta^2 r^2/\ell^2}{1-\zeta^2 r^2/\ell^2} = z(r)\left(-\frac{\zeta^2 r^2}{\ell^2}\right)$:

$$
h(r_2) - h(r_1) = \frac{C_1}{2T_\mathrm{G}}\mathrm{e}^{\frac{\sigma^2}{2}}\left(\ln\frac{z(r_2)-\frac{\sigma^2}{2}}{z(r_1)-\frac{\sigma^2}{2}} + \sum_{k=1}^{\infty}\frac{\left(z(r_2)-\frac{\sigma^2}{2}\right)^k - \left(z(r_1)-\frac{\sigma^2}{2}\right)^k}{k!k}\right)
$$

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$$
\begin{aligned}
&- \frac{C_1}{2T_G}\left(\ln\frac{z(r_2)}{z(r_1)} + \sum_{k=1}^{\infty}\frac{z(r_2)^k - z(r_1)^k}{k!k}\right) \\
&= \frac{C_1}{T_G}e^{\frac{\sigma^2}{2}}\ln\frac{r_2}{r_1} + \frac{C_1}{2T_G}\left(e^{\frac{\sigma^2}{2}} - 1\right)\ln\frac{z(r_2)}{z(r_1)} \\
&+ \frac{C_1}{2T_G}\sum_{k=1}^{\infty}\frac{z(r_2)^k\left(e^{\frac{\sigma^2}{2}}(-\zeta^2 r_2^2/\ell^2)^k - 1\right) - z(r_1)^k\left(e^{\frac{\sigma^2}{2}}(-\zeta^2 r_1^2/\ell^2)^k - 1\right)}{k!k} \\
&\approx \frac{C_1}{T_G}e^{\frac{\sigma^2}{2}}\ln\frac{r_2}{r_1} + \frac{C_1}{2T_G}\left(e^{\frac{\sigma^2}{2}} - 1\right)\ln\frac{z(r_2)}{z(r_1)} + \frac{C_1}{2T_G}\left(e^{\frac{\sigma^2}{2}} - 1\right)(z(r_2) - z(r_1)).
\end{aligned}
\tag{A4}
$$

The final approximate solution as given in Eq. (7) results by re-substituting $z(r)$ with $r_2 = r$ and $r_1 = R$ and inserting $C_1 = -\frac{Q_w}{2\pi}$.

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Table 1. Ensemble input parameters T_G, σ^2 and ℓ and best fit inverse estimation results $\hat{T}_G$, $\hat{\sigma}^2$ and $\hat{\ell}$ with 95 % confidence intervals (in brackets) for ensemble mean $\overline{\langle h(r) \rangle}$ for all generated ensembles.

	T_G	$\hat{T}_G$ $[10^{-4}\,\mathrm{m^2\,s^{-1}}]$	σ^2	$\hat{\sigma}^2$ [–]	ℓ	$\hat{\ell}$ [m]
A	1.0	1.03 (±0.0011)	1.0	1.04 (±0.0022)	10	9.80 (±0.086)
B	1.0	1.08 (±0.0013)	1.0	1.19 (±0.0022)	20	21.6 (±0.127)
C	1.0	1.08 (±0.0021)	2.25	2.49 (±0.0038)	10	10.1 (±0.050)
D	1.0	1.19 (±0.0024)	2.25	2.67 (±0.0039)	20	22.2 (±0.077)
E	1.0	1.16 (±0.0046)	4.0	4.34 (±0.0078)	10	11.0 (±0.042)
F	1.0	1.31 (±0.0088)	4.0	4.27 (±0.0131)	20	22.2 (±0.120)
G	1.5	1.55 (±0.0012)	1.0	1.03 (±0.0016)	10	10.1 (±0.066)
H	1.5	1.62 (±0.0024)	1.0	1.19 (±0.0028)	20	21.2 (±0.158)

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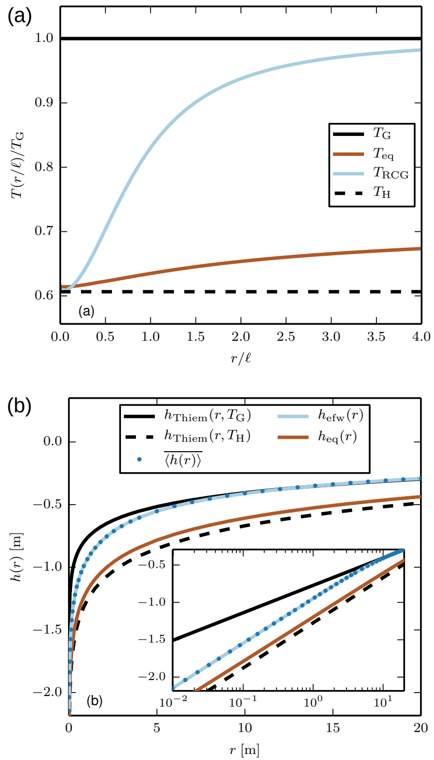

Figure 1. Comparison of equivalent and Radial Coarse Graining approach: **(a)** radially depending transmissivities interpolating between harmonic mean T_H and geometric mean T_G: $T_{RCG}(r)$ from Eq. (4) and $T_{eq}(r)$ calculated based on Thiem's formula Eq. (1) with $h(r) = \overline{\langle h(r) \rangle}$, which is the ensemble mean for Ensemble A (Table 1), **(b)** hydraulic head drawdowns after pumping with: $h_{efw}(r)$ from Eq. (6) as solution of the well flow equation using $T_{RCG}(r)$, $h_{eq}(r)$ as solution of the well flow equation using $T_{eq}(r)$, Thiem's solution with homogeneous substitute values T_G and T_H as well as mean ensemble drawdown $\overline{\langle h(r) \rangle}$.

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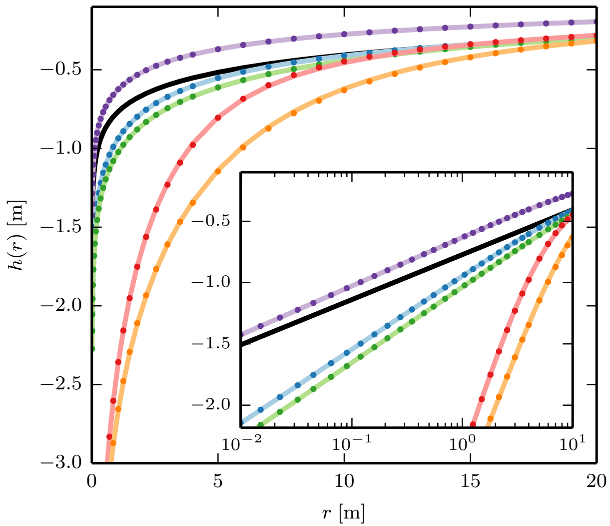

Figure 2. Simulated ensemble means $\overline{\langle h(r)\rangle}$ (dots) and $h_{\mathrm{efw}}(r)$ with best fit estimates (lines) for multiple Ensembles: A (blue), B (green), E (red), F (orange), G (purple). Parameter values are listed in Table 1. Black line shows $h_{\mathrm{Thiem}}(r)$ with $T_G = 10^{-4}\,\mathrm{m^2\,s^{-1}}$.

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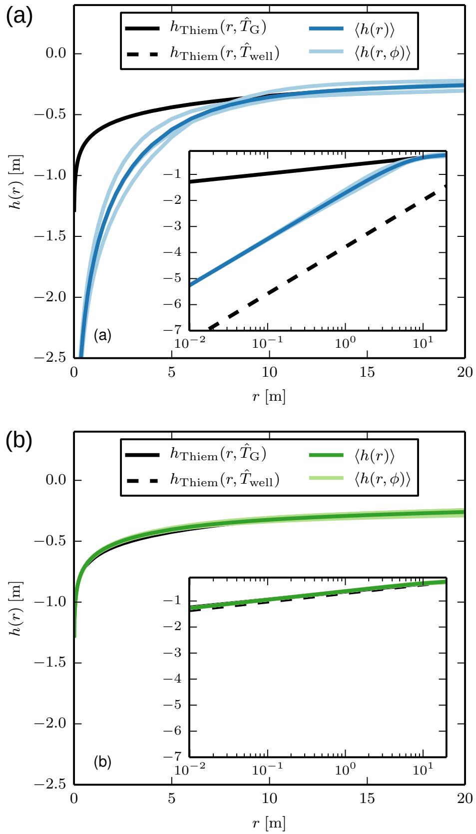

Figure 3. Drawdowns simulated for two individual transmissivity field realizations of Ensemble A ($T_G = 10^{-4}\,\mathrm{m^2\,s^{-1}}$, $\sigma^2 = 1$, $\ell = 10\,\mathrm{m}$): **(a)** realization with $T_{\mathrm{well}} = 0.204 \times 10^{-4}\,\mathrm{m^2\,s^{-1}}$ and **(b)** realization with $T_{\mathrm{well}} = 1.11 \times 10^{-4}\,\mathrm{m^2\,s^{-1}}$. $\langle h(r)\rangle$ (dark color) is the radial mean, $\langle h(r,\phi)\rangle$ (light color) denotes the drawdowns along the four axes ($\phi = 0^\circ, 90^\circ, 180^\circ, 270^\circ$), as well as in black Thiem's solution for homogeneous substitute values.

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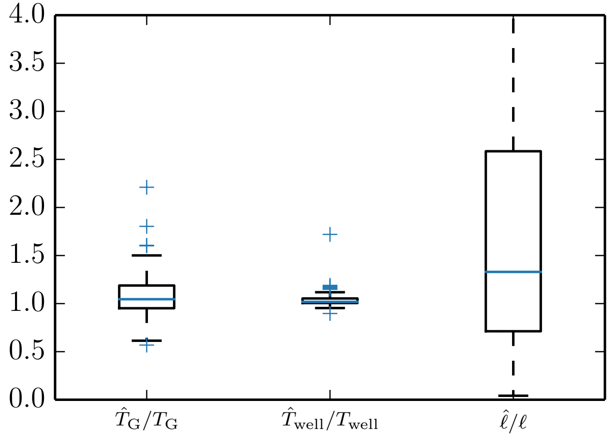

Figure 4. Box plot giving the statistics for $N = 100$ realizations of Ensemble A in normalized form: best fit parameters for every realization $\hat{T}_G$, $\hat{T}_{well}$ and $\hat{\ell}$ are divided by input parameters $T_G = 10^{-4}\,m^2\,s^{-1}$, T_{well} (local transmissivity at the well of individual realization) and $\ell = 10\,m$.

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