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## Response to Editors Comment

*The manuscript has been revised by two reviews. One reviewer is satisfied with the paper, while and the second reviewer raises some issues that require some discussion.*

*The authors are encouraged to explain clearly the approximations associated with the upscaling technique they employ. I think the authors do so at the very beginning of Section 2.2. Here, they mention the idea of "spatial filtering of the flow equation", which is a key point raised by one Reviewer. However, I encourage them to be very clear about possible limitations and approximations introduced by the procedure, so that the reader does not have to become an expert of coarse graining to grasp the extent of the approximations involved. This is only in the spirit of having a self-contained contribution. Indeed, at about line 160 the authors state that they somehow localize the flow equation after coarse graining. However, the message is not crisp and making it clear would benefit the reader. Perhaps the message is diluted by the fact that the authors refer to "coarse-grained transmissivity" and "effective description of the transmissivity" instead of "effective description of the flow field expressed in terms of effective/representative parameters". All these comments are made in the interest of clarity and to maximize impact of the work.*

A few lines were added to specified the limitations and approximations of the procedure at the beginning of section 2.2, especially for readers who are not familiar with the concept of Coarse Graining. The procedure of localization the flow equation after Coarse Graining was explained in the later part of section 2.2. The passage was reworked in order to clarify the message. The sentence using "effective description of the transmissivity" was rephrased.

*An issue that I can see is that in Section 2.4 the authors state that they solve the integral in (3) by plugging into it the transmissivity (4). One can note that (3) is the solution of (2) which, in turn, is not an average flow equation. As such, it looks like the authors are not averaging a flow equation, but considering its solution and then plugging a coarse-grained parameter into it. This was also raised by the most critical Reviewer and the rationale underpinning this procedure must be clearly and convincingly explained.*

The editor clearly summarized our approach: The effective well flow solution is not a solution derived from an effective equation, but is the solution of the deterministic groundwater flow equation under well flow conditions (Eq. 2) in combination with the Radial Coarse Graining transmissivity (4). We tried to strengthen that point in the manuscript by explicitly stating the procedure used (introduction, section 2.4, conclusion).

*As suggested by the Reviewer who worked on the revised manuscript, a few words should be added to address the way randomness of heads, as quantified for example by the variance of heads, can affect estimation errors. I consider this to be an element of importance to be included in a revised version of the manuscript. Additionally, it should be noted that the approach illustrated by Neuman et al. [2004] makes not only use of the (ensemble) average head/drawdown, but also of its variance, thus combining uncertainty within a unique methodology.*

The aspect of "randomness of heads" is discussed in the response to the comment of S.P. Neuman. Associated modifications in the manuscript are stated there as well. The aspect that the approach of Neuman et al. [2004] makes not only use of the ensemble average head, but also of its variance

is included in the manuscript (introduction).

*In summary, I do think the manuscript could still benefit from a set of revisions, which can be defined as minor, along the lines indicated.*

*A list of a few minor comments are following:*

- *Abstract: the authors write "A radially depending description of transmissivity is required, including the parameters of log-transmissivity: mean, variance and correlation length". The way it is written seems that the correlation length, mean and variance, depend on radial coordinate. I am not sure this is the case, so I suggest rephrasing.*

The sentence was rephrased to "An effective description of transmissivity is required, being a function of the radial distance to the well and including the parameters of log-transmissivity: mean, variance and correlation length."

- *Abstract: the sentence: "This is shown making use of virtual pumping tests, for both cases the ensemble mean drawdown and pumping tests at individual transmissivity fields." I am not sure what it means precisely. The authors are required to clarify.*

The passage was rephrased to clarify the message: "For comparison with the effective well flow solution, virtual pumping tests are performed and analyzed for both cases, the ensemble mean drawdown and pumping tests at individual transmissivity fields. Interpretation of ensemble mean drawdowns showed proof of the upscaling method."

- *Abstract: the authors write "Multiple pumping tests at an individual transmissivity fields, combined in a sampling strategy ..." this part is also not clear (How do the tests are combined in a sampling strategy? What kind of strategy are the authors referring to?).*

The sentence was rephrased: "Multiple pumping tests conducted at different locations within an individual transmissivity field are analyzed making use of the effective well flow solution to show that all statistical parameters of aquifer heterogeneity can be inferred under field conditions." The formulation "sampling strategy" was omitted which is misleading in this context.

- *Introduction. The authors state "Geostatistical distributions are generally used ...". How do they define a "geostatistical distribution"? I am not pointing out additional awkward terminology which the authors should revise.*

The sentence was reformulated to "Modelling transmissivity as spatial random function is generally applied to capture the effects of aquifer heterogeneity."

- *Introduction. The authors state: "In the stochastic framework, the solution of the radial flow equation with a log-normal distributed transmissivity is also a random spatial function". What is the radial flow equation the authors are referring to? If they are considering a single realization, then the flow is not radial, but convergent. If the authors are referring to the equation satisfied by the (ensemble) average head, then it is fine. However, this needs to be clearly stated.*

We agree with the editor that the formulation is not clearly stated. Since the statement is

meant in a general way, we substituted the word "radial" by "groundwater".

- *Line 55. I assume the authors are not referring to "a single constant value of transmissivity", but rather to a single constant effective/apparent value of transmissivity.*

The word "effective" was added.

- *Line 75. What is exactly a "homogeneous substitute value"?*

The passage was rephrased to "on effective transmissivity values for describing well flow". The term was replaced by more precise expressions throughout the manuscript.

- *In the Introduction the authors refer to the solution derived by Neuman et al. [2004]. Note that this solution was also applied in an actual field setting by Riva et al. [2009]. I am not stating this to ask the authors to reference this work, but for completeness, since it appears from their phrasing that the solution of Neuman et al. [2004] has only been employed on the basis of synthetic numerical analyses. With reference to the solution of Neuman et al. [2004], note that it is not purely based on Monte Carlo numerical simulations. Otherwise, it draws its root on the analytical solution of the (ensemble) mean flow equation as well as the equation satisfied by the hydraulic head variance/covariance (with its approximations, of course). As such, the authors are encouraged to revise this part. In the context of purely radial (mean) flow, the analytical solutions upon which the type curves of Neuman et al. [2004] are based is that of Riva et al. [2001], which is then combined with the results of numerical Monte Carlo simulations for ease of use. This means that also the third point raised at line 105 is somehow misleading and/or incomplete. All these comments are not made in the spirit of asking the authors to reference the above mentioned manuscript. Otherwise, they are only made for the sake of clarity and complete depiction of the problem.*

We are thankful to the editor for the additional information on the method of Neuman et al. [2004] and its application to an actual field site. The paragraphs in the manuscript were revised accordingly. The application by Riva et al. [2009] was mentioned to avoid the impression of pure synthetic numerical analyses. The sentence stating that the solution of Neuman et al. [2004] based on Monte Carlo numerical simulations was corrected. Point (iii) was shortened to bring to the point the message of different forms of the solution presentation. It is not aimed to evoke the impression that the solution of Neuman et al. [2004] is not derived analytically.

- *Additionally (and as I stated at the beginning of my assessment), it should be noted that the approach illustrated by Neuman et al. [2004] makes not only use of the (ensemble) average head/drawdown, but also of its variance, thus combining uncertainty within a unique methodology. I think this is also an important difference to be highlighted.*

The aspect mentioned by the editor is now noted in the manuscript.

- *The authors concentrate on and refer to an "effective well flow solution". The key point they should explain is whether this solution derives from an effective equation of the kind which can be obtained by applying coarse graining to the flow equation governing the system behavior. This aspect is not unambiguously stated by the authors and I consider eliminating doubts about this part to be essential for final acceptance of the manuscript.*

The "effective well flow solution" is not a solution derived from an effective equation, but is the solution of the deterministic groundwater flow equation under well flow conditions in combination with the Radial Coarse Graining transmissivity. This effective transmissivity description mimics the filtering process taking place during pumping and is obtained by the upscaling procedure Coarse Graining.

Since this aspect seems to be not stated clearly, we tried to eliminate doubts by stating it explicitly in the manuscript (introduction, section 2.4, and conclusions).

- *The authors consider transmissivity to be a function solely of the radial distance (and not of the angular coordinate) from the well in Section 2.1. I am not sure they comment on such an assumption and they are encouraged to do so. My reading of their (2) is that it is a stochastic differential equation (i.e., it should hold for a single realization). As such, I see no particular reason to drop the dependence of transmissivity from the angular coordinate. On the other hand, I see no critical problem in doing so for, say, an ensemble averaged flow equation. This is most likely done in their Section 2.2 but it should be made very clear.*

We fully agree with the editor that purely radial flow is not to be expected in a heterogeneous transmissivity field. In this line it was tried to clearly underline that the equations given in section 2.1 are solely valid for purely radial-depending transmissivity. Since the Radial Coarse Graining Transmissivity was derived in the ensemble picture by *Attinger* [2003] and *Schneider and Attinger* [2008] it does not take angular fluctuations into account. A comment on that was added to the end of section 2.1.

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## Response to Comment of S.P. Neuman

*The modified version of the manuscript is an improvement over the original in both clarity of presentation and demonstration of results. My original concerns, however, have not been alleviated; at the very least, they should be honestly and openly addressed in the manuscript. Though the revised manuscript now acknowledges that the coarse grained transmissivity is random with a reduced variance and enlarged correlation scale (as I had suggested in my original review), the authors still treat it as if it and associated heads were deterministic. At the very least, the authors should address the question how does this randomness impact estimation errors, which appear to be sizable in all their examples. All in all, the manuscript is theoretically on shaky grounds and contains too many ad hoc elements for comfort.*

The concept of Coarse Graining is outlined in several steps in the manuscript. As stated by the reviewer, the Coarse Grained transmissivity derived by Attinger [2003] is random with a reduced variance and enlarged correlation scale. During the procedure of adapting the results to well flow conditions, the Radial Coarse Graining transmissivity was derived (step 3 and 4), which is deterministic. Thus, the associated heads (effective well flow solution) is deterministic. This is now explicitly stated in the manuscript (section 2.3 and 2.4).

Randomness of hydraulic heads can be observed in every simulation of a pumping test in a log-normally distributed transmissivity field as a result of the heterogeneous transmissivity distribution. We agree with the reviewer that this can impact estimation errors. Different strategies were developed to cope with the effect.

The randomness of heads is not a problem in the ensemble analysis (section 3.2). Averaged hydraulic heads are analyzed for a sufficiently large ensemble number to ensure that fluctuations are levelled out except for negligible differences. Estimating the ensemble parameters making use of the deterministic effective well flow solution is not affected by the randomness of heads.

When analyzing pumping tests in individual transmissivity fields the head fluctuations have significant impact on the estimation results using the effective well flow solution. To cope with this issue, an adapted effective well flow solution was developed (section 2.4) which takes the effect of local transmissivity at the well into account. Thereby, the drawdown behaviour at the well can be captured very well, as observable in Figure 3(a). Figure 3(a) also shows that the major impact of head fluctuations is in the transition zone. However, the estimate of the correlation length is determined by the drawdown behaviour in this region giving that  $\ell$  is impacted strongest by the randomness of heads. To minimize this effect, the sampling strategy in section 4 was developed. For individual aquifers, pumping tests are performed at multiple locations and results are interpreted jointly.

The aspect that randomness of hydraulic heads impact on the estimation results of the correlation length is now explicitly stated in the manuscript (sections 2.4, 2.5 and 3.2). A note is also given in section 4 stating the minimization of the impact of head randomness to be one motivation for the sampling strategy.

## Response to Comment of P. Trinchero

*In my first review of this paper I raised some concern about the scientific novelty of the mathematical framework presented in this work, which is just a particularization of a more general analytical solution provided by the same authors in a previous publication [Zech et al., 2012]. Yet, I saw the ground for an interesting technical publication, as the framework provided here could potentially be applied in real applications. So, I suggested the authors to focus on the application to single realizations and to draw importance to the part of the paper where the solution is tested against the ensemble mean.*

*The authors have carefully taken into account my comments and expanded the analysis of single tests to high heterogeneity realizations. Moreover, as I suggested, the authors have extended the analysis of single tests by evaluating the results of the entire ensemble. The authors have also provided convincing explanation on the utility of the section focused on the ensemble pumping test interpretation.*

*I think that this revised version of the manuscript has substantially been improved and that my major comments have been thoroughly addresses. Thus, I think that the paper is worth of publication.*

We appreciate the reviewers evaluation and remain with grateful thanks for the reviewer's comments, which significantly helped to improve the manuscript.

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# Analytical Drawdown Solution for Steady State Pumping Tests in Two-dimensional Isotropic Heterogeneous Aquifers

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## Abstract.

A new method is presented which allows to interpret steady state pumping test in heterogeneous isotropic transmissivity fields. In contrast to mean uniform flow, ~~the~~ pumping test drawdowns in heterogeneous media cannot be described by a single effective or equivalent value of hydraulic transmissivity. ~~A radially depending~~ ~~An effective~~ description of transmissivity is required, ~~being a function of the radial distance to the well and~~ including the parameters of log-transmissivity: mean, variance and correlation length. Such a model is provided by the upscaling procedure Radial Coarse Graining, which describes the transition of near well to far field transmissivity effectively. Based on this approach, an analytical solution for a steady state pumping test drawdown is ~~derived~~ ~~deduced~~. The so-called effective well flow solution is derived for two cases: the ensemble mean of pumping tests and the drawdown ~~at~~ ~~within~~ an individual heterogeneous transmissivity field. The analytical form of the solution allows to inversely estimate the parameters of aquifer heterogeneity. ~~This is shown making use of~~ ~~For comparison with the effective well flow solution,~~ virtual pumping tests ~~are performed and analyzed~~ for both cases, the ensemble mean drawdown and pumping tests at individual transmissivity fields. ~~Interpretation of ensemble mean drawdowns showed proof of the upscaling method.~~ The effective well flow solution reproduces the drawdown for two-dimensional pumping tests in heterogeneous media in contrast to Thiem's solution for homogeneous media. Multiple pumping tests ~~conducted at different locations within~~ an individual transmissivity fields, ~~combined in a sampling strategy,~~ are analyzed making use of the effective well flow solution to show that all statistical parameters of aquifer heterogeneity can be inferred under field conditions. Thus, the presented method is a promising tool to estimate parameters of aquifer heterogeneity, in particular variance and horizontal correlation length of log-transmissivity fields from steady state pumping test measurements.

## 1 Introduction

Pumping tests are a widely used tool to identify horizontal hydraulic conductivity, which is the parameter determining the groundwater flow velocity. Analytical solutions of the radial flow equation are used in practice to analyze measured drawdowns. In general, these solutions assume a constant homogeneous hydraulic conductivity like Thiem's solution for steady state (Thiem, 1906):

$$h_{\text{Thiem}}(r) = -\frac{Q_w}{2\pi DK_h} \ln \frac{r}{R} + h(R). \quad (1)$$

Thiem's solution (1) gives the hydraulic head  $h_{\text{Thiem}}(r)$  depending on the radial distance  $r$  from the well for homogeneous horizontal hydraulic conductivity  $K_h$ . It is valid in a confined aquifer of thickness  $D$  with fully penetrating well and a constant discharge  $Q_w$ ,  $h(R)$  is a known reference head at an arbitrary distance  $R$  from the well.

In large scale pumping tests the vertical extension of the aquifer is negligible compared to horizontal aquifer extend. Thus, flow is assumed to be horizontal and modelled as two-dimensional. Hydraulic conductivity is then replaced by transmissivity, which is defined as the product of conductivity and aquifer thickness  $T = K_h D$ . In the following, transmissivity will be used instead of horizontal hydraulic conductivity, since the focus of the work will be on two-dimensional well flow.

Most natural aquifers exhibit geological heterogeneity in the sedimentary composition, which evolved from the complex geomorphological processes through which they were formed. In particular, transmissivity shows a strong spatial variability. Values measured in the field vary over orders of magnitude (Gelhar, 1993). ~~Geostatistical distributions are generally used~~ Modelling transmissivity as spatial random function is generally applied to capture the effects of aquifer heterogeneities. Transmissivity  $T(\mathbf{x})$  is modelled as log-normally distributed spatial random function:  $\log T(\mathbf{x}) = Y(\mathbf{x})$  is normally distributed with a Gaussian probability density function  $\text{pdf}_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$  in uni-variate form with  $\mu$  and  $\sigma^2$  being the mean and the variance of  $Y$ , respectively. The correlation structure of transmissivity in space is captured by a covariance model  $\text{Cov}[T(\mathbf{x} + \mathbf{s}), T(\mathbf{x})] = \exp(2\mu + \sigma^2 + \text{CV}_Y(\mathbf{s}))$ .

In the stochastic framework, the solution of the ~~radial-groundwater~~ flow equation with a log-normally distributed transmissivity is also a random spatial function. Since the solution of the stochastic differential equation is out of scope, the focus of investigation was on ~~homogeneous substitute-effective transmissivity~~ values for describing well flow ~~effectively~~. As a first approach, Thiem's solution (1) was applied to pumping tests in heterogeneous media. However, this requires a representative transmissivity value  $T$  for the whole range of the depression cone (Matheron, 1967), which does not exist. Effective or equivalent descriptions of transmissivity in pumping tests were investigated, e.g. by Desbarats (1992); Sánchez-Vila et al. (1999); Neuman et al. (2007); Dagan and Lessoff (2007); Schneider and Attinger (2008) and many more. For a detailed review see Sánchez-Vila et al. (2006).

In contrast to mean uniform flow, the pumping test drawdowns in heterogeneous media cannot be described by a single constant **effective** value of transmissivity (Matheron, 1967). Different transmissivities characterize the behavior near and far from the well: the representative value close to the well is the harmonic mean of the log-normal transmissivity  $T_H = \exp(\mu - \frac{1}{2}\sigma^2) = T_G \exp(-\frac{1}{2}\sigma^2)$ . With increasing distance from the well, the drawdown behavior is characterized by the effective transmissivity for uniform flow, which is the geometric mean  $T_G = \exp(\mu)$  for flow in two dimensional isotropic porous media.

It seems obvious, that a representative description of transmissivity for well flow needs to be a radially depending function, which interpolates between the harmonic and the geometric mean. The equivalent transmissivity  $T_{eq}$  is a well established approach of a radially depending description, visualized in Fig. 1a.  $T_{eq}(r) = -\frac{Q_w}{2\pi(h(r)-h(r_w))} \ln \frac{r}{r_w}$  was derived from Thiem's solution (1) (Matheron, 1967; Indelman et al., 1996; Dagan and Lessoff, 2007). In this sense, the equivalent transmissivity  $T_{eq}$  is defined as the value for a homogeneous medium, which reproduces locally the same total outflow as observed in the heterogeneous domain of radius  $r$ .  $T_{eq}$  is strongly impacted by the reference point  $r_w$  and the corresponding head  $h(r_w)$ , which is generally chosen to be the drawdown at the well  $h(r_w)$ . Therefore, the equivalent conductivity stays close to the harmonic mean  $T_H$ , which is representative for the drawdown behavior at the well (Fig. 1a). It takes more than 20 correlation length for  $T_{eq}$  to reach the far field representative value of  $T_G$ .

It is important to mention, that  $T_{eq}$  is not constructed to reproduce the drawdown which was used for calculating  $T_{eq}$ . Strictly speaking, replacing the heterogeneous transmissivity field with the equivalent transmissivity in a single forward model does not give the drawdown, with which  $T_{eq}$  was constructed as visualized in Fig. 1b. Instead  $h_{eq}(r)$  stays close to Thiem's solution with  $T_H$  as **homogeneous substitute constant transmissivity** value.

Schneider and Attinger (2008) introduced a novel approach to describe well flow effectively. They derived a radial adapted transmissivity  $T_{RCG}(r)$  by applying the upscaling technique Coarse Graining to well flow.  $T_{RCG}(r)$  does not only depend on the radial distance  $r$  but also on the statistical parameters of aquifer heterogeneity  $T_G$ ,  $\sigma^2$  and  $\ell$ .  $T_{RCG}(r)$  captures the transition from near well to far field representative transmissivities, based on the radial distance to the well and the parameters of aquifer heterogeneity, as visualized in Fig. 1a.

In this study, an analytical solution for the hydraulic head  $h_{efw}(r)$  is presented, which is based on the Radial Coarse Graining transmissivity  $T_{RCG}(r)$  as an extension to the work of Schneider and Attinger (2008). Similar work has been done by (Zech et al., 2012) for pumping test in three dimensional porous media. The effective well flow solution  $h_{efw}(r)$  describes the mean depression cone of a pumping test in two dimensional heterogeneous media effectively. It can be interpreted as an extension of Thiem's formula (1) to log-normal distributed heterogeneous media. It accounts for the statistical parameters  $T_G$ ,  $\sigma^2$  and  $\ell$  and thus allows to inversely estimate them from measured drawdown data. In contrast to existing head solutions for well flow,  $h_{efw}(r)$  is not limited to low

95 variances, but is applicable to highly heterogeneous media with variances  $\sigma^2 \gg 1$ . More explicitly, the "effective well flow solution" is not a solution derived from an effective equation, but is the solution of the deterministic groundwater flow equation under well flow conditions in combination with an effective transmissivity. As effective transmissivity description the Radial Coarse Graining transmissivity is used because it mimics the filtering process taking place during pumping.

100 In a similar ~~approach~~ line Neuman et al. (2004) presented a graphical approach to estimate the statistical parameters of random transmissivity on the basis of steady state head data. The authors constructed a mathematical description for the apparent transmissivity  $T_a(r)$  as function of the radial distance to the well  $r$  from theoretical findings of near and far field representative transmissivity and a cubic polynomial interpolation in between. The approach makes use of the ensemble average drawdown as well as of its variance, combining uncertainty within a unique methodology. From  $T_a(r)$  the authors constructed type curves for the hydraulic head, depending on the variance  $\sigma^2$  and the correlation length  $\ell$ . Neuman et al. (2004) further gave a multi-point strategy to analyze virtually measured drawdown data by type curve matching including parameter estimation. The solution was also applied in an actual field setting by Riva et al. (2009).

110 The Radial Coarse Graining approach is similar to that of Neuman et al. (2004) in the idea of deriving a solution for the head drawdown for well flow depending on the statistics of the random transmissivity using an effective radial depending transmissivity. Major differences are: (i) the Radial Coarse Graining Transmissivity  $T_{RCG}(r)$  is ~~not based on results of Monte Carlo simulations; but is~~ derived from upscaling with physically motivated approximations; whereas the solution of Neuman et al. (2004) based on the analytical solution of Riva et al. (2001) in combination with numerical Monte Carlo simulations; (ii) the functional form of  $T_{RCG}(r)$  is different from the expression for  $T_a$  of Neuman et al. (2004); (iii) the effective well flow solution  $h_{efw}(r)$  is ~~derived analytically by solving the head equation; providing~~ provided as a closed form mathematical expression instead of type curves; (iv) inverse parameter estimation can be done by minimizing the difference between  
120 the measured drawdown data and  $h_{efw}(r)$  instead of type curves matching. The effective well flow method will be tested in a similar multi-point sampling strategy to analyze measured drawdown data of individual heterogeneous transmissivity fields as done by Neuman et al. (2004) as well as others (Coptý and Findikakis, 2004; Firmani et al., 2006).

The work is organized the following: Sect. 2 is dedicated to the method of Radial Coarse Graining and the derivation of the effective well flow head solution. The concept of Radial Coarse Graining is explained in detail. Furthermore it is distinguished between the effective well flow solution for an ensemble mean and single realizations of heterogeneous transmissivity fields. Section 3 contains the application of the effective well flow solution to simulated pumping tests. It is shown, that  $h_{efw}(r)$  reproduces the drawdown in heterogeneous media and can be used to inversely estimate the statistical parameters of aquifer heterogeneity for both, ensemble mean and single realizations. In Sect. 4, a sampling strategy is presented to infer the parameters of aquifer heterogeneity of an individual trans-

missivity fields from multiple pumping tests at multiple locations making use of  $h_{\text{efw}}(r)$ . Concluding remarks are given in Sect. 5.

## 2 Radial Coarse Graining Transmissivity and Effective Well Flow Head

### 135 2.1 Steady state well flow with radially depending transmissivity

The drawdown of a steady state pumping test with a radially depending transmissivity  $T(r)$  is given as the solution of the differential equation:

$$0 = T(r) \left( \frac{1}{r} \frac{dh}{dr} + \frac{d^2h}{dr^2} \right) + \frac{dT(r)}{dr} \frac{dh}{dr} = \left( \frac{1}{r} + \frac{d \ln T(r)}{dr} \right) \frac{dh}{dr} + \frac{d^2h}{dr^2}. \quad (2)$$

140 The equation can be solved in  $\frac{dh}{dr}$  by separation of variables, resulting in  $\frac{dh}{dr} = C_1 \frac{1}{rT(r)}$ . The hydraulic head  $h(r)$  is then given as the solution of the integral

$$h(r_2) - h(r_1) = C_1 \int_{r_1}^{r_2} \frac{1}{rT(r)} dr. \quad (3)$$

The integration constant  $C_1$  is determined by the boundary condition. Supposing a constant flux boundary condition at the well, gives  $Q_w = -2\pi r_w T(r_w) \frac{dh}{dr}(r_w) = -2\pi r_w T(r_w) \frac{C_1}{r_w T(r_w)}$  and thus,  $C_1 = -\frac{Q_w}{2\pi}$ .

145 Equation (3) is the general solution of the radial flow equation (2) for radially depending transmissivity, independent of the functional form of  $T(r)$ .

When comparing Eq. (3) with the definition of the equivalent transmissivity, it becomes obvious that  $T_{\text{eq}}$  is not constructed to solve the equation. The combination of both formulas results in  $\int_{r_1}^{r_2} \frac{1}{rT(r)} dr = \frac{1}{T_{\text{eq}}(r)} \ln \frac{r_1}{r_2}$ , which is only fulfilled, when  $T(r)$  is constant in  $r$ .

150 In a heterogeneous transmissivity field, radially symmetric flow is not to be expected. However, angular fluctuations level out for ensemble averaged drawdowns. Thus, the assumption of a radial-depending effective transmissivity description  $T(r)$  - independent of the angular coordinate - is well established and reasonable in the context of ensemble analysis. In this line, the Radial Coarse Graining Transmissivity and thus the effective well flow solution are considered to be purely radial-depending referring to ensemble averages.

### 2.2 Concept of Radial Coarse Graining

A radial-depending transmissivity for log-normally distributed media with Gaussian correlation structure was derived by Schneider and Attinger (2008), denoted as  $T_{\text{RCG}}(r)$ . It is based on the upscaling approach Radial Coarse Graining which follows the basic idea of a spatial filtering of the flow equation appropriate to the non-uniform flow character of a pumping test. The method of Radial Coarse Graining is an upscaling method based on mathematically solid filtering of the flow equation and physically reasonable assumptions and approximations. At the current state the procedure is

limited to a multivariate Gaussian random field as a model for the heterogeneity of the transmissivity field.

165 The approach was further developed for three-dimensional well flow by Zech et al. (2012) introducing an effective well flow solution for the hydraulic head. Similarly, the concept of Radial Coarse Graining for two-dimensional well flow will be expanded in the following. The process can be best explained within five major steps:

1. Coarse Graining for uniform flow
- 170 2. Transfer of Coarse Graining to radial flow conditions
3. Overcome non-locality of head equation for non-uniform flow
4. Effective Radial Coarse Graining transmissivity
5. Derivation of effective well flow head

The first three steps will be discussed shortly in the following. Step 4 and 5 will be explained in  
175 detail in Sec.2.3 and Sec. 2.4.

The Coarse Graining approach for uniform flow (step 1) was introduced by Attinger (2003), including derivation, mathematical proof and numerical simulations. The author started at a spatially variable transmissivity field  $T(\mathbf{x})$  and derived a filtered version  $T_{\lambda}^{\text{CG}}(\mathbf{x})$ , where fluctuation smaller than a cut-off length  $\lambda$  are filtered out. The resulting upscaled Coarse Graining transmissivity field  
180  $T_{\lambda}^{\text{CG}}(\mathbf{x})$  represents a log-normal distributed field with a smaller variance  $\langle \sigma^2 \rangle_{\lambda}$ , but larger correlation length  $\langle \ell \rangle_{\lambda}$ . Attinger (2003) showed, that the statistical parameters relate to the parameter of the unfiltered field by  $\langle \sigma^2 \rangle_{\lambda} \equiv \sigma^2 \frac{\ell^2}{\ell^2 + \lambda^2/4}$  and  $\langle \ell \rangle_{\lambda} \equiv (\ell^2 + \lambda^2/4)^{1/2}$ .

The concept of Coarse Graining can similarly be applied to non-uniform flow (step 2). The critical point when transferring the results of Attinger (2003) to radial flow is the Fourier back-  
185 transformation of the filtered head equation after localization. For uniform flow, this can be done due to the reasonable assumption of constant head gradient. For non-uniform flow, this assumption is not valid and thus, localization is not possible straight ahead.

A heuristic approach is taken to overcome the limitation of non-locality for well flow (step 3). Conditions of a quasi-constant head gradient are constructed by adapting the size of the volume  
190 elements over which flow takes place.

~~The head gradient in well flow is proportional to the reciprocal of the distance to the well:  $\nabla h(r) = \frac{h(r) - h(r + \Delta r)}{\Delta r} \propto \frac{1}{r}$ . The gradient is constant for volumes of size proportional to  $r$ . The step can be understood as a change from an equidistant Cartesian coordinate system to a polar coordinate system with cell sizes increasing with distance to the center, where the pumping well is located.~~  
195 ~~Under this adaption, localization can be performed and so the following steps of the Coarse Graining procedure. The changed coordinate system impacts on the scaling procedure and the parameter  $\lambda$ . For uniform flow,  $\lambda$  is constant. Adapted to well flow, the scaling parameter needs to be proportional~~

to  $r$ , because the filter width increases with distance to the well, thus  $\lambda/2 = \zeta r$ . The change in the volume size refers to the filtering procedure and is then realized by an radial-depending scaling parameter  $\lambda(r)$ . The step can be understood as a change from an equidistant Cartesian coordinate system to a polar coordinate system. Technically speaking, the head gradient in well flow is constant for volumes of size proportional to  $r$ , because it is proportional to the reciprocal of the distance to the well:  $\nabla h(r) = \frac{h(r) - h(r + \Delta r)}{\Delta r} \propto \frac{1}{r}$ . Therefore, the scaling parameter is chosen to be proportional to the radial distance  $\lambda = 2\zeta r$ . Then, the filter width increases with distance to the well ensuring that head gradients can be assumed constant over volume elements of increasing in size with distance to the well. Under this adaption, localization can be performed and so the following steps of the Coarse Graining procedure.

The result is an upscaled log-normal distributed field  $T_r^{\text{RCG}}(\mathbf{x})$  with an arithmetic mean  $T_A^{\text{RCG}}(r)$  and a filtered fluctuation term. The step was presented by Schneider and Attinger (2008). It is not performed in a mathematically straight way, but problem adapted to well flow conditions.

### 2.3 Radial Coarse Graining Transmissivity

Spatial heterogeneity is still resolved in  $T_r^{\text{RCG}}(\mathbf{x})$ , although reduced to the amount relevant to the pumping test. A further step of averaging is necessary to derive an effective ~~description of the transmissivity for well flow condition~~ transmissivity which mimics the intrinsic filtering of the flow during pumping and thus reproduces the drawdown behaviour. Thereby, two different aspects are of interest: (i) an effective transmissivity for an ensemble and (ii) effective transmissivity for an individual field. This step includes the transfer from the stochastic picture of transmissivity to a deterministic description.

A result for an effective ensemble description is derived by averaging  $T_r^{\text{RCG}}(\mathbf{x})$  appropriate to well flow condition. The averaging rule is determined by the boundary condition at the well, which is the harmonic mean for two-dimensional well flow (Dagan, 1989). Thus, the effective mean transmissivity, noticed by  $T_{\text{RCG}}(r)$ , is calculated via the theoretical description of the harmonic mean for log-normal distributed fields making use of the variance of the coarsened transmissivity  $\langle \sigma^2 \rangle_r = \frac{\sigma^2}{1 + \zeta^2 r^2 / \ell^2}$ :

$$T_{\text{RCG}}(r) = T_G \exp(-\langle \sigma^2 \rangle_r / 2) = T_G \exp\left(-\frac{1}{2} \frac{\sigma^2}{(1 + \zeta^2 r^2 / \ell^2)}\right), \quad (4)$$

where  $r$  is the distance to the well,  $T_G$  is the geometric mean,  $\sigma^2$  is the variance and  $\ell$  is the correlation length of the log-transmissivity  $T(\mathbf{x})$ .  $\zeta$  is a factor of proportionality, which was determined to be  $\zeta = 1.6$ , as discussed in detail by Zech et al. (2012).

$T_{\text{RCG}}(r)$  can be interpreted as interpolating function between the representative transmissivity at the well  $T_H = T_G \exp(-\frac{1}{2}\sigma^2)$  to the far field value  $T_G$  depending on  $r$ , controlled by the correlation length  $\ell$  (Fig. 1a).

An effective description of well flow transmissivity for an individual field is derived from Eq (4). The behavior of individual fields is different especially at the well due to a lack of ergodicity there. The local transmissivity at the well  $T_{\text{well}}$  is not identical to the harmonic mean  $T_{\text{H}}$  as expected for the ensemble, but refers to the specific value of transmissivity at the well location. An adapted Radial Coarse Graining transmissivity accounts for local effects by replacing the harmonic mean  $T_{\text{H}} = T_{\text{G}} \exp(-\frac{1}{2}\sigma^2)$  by  $T_{\text{well}}$ . In Eq. (4) this refers to substituting the variance by  $-\frac{1}{2}\sigma^2 = \ln T_{\text{well}} - \ln T_{\text{G}}$  and thus,

$$T_{\text{RCG}}^{\text{local}}(r) = T_{\text{G}} \exp\left(\frac{\ln T_{\text{well}} - \ln T_{\text{G}}}{1 + \zeta^2 r^2 / \ell^2}\right) = T_{\text{well}}^{\frac{1}{1 + \zeta^2 r^2 / \ell^2}} T_{\text{G}}^{\frac{\zeta^2 r^2 / \ell^2}{1 + \zeta^2 r^2 / \ell^2}}. \quad (5)$$

$T_{\text{RCG}}^{\text{local}}(r)$  interpolates between the specific transmissivity at the well  $T_{\text{well}}$  and the far field value  $T_{\text{G}}$  depending on the radial distance  $r$  and the correlation length  $\ell$ .

## 2.4 Effective well flow head

An effective drawdown solution is derived by solving the deterministic groundwater flow equation in combination with an effective transmissivity description, namely the Radial Coarse Graining transmissivity. This needs to be clearly distinguished from solving an effective well flow equation. In line with the deterministic nature of the Radial Coarse Graining transmissivity  $T_{\text{RCG}}(r)$ , the associated effective well flow head is deterministic as well.

Explicit results for the hydraulic head drawdown in steady state pumping test ~~with a radially depending transmissivity~~ are achieved by solving the integral in Eq. (3) making use of  $T_{\text{RCG}}(r)$  (Eq. 4). The result is the effective well flow head  $h_{\text{efw}}(r)$  is given by

$$h_{\text{efw}}(r) = -\frac{Q_{\text{w}}}{4\pi T_{\text{G}}} \exp\left(\frac{\sigma^2}{2}\right) \left( \Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 r^2 / \ell^2}{1 + \zeta^2 r^2 / \ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 R^2 / \ell^2}{1 + \zeta^2 R^2 / \ell^2}\right) \right) + \frac{Q_{\text{w}}}{4\pi T_{\text{G}}} \left( \Gamma\left(\frac{\sigma^2}{2} \frac{1}{1 + \zeta^2 r^2 / \ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{1}{1 + \zeta^2 R^2 / \ell^2}\right) \right) + h_{\text{R}}, \quad (6)$$

where  $r$  is the radial distance from the well,  $Q_{\text{w}}$  is the pumping rate,  $T_{\text{G}}$  is the geometric mean,  $\sigma^2$  is the log-transmissivity variance, and  $\ell$  is the correlation length. Again,  $\zeta$  is the factor of proportionality determined to be 1.6 and  $R$  is an arbitrary distance from the well, where the hydraulic head  $h(R) = h_{\text{R}}$  is known.  $\Gamma(x) = \int_{-\infty}^x \frac{\exp(z)}{z} dz$  is the exponential integral function. Details on the derivation of  $h_{\text{efw}}(r)$  can be found in the Appendix.

An approximate solution  $h_{\text{efw}}^{\text{approx}}(r)$  is derived from Eq. (6) by making use of an approximation of  $\Gamma(x)$ . Details are given in the Appendix.

$$h_{\text{efw}}^{\text{approx}}(r) = -\frac{Q_{\text{w}}}{2\pi T_{\text{H}}} \ln \frac{r}{R} - \frac{Q_{\text{w}}}{4\pi T_{\text{G}}} \left( e^{\frac{\sigma^2}{2}} - 1 \right) \cdot \left( \ln \frac{1 + \zeta^2 R^2 / \ell^2}{1 + \zeta^2 r^2 / \ell^2} + \frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 r^2 / \ell^2)} - \frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 R^2 / \ell^2)} \right) + h_{\text{R}}, \quad (7)$$

$h_{\text{efw}}(r)$  is constructed to describe the mean drawdown of a pumping test in two dimensional heterogeneous media effectively. The drawdown curve of  $h_{\text{efw}}(r)$  for a specific choice of parameters (Ensemble A of Table 1) is given in Fig. 1b in comparison to the equivalent drawdown  $h_{\text{eq}}(r)$ , as the solution of the radial flow equation using the equivalent transmissivity  $T_{\text{eq}}$ , based on the same statistical parameters.

The effective well flow solution can be adapted to analyze individual pumping tests by using  $T_{\text{RCG}}^{\text{local}}(r)$  (Eq. 5) instead of  $T_{\text{RCG}}(r)$  (Eq. 4). The local effective well flow solution  $h_{\text{efw}}^{\text{local}}(r)$  is then given by Eq. (6) with  $\frac{\sigma^2}{2}$  substituted by  $-\ln \frac{T_{\text{well}}}{T_{\text{G}}}$  and  $T_{\text{H}}$  substituted by  $T_{\text{well}}$ .

The local effective well flow solution  $h_{\text{efw}}^{\text{local}}(r)$  can be used to analyze drawdowns of single pumping tests in heterogeneous media as encountered in practice. The solution is adapted to the lack of ergodicity at the well, by using transformed parameters  $T_{\text{well}}$ ,  $T_{\text{G}}$  and  $\ell$ . However, the randomness of hydraulic heads can affect the parameter estimation, in particular of the correlation length which is related to drawdown fluctuations in the transition zone. There the impact of heterogeneity is not fully determined by the local transmissivity at the well, but also not fully levelled out as in the far field. The geometric mean  $T_{\text{G}}$  and the correlation length  $\ell$  for a single realization should also therefore be interpreted as local values, not necessarily representing the mean values of the entire field, but those of the pumping well vicinity. Owing to the nature of the pumping test, the drawdown signal does not sample the heterogeneity in transmissivity in a symmetric way, but the shape of the drawdown is mainly determined by the local heterogeneity close to well.

## 2.5 Impact of parameters

The analytical form of  $h_{\text{efw}}(r)$  allows to analyze the impact of the statistical parameters  $T_{\text{G}}$ ,  $\sigma^2$  and  $\ell$  on the drawdown. The drawdown behavior for different choices of parameters can be seen in Fig. 2, which is discussed in detail later on.

Every parameter impacts on the drawdown in a different region. The geometric mean  $T_{\text{G}}$  as representative value for mean uniform flow determines the far field behavior. The variance  $\sigma^2$  determines the drawdown at the well due to the dependence of the near-well asymptotic value  $T_{\text{H}} = T_{\text{G}} \exp(-\frac{1}{2}\sigma^2)$ . The larger the variance the larger are the differences between  $T_{\text{G}}$  and  $T_{\text{H}}$  and the steeper is the drawdown at the well. Whereas, the correlation length  $\ell$  determines the transition from near to far field behavior. Therefore, fluctuation in the hydraulic head in the transition zone can affect estimation errors in the correlation length.

The asymptotic behavior of  $h_{\text{efw}}(r)$  can easily be analyzed using approximate functional description in Eq. (7): for distances close to the well, thus  $r \ll \ell$ ,  $h_{\text{efw}}(r)$  converges to Thiem's solution with  $T_{\text{H}}$  as homogeneous-substitute-constant transmissivity value. All terms, except the first one in Eq. (7), tend to zero or become constant. Thus, they are negligible compared to logarithmic first term

for very small  $r$ ,

$$h_{\text{efw}}^{\text{approx}}(r \ll \ell) \approx -\frac{Q_w}{2\pi T_H} \ln \frac{r}{R} - \frac{Q_w}{4\pi T_G} \left( e^{\frac{\sigma^2}{2}} - 1 \right) \cdot \left( \ln(1 + \zeta^2 R^2 / \ell^2) + \frac{\sigma^2}{2} \right) + h_R$$

$$\approx -\frac{Q_w}{2\pi T_H} \ln \frac{r}{R} + h_R.$$

For large distances from the well, i.e.  $r \gg \ell$ , the solution converges to Thiem's solution with  $T_G$  as  
 300 ~~homogeneous substitute~~ constant transmissivity value. The third and fourth term in Eq. (7) tend to zero and cancel out. The ones in the second term can be neglected, thus

$$h_{\text{efw}}^{\text{approx}}(r \gg \ell) \approx -\frac{Q_w}{2\pi T_H} \ln \frac{r}{R} - \left( \frac{Q_w}{4\pi T_H} - \frac{Q_w}{4\pi T_G} \right) \cdot \ln \left( \frac{\zeta^2 R^2 / \ell^2}{\zeta^2 r^2 / \ell^2} \right) + h_R \approx -\frac{Q_w}{2\pi T_G} \ln \frac{r}{R} + h_R.$$

The larger the correlation length  $\ell$  the longer takes the transition of the drawdown from near well to far field behavior. The influence of  $\ell$  on  $h_{\text{efw}}(r)$  vanishes quickly with increasing distance to the well.

305 The drawdown reaches the far field behavior after approximately two correlation lengths  $h_{\text{efw}}(r > 2\ell) = h_{\text{Thiem}}(r > 2\ell)$  with  $T_G$  as ~~homogeneous substitute~~ constant transmissivity value (Fig. 1b). These findings are in line with the results of Neuman et al. (2004).

### 3 Robust estimation of statistical parameters

#### 3.1 Numerical pumping tests

310 Numerical pumping tests in heterogeneous porous media were generated as artificial measurements. They were used to test the capability of  $h_{\text{efw}}(r)$  in reproducing the mean drawdown and in estimating the underlying parameters of heterogeneity. Pumping tests were simulated using the finite element software OpenGeoSys. The software was successfully tested against a wide range of benchmarks (Kolditz et al., 2012). Results of a steady state simulation with homogeneous transmissivity were in  
 315 perfect agreement with Thiem's analytical solution Eq. (1).

The numerical grid was constructed as a square of  $256 \times 256$  elements with a uniform grid cell size of 1 m except for cells in the vicinity of the pumping well. The mesh was refined in the range of 4 m around the well, which ensures a fine resolution of the steep head gradients at the well. The well in the center of the mesh was included as a hollow cylinder with radius  $r_w = 0.01$  m. The constant  
 320 pumping rate of  $Q_w = -10^{-4} \text{ m}^3 \text{ s}^{-1}$  was distributed equally to all elements at the well. At the radial distance  $R = 128$  m a constant head of  $h(R) = 0$  m was applied giving a circular outer head boundary condition.

Log-normally distributed, Gaussian correlated transmissivity fields were generated using a statistical field generator based on the randomization method (Heße et al., 2014). Multiple ensembles  
 325 with different statistical parameter values were generated, including high variances up to  $\sigma^2 = 4$  (Table 1). Ensemble A with  $T_G = 10^{-4} \text{ m}^2 \text{ s}^{-1}$ ,  $\sigma^2 = 1$  and  $\ell = 10$  m served as reference ensemble for specific cases. Every ensemble consists of  $N = 5000$  realizations, which was tested as sufficiently large to ensure ensemble convergence.

Pumping test simulations were post-processed by performing an angular and an ensemble average.

330 For every realization  $i$ , the simulated drawdown  $\langle h_i(r, \phi) \rangle$  at the radial and angular location  $(r, \phi)$  in polar coordinates was averaged over the four axial directions:  $\langle h_i(r) \rangle = \sum_{\phi_j} \langle h_i(r, \phi_j) \rangle$ . The ensemble mean was the sum over the angular mean of all individual realizations:  $\overline{\langle h(r) \rangle} = \sum_{i=1}^N \langle h_i(r) \rangle$ .

Non-linear regression was used to find the best fitting values for the statistical parameters, denoted by  $\hat{T}_G$ ,  $\hat{\sigma}^2$ , and  $\hat{\ell}$ . The best fitting estimates were determined by minimizing the mean square error  
 335 of the difference between the analytical solution  $h_{\text{efw}}(r)$  and the measured drawdown samples  $h(r)$ :  $\min_{T_G, \sigma^2, \ell} \sum_r (h(r) - h_{\text{efw}}(r))^2$  making use of the Levenberg–Marquardt algorithm. The reliability of the estimated parameters was evaluated using 95%-confidence intervals.

The estimation procedure was applied to the head measurements at every meter distance starting at the well up to a distance of 80m. The range beyond 80m was not taken into account to avoid  
 340 boundary effects. The range of 80m includes at least 4 correlation lengths for all tested ensembles, which is sufficient to ensure convergence to the far field behavior. The question of the applicability of  $h_{\text{efw}}(r)$  on limited head data is of quite complex nature. For a detailed discussion on that issue the reader is referred to Zech et al. (2015).

### 3.2 Ensemble pumping test interpretation

345 First, the simulated ensemble means were analyzed making use of the ensemble version of  $T_{\text{RCG}}(r)$  and  $h_{\text{efw}}(r)$  (Eqs. 4 and 6). Simulated ensemble means  $\overline{\langle h(r) \rangle}$  for multiple choices of statistical parameters  $T_G$ ,  $\sigma^2$  and  $\ell$  are visualized in Fig. 2 in combination with  $h_{\text{efw}}(r)$  for the best fitting parameter estimates  $\hat{T}_G$ ,  $\hat{\sigma}^2$ , and  $\hat{\ell}$ . Input parameters as well as inverse estimation results for all tested ensembles are listed in Table 1.

350 The best fitting estimates show, that all three parameters could be inferred from the ensemble mean with a high degree of accuracy. The deviation of the geometric mean from the input value is in general less than 10%, only for high variances the deviations are up to 30%. Variances deviate in a range of 20% and estimated correlation lengths are accurate within 10% of the initial input parameter.

355 The confidence intervals of the estimates  $\hat{T}_G$  and  $\hat{\sigma}^2$  are very small, showing a high sensitivity of the effective well flow solution  $h_{\text{efw}}(r)$  towards geometric mean and variance. The confidence intervals of the correlation length are larger due to the dependence of the estimate of  $\hat{\ell}$  on the estimates  $\hat{T}_G$  and  $\hat{\sigma}^2$ . This is due to the fact, that the correlation length determines the transition from  $\hat{T}_{\text{well}} = \hat{T}_G \exp(-\frac{1}{2}\hat{\sigma}^2)$  to  $\hat{T}_G$ , which results in larger uncertainties in the estimates of  $\hat{\ell}$ .

### 360 3.3 Individual pumping test interpretation

In the following, pumping test drawdowns of individual transmissivity fields are interpreted based on the adaption version  $h_{\text{efw}}^{\text{local}}(r)$  as discussed in Sect. 2.4. The drawdowns along the four axial

directions as well as the radial mean for two realizations from Ensemble A ( $T_G = 10^{-4} \text{ m}^2 \text{ s}^{-1}$ ,  $\sigma^2 = 1$ ,  $\ell = 10 \text{ m}$ ) are visualized in Fig. 3a and b.

365 Both realizations from Fig. 3a and b differ significantly in the value of the local transmissivity at the well. The analysis of the transmissivity fields at the well gave sampled values of  $\langle T_{\text{well}}^{(a)} \rangle = 0.204 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  and  $\langle T_{\text{well}}^{(b)} \rangle = 1.11 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ , which is in both cases far from the theoretical harmonic mean value  $T_H = 0.61 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  as being the representative value for the near well behavior.

370 Inverse estimation results for the realization in Fig. 3a differ for the drawdowns along the four axial directions  $\langle h(r, \phi_j) \rangle$  and the radial mean  $\langle h(r) \rangle$ : the estimated geometric mean ranges between  $1.03 \times 10^{-4}$  and  $1.45 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  for the four axial directions, with an average value of  $\hat{T}_G = 1.17 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ . The estimates for the local transmissivity at the well are between  $0.195 \times 10^{-4}$  and  $0.212 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ , with an average value of  $\hat{T}_{\text{well}} = 0.204 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ , which  
 375 is exactly the sampled local transmissivity  $\langle T_{\text{well}}^{(a)} \rangle$ . The value of  $\hat{T}_{\text{well}} = 0.204 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  is equivalent to a local variance of  $\hat{\sigma}^2 = 3.49$ . The estimated correlation length ranges between 7.95 and 18.15 m, with an average of  $\hat{\ell} = 12.77 \text{ m}$ . It shows that the randomness of hydraulic heads due to the heterogeneity of transmissivity impact on the estimation results of the correlation length. The differences in the estimates for the drawdowns in different direction for the same realization  
 380 of transmissivity shows that the parameter estimates reflect local heterogeneity in the vicinity of the well rather than the global statistical parameters of the transmissivity field. This was studied and discussed in detail for pumping tests in three dimensional heterogeneous media by Zech et al. (2015).

The realization in Fig. 3b does not allow to infer the parameters of variance and correlation length,  
 385 due to the similarity of  $T_{\text{well}}$  and  $T_G$ . Near and far field representative transmissivities are nearly identical, thus the pumping test appears to behave like in a homogeneous medium (Fig. 3b). However, the behavior is not representative but a result of the coincidental choice of the location of the pumping well.

A statistical analysis of the estimation results is presented in Fig. 4 for all 5000 realizations of  
 390 Ensemble A. Histogram on the best fitting estimates in normalized form are shown, where normalization of results means that they were divided by the input parameters. It can be inferred that the estimate of the geometric mean  $\hat{T}_G$  is in general close to the input value  $T_G$ . The estimate of the local transmissivity at the well  $\hat{T}_{\text{well}}$  is very close to the sampled values  $\langle T_{\text{well}} \rangle$  for nearly all realizations. Thus, the method reproduced very well the local transmissivity at the well. However, the  
 395 local value  $T_{\text{well}}$  of every realization can be far from the theoretical value of  $T_H$ , where both realizations in Fig. 3 gave example. The estimates of the correlation length show a very large scatter. Exceptionally large and small value for  $\hat{\ell}$  refer to realizations, where it was nearly impossible to infer it due to the similarity of  $T_{\text{well}}$  and  $T_G$ , as for the realization of Fig. 3b. The large range of estimated correlation lengths also point towards the fact that  $\hat{\ell}$  of a single drawdown needs to be interpreted as

400 a local value, which is determined by the transmissivity distribution in the vicinity of the well rather than the distribution of the entire field. However, the median of the normalized estimated correlation lengths is close to one, pointing to the fact that representative values can be inferred by taking the mean from multiple pumping tests.

#### 4 Application Example: Single Aquifer Analysis

405 Pumping test campaigns in the field often include the performance of multiple pumping tests within one aquifer. Drawdown measurements at multiple test locations can be used to gain representative parameters of the heterogeneous transmissivity field. The sampled area increases and the effect of local heterogeneity [through randomness of heads](#) reduces. In the following, it is shown, how mean  $T_G$ , variance  $\sigma^2$  and the correlation length  $\ell$  of an individual transmissivity fields can be inferred  
410 making use of a sampling strategy in combination with  $h_{\text{efw}}^{\text{local}}(r)$ .

##### 4.1 Sampling Strategy

The sampling strategy was constructed as pumping test campaign in a virtual aquifer with heterogeneous transmissivity. A series of steady state pumping tests was performed at  $n$  different wells. For each test, drawdowns were measured at all  $n$  wells and at  $m$  additional observation wells. A similar  
415 sampling strategy to infer the aquifer statistics from drawdown measurements have been pursued by e.g. Neuman et al. (2004); Copty and Findikakis (2004); Firmani et al. (2006).

The used sampling strategy includes  $n = 8$  pumping wells and  $m = 4$  observation wells. The specific location of all wells are indicated in Fig. 5. All 8 pumping wells are located within a distance of 18 m. The observation wells are located at larger distances and in all four directions. The well  
420 locations were designed to gain numerous drawdown measurements in the vicinity of each pumping well to allow a reliable estimation of  $T_{\text{well}}$  (or  $\sigma_{\text{local}}^2$ , respectively) and  $\ell$  [by reducing the impact of head fluctuations on the estimation results](#). The additional observation wells provide head observations in the far field to gain a representative value for  $T_G$ . The choice of the well locations does not interfere with the refinement of the numerical grid at the pumping well.

425 Each of the 8 pumping tests was analyzed with  $h_{\text{efw}}^{\text{local}}(r)$  (Sect. 2.4). The best fitting estimates  $\hat{T}_G$ ,  $\hat{T}_{\text{well}}$ , and  $\hat{\ell}$  for all tests were inferred by minimizing the difference between the analytical solution and the 12 measurements. Additionally, parameter estimates were inferred by analyzing the drawdown measurements of all tests jointly.

##### 4.2 Aquifer Analysis

430 The sampling strategy was applied to fields of all ensembles A-G (Table 1). Results are presented for two fields: D1 out of Ensemble D ( $\sigma^2 = 2.25$ ,  $\ell = 20$  m) and E1 out of Ensemble E ( $\sigma^2 = 4.0$ ,  $\ell = 10$  m). Each field was generated according to the theoretical values defined for the particular

ensemble and afterwards analyzed geostatistically to determine the sampled values. The fields D1 and E1 are visualized in Fig. 5. The drawdown measurements for all 8 pumping tests at both fields are given in Fig. 6. The inverse estimates as well as theoretical input and sampling values for the statistical parameters are summarized in Table 2.

Analyzing the data from all 8 pumping tests at field D1 jointly yields very close estimates of all parameters  $\hat{T}_G$ ,  $\hat{T}_{\text{well}}$  (corresponding to  $\hat{\sigma}^2 = 2.255$ ), and  $\hat{\ell}$  to the theoretical and sampled values. The geometric mean estimate is similar for all of the 8 individual pumping tests. In contrast, the values of  $\hat{T}_{\text{well}}$  vary within one order of magnitude. This behavior was expected, since  $\hat{T}_{\text{well}}$  represents the local transmissivity value at the pumping well. The wide range of estimates is a result of the high variance of the transmissivity field. The estimates of the correlation length  $\hat{\ell}$  differ between the individual tests within a reasonable range of a few meters. The only exception is the estimate for pumping at PW<sub>5</sub>. For this specific pumping test is highly uncertain due to the coincidence of the values of  $\hat{T}_{\text{well}}$  and  $\hat{T}_G$ , similar to the realizations in Fig. 3b, as discussed in section 3.3. However, the mean value over the individual tests as well as the estimate from the joint analysis of all measurements gave reliable estimates for the correlation length.

The analysis of the sampling strategy at field E1 yields similar results as for D1. The geometric mean values  $\hat{T}_G$  differ little among the 8 individual pumping tests and for the joint analysis. The mean value is double the value as the theoretical one, but close to the sampled geometric mean (Table 2). The local transmissivities  $\hat{T}_{\text{well}}$  again vary within one order of magnitude, reflecting the high variance of the field. The mean and jointly estimated values are higher than theoretical one, which is in correspondence to the difference in the geometric mean. The estimates of the correlation length  $\hat{\ell}$  deviate in a reasonable range of a few meters, which reflects the impact of the location of the pumping well with regard to the shape of the correlation structure around the well.

Finally, the analysis shows that representative values of the statistical parameters can be determined by performing pumping test at multiple locations of an individual transmissivity field. It was shown, that  $h_{\text{efw}}(r)$  is feasible to interpret steady state pumping tests in highly heterogeneous fields.

## 5 Conclusions

The analytical effective well flow solution  $h_{\text{efw}}(r)$  is presented, which can be interpreted as extension of Thiem's equation to heterogeneous media.  $h_{\text{efw}}(r)$  depends on the statistical parameters of log-normal distributed transmissivity: geometric mean  $T_G$ , variance  $\sigma^2$  and correlation length  $\ell$ .  $h_{\text{efw}}(r)$  was derived based on the Radial Coarse Graining transmissivity  $T_{\text{RCG}}(r)$  introduced by Schneider and Attinger (2008), which interpolates between the near well and far field representative transmissivities for well flow. **The effective well flow solution does not refer to an effective well flow equation directly, but is an analytical solution of the groundwater flow equation under well flow conditions in combination with the Radial Coarse Graining transmissivity as an effective transmissivity.** Simula-

tion of pumping tests were performed in log-normally distributed transmissivity fields and compared with  $h_{\text{efw}}(r)$ . Based on the results, the following conclusions can be drawn:

- 470 1.  $h_{\text{efw}}(r)$  describes the mean drawdown of a pumping test in two dimensional heterogeneous isotropic media effectively. It is not limited to small variance, but is tested to reproduce ensemble means for highly heterogeneous media with variances up to  $\sigma^2 = 4$ .
2. The analytical character of  $h_{\text{efw}}(r)$  allows to perform inverse estimation of the statistical parameters of the transmissivity fields from measured drawdowns. Geometric mean  $T_G$ , vari-  
475 ances  $\sigma^2$  and correlation length  $\ell$  can be estimated for a wide range of parameters with a high accuracy and certainty from ensemble mean drawdowns.
3. Parameter estimates from individual drawdowns reflect local heterogeneity at the well rather than the global statistical parameters of the transmissivity field.
4. Representative values of geometric mean, variance and correlation length for an individual  
480 field of transmissivity can be determined by performing pumping test at multiple locations of that field, estimating the parameters for every test separately and than performing a statistical analysis of the results.

$h_{\text{efw}}(r)$  is a promising tool to interpret steady state pumping tests in order to infer the statistical parameters of the underlying transmissivity field without time- and cost-intensive laboratory inves-  
485 tigations. Future steps will include the expansion of the method to interpret transient pumping test data.

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490 The effective well flow head  $h_{\text{efw}}(r)$  as solution of the well flow equation (2) is derived by solving the integral (Eq. 3) with the analytical expression of  $T_{\text{RCG}}(r)$  from Eq. (4).

$$h(r_2) - h(r_1) = \frac{C_1}{T_G} \int_{r_1}^{r_2} \frac{1}{r} \exp\left(\frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 r^2 / \ell^2)}\right) dr. \quad (\text{A1})$$

The integral is evaluated analytically by making use of the exponential integral function

$$\Gamma(x) - \Gamma(X) = \int_X^x \frac{\exp(z)}{z} dz = \ln \frac{x}{X} + \sum_{k=1}^{\infty} \frac{x^k - X^k}{k!k}. \quad (\text{A2})$$

495 The argument in the exponent in Eq. (A1) is substituted by  $z(r) = \frac{\sigma^2}{2} \frac{1}{(1+\zeta^2 r^2/\ell^2)}$  with integrator  $dr = -\frac{\ell\sigma^2}{4\zeta z^2} \left(\frac{\sigma^2}{2z} - 1\right)^{-\frac{1}{2}} dz$ , furthermore partial fraction decomposition is used, resulting in

$$\begin{aligned}
h(r_2) - h(r_1) &= \frac{C_1 \sigma^2}{T_G} \frac{1}{4} \int_{z(r_1)}^{z(r_2)} \frac{\exp(z)}{z(z - \frac{\sigma^2}{2})} dz \\
&= \frac{C_1}{2T_G} \int_{z(r_1) - \frac{\sigma^2}{2}}^{z(r_2) - \frac{\sigma^2}{2}} \frac{\exp\left(z + \frac{\sigma^2}{2}\right)}{z} dz - \frac{C_1}{2T_G} \int_{z(r_1)}^{z(r_2)} \frac{\exp(z)}{z} dz \\
&= \frac{C_1}{2T_G} e^{\frac{\sigma^2}{2}} \left( \Gamma\left(z(r_2) - \frac{\sigma^2}{2}\right) - \Gamma\left(z(r_1) - \frac{\sigma^2}{2}\right) \right) \\
500 \quad &- \frac{C_1}{2T_G} (\Gamma(z(r_2)) - \Gamma(z(r_1))).
\end{aligned} \tag{A3}$$

The final solution for the effective well flow head as given in Eq. (6) results by re-substituting the abbreviation  $z(r) = \frac{\sigma^2}{2} \frac{1}{(1+\zeta^2 r^2/\ell^2)}$  with  $r_2 = r$  and  $r_1 = R$  and inserting  $C_1 = -\frac{Q_w}{2\pi}$  as derived from the constant flux boundary condition (Sect. 2.1).

An approximate formulation of Eq. (A3) can be derived by using the definition of the exponential  
505 integral function as infinite sum, given in Eq. (A2) in combination with the relationship  $z(r) - \frac{\sigma^2}{2} = \frac{\sigma^2}{2} \frac{-\zeta^2 r^2/\ell^2}{1-\zeta^2 r^2/\ell^2} = z(r) \left(-\frac{\zeta^2 r^2}{\ell^2}\right)$ :

$$\begin{aligned}
h(r_2) - h(r_1) &= \frac{C_1}{2T_G} e^{\frac{\sigma^2}{2}} \left( \ln \frac{z(r_2) - \frac{\sigma^2}{2}}{z(r_1) - \frac{\sigma^2}{2}} + \sum_{k=1}^{\infty} \frac{\left(z(r_2) - \frac{\sigma^2}{2}\right)^k - \left(z(r_1) - \frac{\sigma^2}{2}\right)^k}{k!k} \right) \\
&\quad - \frac{C_1}{2T_G} \left( \ln \frac{z(r_2)}{z(r_1)} + \sum_{k=1}^{\infty} \frac{z(r_2)^k - z(r_1)^k}{k!k} \right) \\
&= \frac{C_1}{T_G} e^{\frac{\sigma^2}{2}} \ln \frac{r_2}{r_1} + \frac{C_1}{2T_G} \left( e^{\frac{\sigma^2}{2}} - 1 \right) \ln \frac{z(r_2)}{z(r_1)} \\
510 \quad &+ \frac{C_1}{2T_G} \sum_{k=1}^{\infty} \frac{z(r_2)^k \left( e^{\frac{\sigma^2}{2}} (-\zeta^2 r_2^2/\ell^2)^k - 1 \right) - z(r_1)^k \left( e^{\frac{\sigma^2}{2}} (-\zeta^2 r_1^2/\ell^2)^k - 1 \right)}{k!k} \\
&\approx \frac{C_1}{T_G} e^{\frac{\sigma^2}{2}} \ln \frac{r_2}{r_1} + \frac{C_1}{2T_G} \left( e^{\frac{\sigma^2}{2}} - 1 \right) \ln \frac{z(r_2)}{z(r_1)} + \frac{C_1}{2T_G} \left( e^{\frac{\sigma^2}{2}} - 1 \right) (z(r_2) - z(r_1)).
\end{aligned} \tag{A4}$$

The final approximate solution as given in Eq. (7) results by re-substituting  $z(r)$  with  $r_2 = r$  and  $r_1 = R$  and inserting  $C_1 = -\frac{Q_w}{2\pi}$ .

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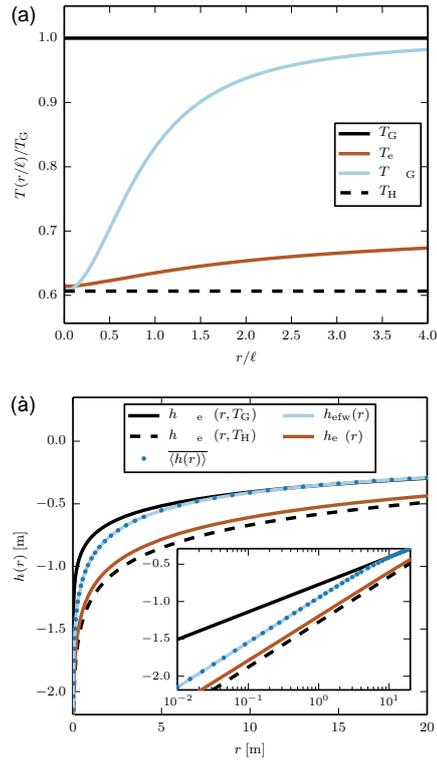
**Table 1.** Ensemble input parameters  $T_G$ ,  $\sigma^2$  and  $\ell$  and best fitting inverse estimation results  $\hat{T}_G$ ,  $\hat{\sigma}^2$  and  $\hat{\ell}$  with 95% confidence intervals (in brackets) for ensemble mean  $\overline{\langle h(r) \rangle}$  for all generated ensembles.

	$T_G$	$\hat{T}_G$ [ $10^{-4} \text{ m}^2 \text{ s}^{-1}$ ]	$\sigma^2$	$\hat{\sigma}^2$ [-]	$\ell$	$\hat{\ell}$ [m]
A	1.0	1.03 ( $\pm 0.0011$ )	1.0	1.04 ( $\pm 0.0022$ )	10	9.80 ( $\pm 0.086$ )
B	1.0	1.08 ( $\pm 0.0013$ )	1.0	1.19 ( $\pm 0.0022$ )	20	21.6 ( $\pm 0.127$ )
C	1.0	1.08 ( $\pm 0.0021$ )	2.25	2.49 ( $\pm 0.0038$ )	10	10.1 ( $\pm 0.050$ )
D	1.0	1.19 ( $\pm 0.0024$ )	2.25	2.67 ( $\pm 0.0039$ )	20	22.2 ( $\pm 0.077$ )
E	1.0	1.16 ( $\pm 0.0046$ )	4.0	4.34 ( $\pm 0.0078$ )	10	11.0 ( $\pm 0.042$ )
F	1.0	1.31 ( $\pm 0.0088$ )	4.0	4.27 ( $\pm 0.0131$ )	20	22.2 ( $\pm 0.120$ )
G	1.5	1.55 ( $\pm 0.0012$ )	1.0	1.03 ( $\pm 0.0016$ )	10	10.1 ( $\pm 0.066$ )

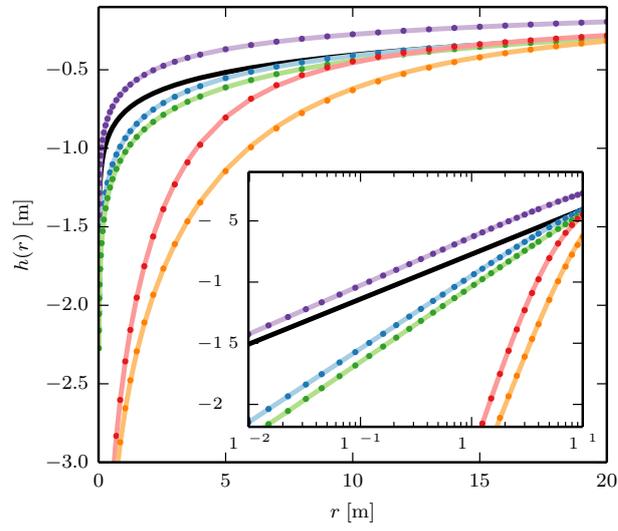
**Table 2.** Parameter estimates of geometric mean transmissivity  $\hat{T}_G$  [ $10^{-4} \text{ m}^2/\text{s}$ ], local transmissivity at the well  $\hat{T}_{\text{well}}$  [ $10^{-4} \text{ m}^2/\text{s}$ ] and correlation length  $\hat{\ell}$  [m] for the 8 pumping tests at fields D1 (from Ensemble D,  $\sigma^2 = 2.25$ ) and E1 (from Ensemble E,  $\sigma^2 = 4.0$ ). Additionally, the theoretical and the sampled values ( $T_{\text{well}} \equiv T_H$ ) are given.

	D1			E1		
	$\hat{T}_G$	$\hat{T}_{\text{well}}$	$\hat{\ell}$	$\hat{T}_G$	$\hat{T}_{\text{well}}$	$\hat{\ell}$
PW <sub>0</sub>	1.025	0.434	29.51	1.945	0.313	9.56
PW <sub>1</sub>	1.023	0.362	27.23	2.202	0.445	11.55
PW <sub>2</sub>	1.076	0.220	23.68	2.093	0.437	10.03
PW <sub>3</sub>	0.898	1.057	9.51	2.052	0.520	15.34
PW <sub>4</sub>	1.001	0.147	20.53	2.174	1.847	12.30
PW <sub>5</sub>	0.889	1.071	5.33	1.980	1.117	5.43
PW <sub>6</sub>	1.038	0.177	20.39	1.840	0.148	8.78
PW <sub>7</sub>	0.901	1.700	16.48	1.969	0.476	17.04
Mean of 8	0.981	0.646	19.08	2.032	0.663	9.90
Jointly	1.013	0.328	22.38	2.010	0.409	9.97
Theory	1.0	0.325	20.0	1.0	0.135	10.0
Sampled	0.985	0.333	23.43	1.999	0.491	12.66

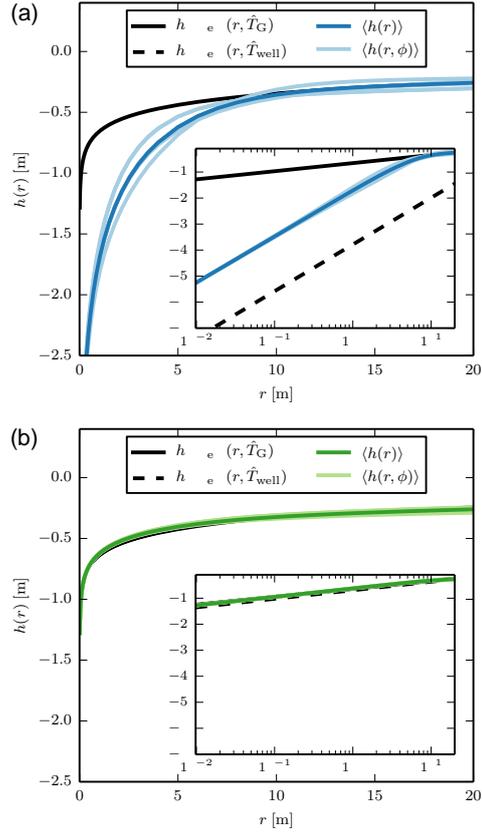
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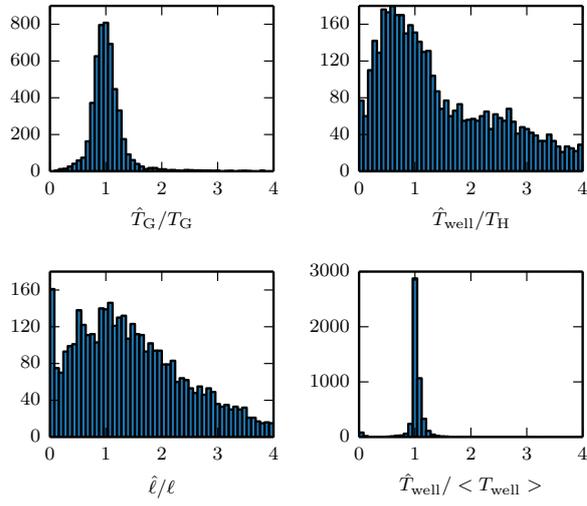
**Figure 1.** Comparison of equivalent and Radial Coarse Graining approach: **(a)** radially depending transmissivities interpolating between harmonic mean  $T_H$  and geometric mean  $T_G$ :  $T_{RCG}(r)$  from Eq. (4) and  $T_{eq}(r)$  calculated based on Thiem's formula Eq. (1) with  $h(r) = \overline{h(r)}$ , which is the ensemble mean for Ensemble A (Table 1), **(b)** hydraulic head drawdowns after pumping with:  $h_{efw}(r)$  from Eq. (6) as solution of the well flow equation using  $T_{RCG}(r)$ ,  $h_{eq}(r)$  as solution of the well flow equation using  $T_{eq}(r)$ , Thiem's solution with **homogeneous substitute constant** values  $T_G$  and  $T_H$  as well as mean ensemble drawdown  $\overline{h(r)}$ .



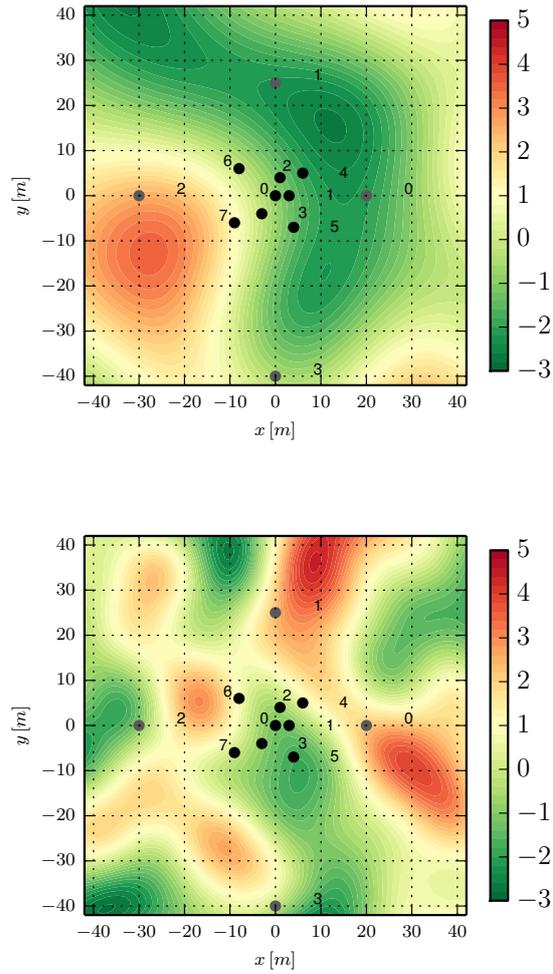
**Figure 2.** Simulated ensemble means  $\overline{h(r)}$  (dots) and  $h_{\text{efw}}(r)$  with best fitting estimates (lines) for multiple Ensembles: A (blue), B (green), E (red), F (orange), G (purple). Parameter values are listed in Table 1. Black line shows  $h_{\text{Thiem}}(r)$  with  $T_G = 10^{-4} \text{ m}^2 \text{ s}^{-1}$ .



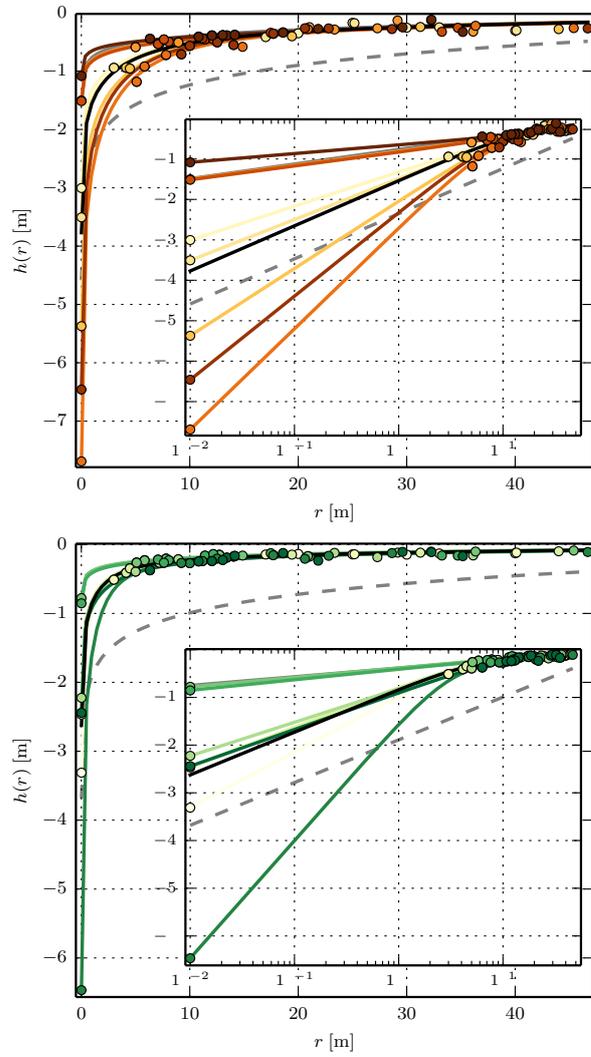
**Figure 3.** Drawdowns simulated for two individual transmissivity field realizations of Ensemble A ( $T_G = 10^{-4} \text{ m}^2 \text{ s}^{-1}$ ,  $\sigma^2 = 1$ ,  $\ell = 10 \text{ m}$ ): **(a)** realization with  $T_{\text{well}} = 0.204 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  and **(b)** realization with  $T_{\text{well}} = 1.11 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ .  $\langle h(r) \rangle$  (dark color) is the radial mean,  $\langle h(r, \phi) \rangle$  (light color) denotes the drawdowns along the four axes ( $\phi = 0^\circ, 90^\circ, 180^\circ, 270^\circ$ ), as well as in black Thiem's solution for **homogeneous substitute-constant** values.



**Figure 4.** Histogram on the best fitting estimates ( $\hat{T}_G$ ,  $\hat{T}_{\text{well}}$ ,  $\hat{\ell}$ ) versus the theoretical input values ( $T_G$ ,  $T_H, \ell$ ) and the sampled transmissivity at the pumping well ( $\langle T_{\text{well}} \rangle$ ) for the  $N = 5000$  realizations of Ensemble A.



**Figure 5.** Spatial distribution of log-transmissivity for fields (a) D1 and (b) E1 and locations of the eight pumping wells ( $PW_0, \dots, PW_7$  in black) and the four observation wells ( $OW_0, \dots, OW_3$  in gray).



**Figure 6.** Simulated drawdown measurements (dots) and fitted effective well flow solution  $h_{efw}(r)$  (lines) for eight pumping tests within the heterogeneous transmissivity fields (a) D1 and (b) E1. Colours indicate the results for the individual pumping tests at  $PW_0, \dots, PW_7$  (from light to dark). The black line denotes the effective well flow solution  $h_{efw}(r)$  fitted to all measurements jointly. Gray lines denote Thiem's solution for  $\hat{T}_G$  (solid) and for  $\hat{T}_{well}$  (dashed). Statistical parameters are given in Table 2.