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December 22, 2015

Dear Prof. Riva,

please find enclosed the revisions on the manuscript "Technical Note: Analytical Solution for the Mean Drawdown of Steady State Pumping Tests in Two-dimensional Isotropic Heterogeneous Aquifers" (hess-2015-252) including a point-by-point response to the reviews, a list of all relevant changes made in the manuscript, and a marked-up manuscript version.

First, we want to thank the reviewer(s) for the fruitful comments, which helped to improve the manuscript a lot.

The reviewer's comments are quoted verbatim in italics, in the order provided. Responses are given point-by-point including locator showing where the action can be viewed in the marked-up manuscript. Line numbers given by the reviewers refer to the line numbering of the previous submitted draft version. Line numbers in the response refer to those in the marked-up manuscript version.

We hope you will find that our responses meet your expectations.

Sincerely,

Abranne Zect

Alraune Zech

encl: Detailed responses Marked-up manuscript version

Response to Comment of S.P. Neuman

We thank S.P. Neuman for the valuable comments. His response shows that the derivation of the Radial Coarse Graining (RCG) approach was not displayed sufficiently clear in the manuscript and previous publications on that topic. Therefore the concept is displayed in more detail in the manuscript, including theoretical background, approximations as well as heuristic steps.

The referee objects versus the publication of the work due to "fundamental inconsistencies" in the basic approach of RCG. In order to clarify that point, we start by a detailed description of the RCG approach in general. Afterwards we give a point by point reply to the comments of the referee with reference to the detailed concept description and changes in the manuscript.

Radial Coarse Graining Approach

The concept of Radial Coarse Graining can be best explained within five major steps:

- 1. Concept of Coarse Graining for uniform flow
- 2. Transfer of Coarse Graining to radial flow
- 3. Overcome non-locality for non-uniform flow
- 4. Derive effective hydraulic conductivity for well flow
- 5. Derivation of effective well flow head

Step 1: Coarse Graining for Uniform Flow

The basic concept of Coarse Graining for flow in porous media was introduced by Attinger [2003]. The author describes the upscaling procedure for uniform flow. The result is an upscaled log-normally distributed hydraulic conductivity field $K_{\lambda}^{\text{CG}}(\vec{x})$, which is coarsened according to a cut-off value λ . Fluctuation smaller than λ are filtered out, fluctuation at a scale larger than the filter width λ are still resolved. The derivation included multiple points:

- 1. Starting point is the steady state head distribution for single-phase, incompressible flow through a heterogeneous medium: $-\nabla (K(\vec{x})\nabla\phi(\vec{x})) = \rho(\vec{x})$, with $K(\vec{x})$ being the hydraulic conductivity, $\phi(\vec{x})$ the hydraulic head and $\rho(\vec{x})$ the source/sink term in *d*-dimensional space.
- 2. Hydraulic conductivity is modelled as spatial random function with a log-normal distribution: $K(\vec{x}) = K_0 \exp f(\vec{x})$, with $f(\vec{x})$ being normally distributed. It is further assumed, that $K(\vec{x}) = \bar{K} + \tilde{K}(\vec{x})$ can be separated into a constant mean value \bar{K} and a spatially depending fluctuation term $\tilde{K}(\vec{x})$ with zero mean.
- 3. A filter function $\langle . \rangle_{\lambda}$ is defined based on the parameter λ , which resolves all fluctuation larger than λ and filters out those smaller than λ .
- 4. The filter is applied to the head equation and thus to the hydraulic head distribution, which results from the stochastic description of the hydraulic conductivity. The filtered head equation is than solved in Fourier space.
- 5. After mathematical treatment, the author results in an expression whose "Fourier-back-transform [...] yields a non-local resolution dependent hydraulic conductivity tensor as found as well by Neuman and Orr [1993]". The expression is simplified by evaluating the hydraulic conductivity tensor at a specific point in Fourier space $\vec{q} = 0$, "which corresponds to localization in the work of Neuman and Orr [1993]."
- 6. The Fourier back-transformation of the filtered head equation reads in real space after localization:

$$-\nabla\left(\bar{K} + \langle \tilde{K}(\vec{x}) \rangle_{\lambda}\right) \nabla \langle \phi(\vec{x}) \rangle_{\lambda} + \delta \hat{K}^{\text{eff}}(\vec{q} = 0, \lambda) \nabla \langle \phi(\vec{x}) \rangle_{\lambda} = \langle \rho(\vec{x}) \rangle_{\lambda} \quad , \tag{1}$$

where $\delta \hat{K}^{\text{eff}}$ is the scale-dependent effective hydraulic conductivity tensor which is induced by small scale heterogeneities varying on typical scales smaller than λ .

7. The upscaled hydraulic conductivity for the filtered head equation reads $K_{\lambda}^{\text{CG}}(\vec{x}) = K^{\text{eff}}(\lambda) +$

 $\langle \tilde{K}(\vec{x}) \rangle_{\lambda}$, where the effective mean value is $K^{\text{eff}}(\lambda) = \bar{K} + \delta \hat{K}^{\text{eff}}(\vec{q} = 0, \lambda)$ in lowest order perturbation.

8. Explicit results for $\delta \hat{K}^{\text{eff}}$ are first evaluated in lowest order perturbation. Then, renormalization group analysis is applied by extending the calculations to higher order perturbation theory. Attinger [2003] results in the closed form expression for the effective mean value

$$K^{\text{eff}}(\lambda) = K_{\text{G}} \exp\left(\sigma^2\left(\frac{1}{2} - \frac{1}{d}\right)\right) \exp\left(\frac{1}{d}\sigma^2\left(\frac{\ell^2}{\ell^2 + \lambda^2/4}\right)^{d/2}\right)$$
(2)

where σ^2 is the variance and ℓ is the correlation length of the unfiltered hydraulic conductivity distribution $K(\vec{x})$; d is the dimension of space.

9. The filtered hydraulic conductivity distribution $K_{\lambda}^{\text{CG}}(\vec{x})$ is still a log-normal distributed quantity. Attinger [2003] showed, that the variance $\langle \sigma^2 \rangle_{\lambda}$ and the correlation length $\langle \ell \rangle_{\lambda}$ of the filtered field $K_{\lambda}^{\text{CG}}(\vec{x})$ can be expressed by the variance σ^2 and the correlation length ℓ of the unfiltered conductivity field $K(\vec{x})$:

$$\langle \sigma^2 \rangle_{\lambda} \equiv \sigma^2 \left(\frac{\ell^2}{\ell^2 + \lambda^2/4} \right)^{d/2} \quad \langle \ell \rangle_{\lambda} \equiv \left(\ell^2 + \lambda^2/4 \right)^{1/2} \tag{3}$$

Step 2: Transfer of Coarse Graining Approach to Radial Flow and the Problem of Non-locality for Non-uniform Flow

The basic concept of Coarse Graining can be applied to non-uniform flow straight ahead. The critical step when transferring the results of *Attinger* [2003] to radial flow is the localization of the non-local resolution dependent hydraulic conductivity tensor $\delta \hat{K}^{\text{eff}}$ (point 5, step 1).

For uniform flow, the assumption of constant pressure gradient $\nabla \phi(x)$ holds at least at the coarser scale. The gradient term and the non-local resolution dependent hydraulic conductivity tensor can be separated. And localization can be applied to $\delta \hat{K}^{\text{eff}}$. For non-uniform flow, the basic assumption of constant flux is not valid any more.

Step 3: Heuristic Approach to Overcome Non-Locality

A heuristic approach is taken to overcome the limitation of non-locality. The basic idea it to ask, how to achieve a quasi-constant pressure gradient, in order to allow a localization. For well flow, the answer is given by adapting the size of the volume elements over which flow takes place, which corresponds to a change in the coordinate system.

The head gradient in well flow is proportional to the reciprocal of the distance r to the well: $\nabla h(r) = \frac{h(r)-h(r+\Delta r)}{\Delta r} \propto \frac{1}{r}$. The gradient is constant for volumes of size proportional to r. Thus, a new coordinate system with a constant head gradient must have cells with increasing volume relative to r. Figure 1 gives an illustration of the modified coordinate system.

The changed coordinate system impacts on the scaling procedure and the parameter λ . For uniform flow, λ is constant in a equidistant Cartesian coordinate system. Adapted to well flow, the scaling parameter needs to be proportional to r, because the filter width increases with distance to the well.

Under the assumption of the adapted coordinate system the localization can be performed and so the following steps of the Coarse Graining procedure. The consequence for the upscaled hydraulic conductivity field is a change in the scaling parameter $\lambda = 2\zeta r$, where ζ is a factor of proportionality. The mean value $K^{\text{eff}}(r)$ is radially depending and the filtered log-normal distributed field in *d*-dimensions reads

$$K_r^{\text{RCG}}(\vec{x}) = K^{\text{eff}}(r) + \langle \tilde{K}(\vec{x}) \rangle_r = K_{\text{G}} \exp\left(\sigma^2 \left(\frac{1}{2} - \frac{1}{d}\right)\right) \exp\left(\frac{1}{d}\sigma^2 \left(\frac{1}{1 + \zeta^2 r^2/\ell^2}\right)^{d/2}\right) + \langle \tilde{K}(\vec{x}) \rangle_r \quad (4)$$

The filtered field $K_r^{\text{RCG}}(\vec{x})$ can be understood as an upscaled hydraulic conductivity, which gives the same drawdown behaviour under well flow conditions as the originally unfiltered hydraulic conductivity field $K(\vec{x})$. The filtered field $K_r^{\text{RCG}}(\vec{x})$ still contains spatial heterogeneity and local fluctuations, but



Figure 1: Right: Cartesian coordinate system representation; hydraulic head gradients can be assumed constant at every cell for uniform flow; scaling parameter λ is constant. Left: Well flow adapted coordinate system representation with increasing cell size relative to the distance to the well r; hydraulic head gradients can be assumed constant at every cell for well flow.

reduced to the amount relevant to the pumping test. The coarsening is constructed to filter only those information out, which are not seen by the pumping test.

The step was presented by *Schneider and Attinger* [2008]. It is not performed in a mathematically straight way, but problem adapted to well flow conditions.

Step 4: From Spatially Variable towards Effective Hydraulic Conductivity

Spatial heterogeneity is still resolved in $K_r^{\text{RCG}}(\vec{x})$, although reduced to the amount relevant to the pumping test. Aiming at deriving an effective description of the hydraulic conductivity for well flow condition, a further step of averaging is necessary. Thereby two different aspects are of interest: (i) an effective description of well flow conductivity for an ensemble and (ii) effective description of well flow conductivity for an individual field.

A result for an effective ensemble description can be derived by averaging $K_r^{\text{RCG}}(\vec{x})$ appropriate to well flow condition. The averaging rule is determined by the boundary condition at the well.

In the following, we focus on 2D since the manuscript refers to two-dimensional well flow. Hydraulic conductivity is replaced by its depth average, the transmissivity T. The averaging rule at the well is given by the harmonic mean [Dagan, 1989]. The harmonic mean RCG-transmissivity can be calculated via the theoretical description making use of the variance of the coarsened transmissivity $\langle \sigma^2 \rangle_r$ from Eq. (3) adapted to well flow with $\lambda = 2\zeta r$:

$$T_H^{\rm RCG}(r) = T_{\rm G} \exp\left(-\langle \sigma^2 \rangle_r / 2\right) = T_{\rm G} \exp\left(-\frac{1}{2}\frac{\sigma^2}{1+\zeta^2 r^2/\ell^2}\right) \ . \tag{5}$$

The general procedure is identical for three-dimensional anisotropic hydraulic conductivity as discussed in Zech et al. [2012].

An effective description of well flow transmissivity for an individual field can be derived from $T_H^{\text{RCG}}(r)$. The behaviour of individual fields is different especially at the well due to a lack of ergodicity there. The local transmissivity at the well T_{well} is not identical to the harmonic mean T_{H} as expected for the ensemble, but refers to the specific value of transmissivity at the well location. An adapted radial coarse graining transmissivity accounts for local effects by replacing the harmonic mean $T_{\text{H}} = T_{\text{G}} \exp\left(-\frac{1}{2}\sigma^2\right)$ by T_{well} . In Eq. (5) this refers to substituting the variance by $-\frac{1}{2}\sigma^2 = \ln T_{\text{well}} - \ln T_{\text{G}}$.

Step 5: Effective Well Flow Head

The final step of the RCG approach is the derivation of the effective well flow head $h^{\text{efw}}(r)$. The effective RCG-transmissivity (Eq. 5) is inserted to the deterministic well head equation,

$$0 = \left(\frac{1}{r} + \frac{\mathrm{d}\ln T_H^{\mathrm{RCG}}(r)}{\mathrm{d}r}\right) \frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2 h}{\mathrm{d}r^2} \ . \tag{6}$$

The analytical solution of Eq. (6) for two-dimensional flow is given by

$$h_{\rm efw}(r) = -\frac{Q_w}{4\pi T_{\rm G}} \exp\left(\frac{\sigma^2}{2}\right) \left(\Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 R^2/\ell^2}{1+\zeta^2 R^2/\ell^2}\right)\right) + \frac{Q_w}{4\pi T_{\rm G}} \left(\Gamma\left(\frac{\sigma^2}{2} \frac{1}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{1}{1+\zeta^2 R^2/\ell^2}\right)\right) + h_R \quad (7)$$

where Q_w is the pumping rate, T_G is the geometric mean, σ^2 is the log-transmissivity variance, and ℓ is the correlation length; ζ is the factor of proportionality determined to be 1.6 and R is an arbitrary distance to the well, where the hydraulic head $h(R) = h_R$ is known. Γ is the exponential integral $\Gamma(x) = \int_{\infty}^x \frac{\exp(z)}{z} dz$.

The effective well flow head for the ensemble behaviour can be adapted to single realizations similar to the RCG transmissivity. The harmonic mean $T_{\rm H}$ is replaced by the local transmissivity at the well $T_{\rm well}$ by substituting $-\frac{1}{2}\sigma^2 = \ln T_{\rm well} - \ln T_{\rm G}$ in Eq. (7).

Step 5 derived for two-dimensional well flow (Eq. 7) represents the achievement of the Technical Note under consideration (manuscript section 2.3, appendix). Steps 1 is published in *Attinger* [2003], steps 2-4 are presented in the work of *Schneider and Attinger* [2008] for 2D. Steps 4 and 5 are discussed in the work of *Zech et al.* [2012] for well flow in 3D. The latter were able to show the appropriateness of the approach by comparison with numerical pumping test simulations. The same for 2D well flow is one major issue of the current manuscript.

Detailed Response to Referee's Comments

In this technical note the authors (a) develop an analytical solution for mean steady state drawdown under horizontal flow to a well withdrawing water from a randomly heterogeneous aquifer at a constant rate and (b) suggest ways to evaluate properties of aquifer transmissivity on the basis of measured drawdowns. Their analysis is based on a Radial Coarse Graining (RCG) approach described in Schneider and Attinger [2008]. It considers two versions of coarse grained transmissivity, termed ensemble and local, given in parametric form as functions of radial distance to the well. The authors then propose ways to determine the corresponding parameters on the basis of measured drawdowns.

To properly review this note for HESS I found it necessary to study the above work of Schneider and Attinger (SA). Here I discovered what appear to be fundamental inconsistencies in their RCG approach. The development in SA starts with a stochastic representation of 2D steady flow in a random transmissivity field toward the well, subject to deterministic inner and outer boundary conditions. As we all know, this stochastic head equation embodies two physical principles, conservation of (incompressible) water volume and Darcy's law. RCG a la SA consists of upscaling transmissivities through weighted spatial averaging with a weight function that depends on radial distance. The resulting spatially averaged transmissivity is considered to be deterministic. Replacing transmissivity in the original stochastic equation with its upscaled version thus renders this equation deterministic in what the authors consider, and label, RCG drawdown. It is this "RCG" equation that Zech and Attinger rely on in the technical note under review.

Unfortunately, the latter RCG equation is not consistent with the two physical principles on which the original stochastic equation rests. To preserve these principles SA should have applied RCG to the original stochastic head equation, not just to transmissivity. Averaging the original equation would have resulted in a modified head equation, preserving the underlying physics, but including a new integro-differential term with an integrand that contains both transmissivity and hydraulic gradient. This non-local cross term would be equivalent to the residual flux term in the probabilistically averaged stochastic head equation of Neuman and Orr [1993]. By (inadvertently?) dropping this mixed integro-differential term, SA have introduced a bias into their resulting RCG head equation the magnitude of which could be large or small, depending on circumstances. We know from subsequent numerical solutions of the Neuman and Orr stochastic moment equations that ignoring their residual flux, as has been common in the stochastic literature, may result in unjustifiably large biases.

With the detailed recapitalization of the concept of RCG we aim to elucidate the "fundamental inconsistencies" of the Radial Coarse Graining approach derived by *Schneider and Attinger* [2008] as mentioned by the referee. As the referee describes, the derivation of the RCG-transmissivity starts with a stochastic representation of transmissivity as spatial random function with a log-normal distribution. The major step consists of upscaling transmissivities through weighted spatial averaging with a weight function that depends on radial distance, resulting in a deterministic spatially averaged transmissivity.

As a first point, we want to specify that "the spatially averaged transmissivity" is not replaced in the original stochastic equation. The RCG-drawdown is the result of the deterministic head equation considering the transmissivity not constant, but radial depending. Thus, the "RCG-equation" as called by the referee, is an independent physical equation and thereby embodies the two physical principles, conservation of water volume and Darcy's law. The equation is not meant to replace the original stochastic equation.

The procedure of upscaling was applied to the head equation with spatially variable transmissivity (step 1). As mentioned by the referee, the averaging procedure leads to a new non-local integro-differential term (step 2). This term is not dropped, but treated in an heuristic way (step 3) to result in a filtered transmissivity field, which still resolves spatial heterogeneity, but at coarser scale. In order to gain an effective well flow transmissivity, spatial averaging is applied. The result is a mean RCG-transmissivity depending on the distance to the well, but not containing local fluctuation (step 4). The mean RCG transmissivity is not meant to fulfil the original head equation, but is constructed to reproduce the ensemble mean drawdown of pumping tests in heterogeneous media, in dependence of the statistical parameters of the underlying log-normally distributed hydraulic conductivity/transmissivity fields. This drawdown description (effective well flow solution) is derived by solving the head equation under well flow condition with the effective well flow head enables to estimate the parameters of aquifer statistics by comparison with simulated and/or measured drawdowns.

We are aware that the adaption of Coarse Graining from uniform to radial flow conditions was not performed in a rigorous mathematical way, but includes heuristics steps, which are well-considered and adapted to the problem at hand. The assumptions made are physically motivated and verified by numerical proof. The effective well flow head is compared with ensemble mean drawdowns of simulated pumping tests in heterogeneous media. The very well match confirmed the appropriateness of the conjectures taken in the RCG-approach.

Since it appears that the derivation of the Radial Coarse Graining approach was not displayed sufficiently clear in the manuscript and previous publications, the manuscript was extended by a condensed form of the concept description as given in the previous section. Section 2 was expanded with an additional subsection on the basic concept of coarse graining (section 2.2) and a modification of section "Radial coarse graining transmissivity" (section 2.3). Accordingly, minor adaption were performed in the introduction and in subsection 2.4.

A lesser but not insignificant issue with RCG is the treatment of RCG transmissivity as deterministic: there is nothing in the SA approach to guarantee that weighted volume averaging of randomly varying transmissivity would itself not be random, albeit with a lesser variance (but longer correlation scales).

We agree with the referee, that the weighted volume average of a randomly varying transmissivity is itself random, with a lesser variance but longer correlation scales. This is exactly what is stated by *Attinger* [2003] (step 1, point 9). A note on that is added to the manuscript (page 6, lines 70 - 72). The same is valid for Radial Coarse Graining (step 3). However, the derivation of an effective mean transmissivity for well flow includes averaging (step 4), which renders the RCG-transmissivity deterministic. An explanation on this issue is added to the manuscript (page 6 and 7, first two paragraphs of section 2.3).

On a minor note, it would have been fair for Zech and Attinger to juxtapose their proposed pumping test interpretation method with that of Neuman et al. [2004].

We thank the referee for the advise to recapitulate the results of *Neuman et al.* [2004] and juxtapose the work to ours. It helped to improve the manuscript significantly, especially with respect to the multi-point strategy to analyze 'measured' drawdown data. It inspired us to do a similar analysis, which is added to the manuscript as section 4. Details are provided in the response to the other referee.

The work of Neuman et al. [2004] was juxtaposed to the proposed pumping test interpretation method in the introduction by stating shortly the approach of Neuman et al. [2004] and the differences to our approach (page 3, lines 94 – page 4, line 12). Furthermore, Neuman et al. [2004] stated similar regions of impact for variance and correlation length. Their results support our findings, that $T(r) = T_{\rm G}$ for $r \ge 2\ell$ and $T(r) \rightarrow T_{\rm H}$ for $r \rightarrow r_w$. A statement on that was added to section 2.5 (page 9, lines 89–90).

I regret that, given the above fundamental inconsistencies, I cannot recommend publication of the technical note by Zech and Attinger in HESS.

We hope we could clarify the points mentioned by the referee as "fundamental inconsistencies". We are aware of the fact, that the RCG-approach is not a mathematical rigorous one, but a problemadapted based on reasonable physical assumptions. The comparison with numerical simulations showed an extremely good match between the RCG-hydraulic head solution and the simulated ensemble mean drawdown, which we feel underlines, that our approach has its value for 2D pumping test interpretation.

Response to Comment of P. Trinchero

First, we thank the referee for his fruitful comments, which helped to improve the manuscript a lot.

This version of the paper is mostly focused on assessing the accuracy of the estimates obtained when applying the solution over the ensemble. It is just a personal opinion, but I do not find this part of the document particularly interesting as (i) effective flow parameters have been extensively studied by lots of previous works (e.g. Sanchez-Vila et al. [2006] and references therein) and (ii) the estimation of variance and correlation length from the ensemble is a nice exercise but has no real applicability.

We agree with the reviewer, that the application of the solution to single realizations is more interesting with regard to the interpretation of field pumping test. The part of the paper, where the solution is tested against the ensemble mean, is rather of technical nature and therefore kept short. The corresponding section 3.2 in the manuscript aims to confirm the appropriateness of the Radial Coarse Graining (RCG) approach for interpreting pumping test in heterogeneous media by showing the agreement with well known effective parameters for well flow. It can be understood as the numerical proof of the hypothesis taken in the derivation of the RCG approach. We feel, that this is necessary, specifically with regard to the comment of the other referee S.P. Neuman. It is further aimed to show how the stochastic parameters of the log-normal distributed media can be directly estimated from drawdown data without going a detour on effective or equivalent transmissivity descriptions or using type curves.

As I said, I think that the real added value of this work is when it is applied to single realizations. Thus, I think that the examples presented in the document are not really exhaustive. For instance, the solution is tested only over a few realizations of set A (Table 1), which has a relatively small variance. What would happen with more challenging realizations (e.g. set C/D or even E/F)? Also, from the two selected realizations we observe some obvious (but still interesting) effect; i.e. when the contrast of transmissivity between the near and far field is modest, almost no information can be inferred whereas when this contrast increases, the accuracy of the estimation also increases. I think that this need to be analyzed in a more rigorous way for instance by using (individually) the whole set of realizations. Scatter plots of T_{well}/T_G vs. $\hat{\ell}/\ell$ would help to get insight into the range of applicability of the solution and its dependence on the contrasts of transmissivity.

As recommended by the reviewer, we expanded the analysis of single realizations, in particular for highly heterogeneous media (Ensemble D with $\sigma^2 = 2.25$ and Ensemble E with $\sigma^2 = 4$). Inspired by the analysis of virtual pumping test campaigns as done e.g. by Neuman [2004], Copty and Findikakis [2004], Firmani

[2006], we developed and tested a sampling strategy. Additional pumping test simulations were conducted to test the feasibility of the effective well flow method for interpreting a series of steady state pumping tests within a single aquifer.

The procedure as well as results are presented in the newly created section 4, including new Figures 5 and 6 and Table 2. Accordingly, minor adaption were necessary in the introduction (page 4, lines 19–21). The last paragraph of section 3.3 was removed, since the general statement is exactly what is shown in detail in section 4.

The referee further suggests to present results for the whole set of realizations, e.g. by scatter plots. We see the point, that effects observed in single realizations are difficult to interpret with respect to the entire ensemble. We aimed to give credit to that point by presenting a boxplot of the estimation results for 100 realizations in Figure 4. However, the plot might not have provided as much information as we wanted it to. We tested the proposed scatter plots, but they are difficult to interpret and do not provide additional information. Instead we tested histogram plots for the estimation results of the entire ensemble of N = 5000 realizations. The histogram plots better support the discussion in section 3.3. Therefore, we modified Figure 4 and substituted the barplot by the histogram plots (page 22). Minor adaption were made in the corresponding paragraph in section 3.3 and the caption of Figure 4.

I have also two minor comments:

- a differential operator is missing in eq.(2) and
- I think that set H of Table 1 is never used.

The differential operator was added in Eq. (2) (page 4, line 27) and set H was removed from Table 1 (page 18).

Additional Major Changes

We aim to adapted the title: "Analytical Drawdown Solution for Steady State Pumping Tests in Twodimensional Isotropic Heterogeneous Aquifers" instead of "Technical Note: Analytical Solution for the Mean Drawdown of Steady State Pumping Tests in Two-dimensional Isotropic Heterogeneous Aquifers". Manuscript prepared for Hydrol. Earth Syst. Sci. with version 2014/09/16 7.15 Copernicus papers of the LATEX class copernicus.cls. Date: 22 December 2015

Technical Note: Analytical **Drawdown** Solution for the Mean Drawdown of Steady State Pumping Tests in Two-dimensional Isotropic Heterogeneous Aquifers

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Abstract.

A new method is presented which allows to interpret steady state pumping test in heterogeneous isotropic transmissivity fields. In contrast to mean uniform flow, the pumping test drawdowns in heterogeneous media cannot be described by a single effective or equivalent value of hydraulic trans-

- 5 missivity. A radially depending description of transmissivity is required, including the parameters of aquifer heterogeneitylog-transmissivity: mean, variance and correlation length. Such a model is provided by the upscaling procedure Radial Coarse Graining, which describes the transition of near well to far field transmissivity effectively. Based on the Radial Coarse Graining Transmissivitythis approach, an analytical solution for a steady state pumping test drawdown is derived. The so-called
- 10 effective well flow solution is derived for two cases: the ensemble mean of pumping tests and the drawdown at an individual heterogeneous transmissivity field. The analytical form of the solution allows to inversely estimate the parameters of aquifer heterogeneity from pumping test data. This is shown making use of virtual pumping test datatests, for both cases the ensemble mean drawdown and pumping tests at individual transmissivity fields. The effective well flow solution reproduces the
- 15 drawdown for two-dimensional pumping tests in heterogeneous media and-in contrast to Thiem's solution for homogeneous media. Multiple pumping tests at an individual transmissivity fields, combined in a sampling strategy, are analyzed making use of the effective well flow solution to show that all statistical parameters of aquifer heterogeneity can be inferred under field conditions. Thus, the presented method is a promising tool to estimate parameters of aquifer heterogeneity, in partic-
- 20 ular for the variance and horizontal correlation length of log-transmissivity fields from steady state pumping test measurements.

1 Introduction

head at an arbitrary distance R from the well.

Pumping tests are a widely used tool to identify horizontal hydraulic conductivity, which is the parameter determining the groundwater flow velocity. Analytical solutions of the radial flow equation

25 are used in practice to analyze measured drawdowns. In general, these solutions assume a constant homogeneous hydraulic conductivity like Thiem's solution for steady state (Thiem, 1906):

$$h_{\text{Thiem}}(r) = -\frac{Q_{\text{w}}}{2\pi D K_{\text{h}}} \ln \frac{r}{R} + h(R).$$
(1)

Thiem's solution (1) gives the hydraulic head $h_{\text{Thiem}}(r)$ depending on the radial distance r from the well for homogeneous horizontal hydraulic conductivity K_{h} . It is valid in a confined aquifer of thickness D with fully penetrating well and a constant discharge Q_{w} , h(R) is a known reference

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In large scale pumping tests the vertical extension of the aquifer is negligible compared to horizontal aquifer extend. Thus, flow is assumed to be horizontal and modelled as two-dimensional. Hydraulic conductivity is then replaced by transmissivity, which is defined as the product of con-

35 ductivity and aquifer thickness $T = K_h D$. In the following, transmissivity will be used instead of horizontal hydraulic conductivity, since the focus of the work will be on two-dimensional well flow.

Most natural aquifers exhibit geological heterogeneity in the sedimentary composition, which evolved from the complex geomorphological processes through which they were formed. In particular, transmissivity shows a strong spatial variability. Values measured in the field vary over orders

of magnitude (Gelhar, 1993). Geostatistical distributions are generally used to capture the effects of aquifer heterogeneities. Transmissivity T(x) is modelled as log-normally distributed spatial random function: log T(x) = Y(x) is normally distributed with a Gaussian probability density function pdf_Y(x) = 1/(2πσ²) exp (-(x-μ)²/(2σ²)) in uni-variate form with μ and σ² being the mean and the variance of Y, respectively. The correlation structure of transmissivity in space is captured by a co-variance model Cov [T(x + s), T(x)] = exp (2μ + σ² + CV_Y(s)).

In the stochastic framework, the solution of the radial flow equation with a log-normal distributed transmissivity is also a random spatial function. Since the solution of the stochastic differential equation is out of scope, the focus of investigation was on homogeneous substitute values for describing well flow effectively. As a first approach, Thiem's solution (1) was applied to pumping tests in het-

- 50 erogeneous media. However, this requires a representative transmissivity value T for the whole range of the depression cone (Matheron, 1967), which does not exist. Effective or equivalent descriptions of transmissivity in pumping tests were investigated, e.g. by Desbarats (1992); Sánchez-Vila et al. (1999); Neuman et al. (2007); Dagan and Lessoff (2007); Schneider and Attinger (2008) and many more. For a detailed review see Sánchez-Vila et al. (2006).
- 55 In contrast to mean uniform flow, the pumping test drawdowns in heterogeneous media cannot be described by a single constant value of transmissivity (Matheron, 1967). Different transmissivities characterize the behavior near and far from the well: the representative value close to the well is

the harmonic mean of the log-normal transmissivity $T_{\rm H} = \exp\left(\mu - \frac{1}{2}\sigma^2\right) = T_{\rm G}\exp\left(-\frac{1}{2}\sigma^2\right)$. With increasing distance from the well, the drawdown behaviour behavior is characterized by the effective transmissivity for uniform flow, which is the geometric mean $T_{\rm G} = \exp(\mu)$ for flow in two

dimensional isotropic porous media.

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It seems obvious, that a representative description of transmissivity for well flow needs to be a radially depending function, which interpolates between the harmonic and the geometric mean. The equivalent transmissivity T_{eq} is a well established approach of a radially depending description, visu-

- alized in Fig. 1a. $T_{eq}(r) = -\frac{Q_w}{2\pi(h(r) h(r_w))} \ln \frac{r}{r_w}$ was derived from Thiem's solution (1) (Matheron, 1967; Indelman et al., 1996; Dagan and Lessoff, 2007). In this sense, the equivalent transmissivity T_{eq} is defined as the value for a homogeneous medium, which reproduces locally the same total outflow as observed in the heterogeneous domain of radius r. T_{eq} is strongly impacted by the reference point r_w and the corresponding head $h(r_w)$, which is generally chosen to be the drawdown at
- 70 the well $h(r_w)$. Therefore, the equivalent conductivity stays close to the harmonic mean T_H , which is representative for the drawdown behaviour behavior at the well (Fig. 1a). It takes more than 20 correlation length for T_{eq} to reach the far field representative value of T_G .

It is important to mention, that T_{eq} is not constructed to reproduce the drawdown which was used for calculating T_{eq} . Strictly speaking, replacing the heterogeneous transmissivity field with the

75 equivalent transmissivity in a single forward model does not give the drawdown, with which T_{eq} was constructed as visualized in Fig. 1b. Instead $h_{eq}(r)$ stays close to Thiem's solution with T_{H} as homogeneous substitute value.

Schneider and Attinger (2008) introduced a novel approach to describe well flow effectively. They derived a radial adapted transmissivity $T_{RCG}(r)$ by applying the upscaling technique Coarse Grain-

80 ing to well flow. $T_{\text{RCG}}(r)$ does not only depend on the radial distance r but also on the statistical parameters of aquifer heterogeneity T_{G} , σ^2 and ℓ . $T_{\text{RCG}}(r)$ captures the transition from near well to far field representative transmissivities, based on the radial distance to the well and the parameters of aquifer heterogeneity, as visualized in Fig. 1a.

In this study, an analytical solution for the hydraulic head $h_{efw}(r)$ is presented, which is based on the Radial Coarse Graining transmissivity $T_{RCG}(r)$ as an extension to the work of Schneider and Attinger (2008). Similar work has been done by (Zech et al., 2012) for pumping test in three dimensional porous media. The effective well flow solution $h_{efw}(r)$ describes the mean depression cone of a pumping test in two dimensional heterogeneous media effectively. It can be interpreted as an extension of Thiem's formula (1) to log-normal distributed heterogeneous media. It accounts for

90 the statistical parameters $T_{\rm G}$, σ^2 and ℓ and thus allows to inversely estimate them from measured drawdown data. In contrast to existing head solutions for well flow, $h_{\rm efw}(r)$ is not limited to low variances, but is applicable to highly heterogeneous media with variances $\sigma^2 \gg 1$.

In a similar approach Neuman et al. (2004) presented a graphical approach to estimate the statistical parameters of random transmissivity on the basis of steady state head data. The authors 95 constructed a mathematical description for the apparent transmissivity T_a(r) as function of the radial distance to the well r from theoretical findings of near and far field representative transmissivity and a cubic polynomial interpolation in between. From T_a(r) the authors constructed type curves for the hydraulic head, depending on the variance σ² and the correlation length *l*. Neuman et al. (2004) further gave a multi-point strategy to analyze virtually measured drawdown
100 data by type curve matching including parameter estimation.

The Radial Coarse Graining approach is similar to that of Neuman et al. (2004) in the idea of deriving a solution for the head drawdown for well flow depending on the statistics of the random transmissivity using an effective radial depending transmissivity. Major differences are: (i) the Radial Coarse Graining Transmissivity $T_{\text{RCG}}(r)$ is not based on results of Monte Carlo simulations,

- 105 but is derived from upscaling with physically motivated approximations; (ii) the functional form of $T_{\text{RCG}}(r)$ is different from the expression for T_{a} of Neuman et al. (2004); (iii) the effective well flow solution $h_{\text{efw}}(r)$ is derived analytically by solving the head equation; providing a closed form mathematical expression instead of type curves; (iv) inverse parameter estimation can be done by minimizing the difference between the measured drawdown data and $h_{\text{efw}}(r)$ instead of type curves
- 110 matching. The effective well flow method will be tested in a similar multi-point sampling strategy to analyze measured drawdown data of individual heterogeneous transmissivity fields as done by Neuman et al. (2004) as well as others (Copty and Findikakis, 2004; Firmani et al., 2006).

The work is organized the following: Sect. 2 is dedicated to the method of Radial Coarse Graining and the derivation of the effective well flow head solution. Thereby, The concept of Radial Coarse

- 115 Graining is explained in detail. Furthermore it is distinguished between the effective well flow solution for an ensemble mean and single realizations of heterogeneous transmissivity fields. Section 3 contains the application of the effective well flow solution to simulated pumping tests. It is shown, that $h_{efw}(r)$ reproduces the drawdown in heterogeneous media and can be used to inversely estimate the statistical parameters of aquifer heterogeneity for both, ensemble mean and single realizations. In
- 120 Sect. 4, a sampling strategy is presented to infer the parameters of aquifer heterogeneity of an individual transmissivity fields from multiple pumping tests at multiple locations making use of $h_{\text{efw}}(r)$. Concluding remarks are given in Sect. 5.

2 Radial coarse graining transmissivity and effective well flow head

2.1 Steady state well flow with radially depending transmissivity

125 The drawdown of a steady state pumping test with a radially depending transmissivity T(r) is given as the solution of the differential equation:

$$0 = T(r)\left(\frac{1}{r}\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2h}{\mathrm{d}r^2}\right) + \frac{\mathrm{d}T(r)}{\mathrm{d}r}\frac{\mathrm{d}h}{\mathrm{d}r} = \left(\frac{1}{r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}r}\right)\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\mathrm{d}^2h}{\mathrm{d}r^2}.$$
(2)

The equation can be solved in $\frac{dh}{dr}$ by separation of variables, resulting in $\frac{dh}{dr} = C_1 \frac{1}{rT(r)}$. The hydraulic head h(r) is then given as the solution of the integral

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$$h(r_2) - h(r_1) = C_1 \int_{r_1}^{r_2} \frac{1}{rT(r)} dr.$$
 (3)

The integration constant C_1 is determined by the boundary condition. Supposing a constant flux boundary condition at the well, gives $Q_w = -2\pi r_w T(r_w) \frac{dh}{dr}(r_w) = -2\pi r_w T(r_w) \frac{C_1}{r_w T(r_w)}$ and thus, $C_1 = -\frac{Q_w}{2\pi}$.

Equation (3) is the general solution of the radial flow equation (2) for radially depending trans-135 missivity, independent of the functional form of T(r).

When comparing Eq. (3) with the definition of the equivalent transmissivity, it becomes obvious that T_{eq} is not constructed to solve the equation. The combination of both formulas results in $\int_{r_1}^{r_2} \frac{1}{rT(r)} dr = \frac{1}{T_{eq}(r)} \ln \frac{r_1}{r_2}$, which is only fulfilled, when T(r) is constant in r.

2.2 Concept of Radial Coarse Graining

- 140 A radially depending transmissivity for log-normal distributed media with Gaussian correlation structure was derived by Schneider and Attinger (2008), denoted as the Radial Coarse Graining Transmissivity $T_{RCG}(r)$. It was derived making use of the upscaling procedure Coarse Graining, introduced for uniform flow by Attinger (2003). The basic idea of Radial Coarse Graining is to perform a spatial filtering on the flow equation which is appropriate to the non-uniform flow character
- 145 of a pumping test. The filter was chosen proportional to the radial distance from the well. Hence, the filter length is very small close to the well, so nearly no filtering is applied and the heterogeneity of the local transmissivities is resolved. Far away from the well, the filter volumes are very large and the local heterogeneous transmissivity values are replaced by the effective value for uniform flow. Detailed discussions on Radial Coarse Graining can be found in Schneider and Attinger (2008)and
- 150 Zech et al. (2012).

A radial-depending transmissivity for log-normally distributed media with Gaussian correlation structure was derived by Schneider and Attinger (2008), denoted as $T_{RCG}(r)$. It is based on the upscaling approach Radial Coarse Graining which follows the basic idea of a spatial filtering of the flow equation appropriate to the non-uniform flow character of a pumping test.

- 155 The approach was further developed for three-dimensional well flow by Zech et al. (2012) introducing an effective well flow solution for the hydraulic head. Similarly, the concept of Radial Coarse Graining for two-dimensional well flow will be expanded in the following. The process can be best explained within five major steps:
 - 1. Coarse Graining for uniform flow
- 160 2. Transfer of Coarse Graining to radial flow conditions

- 3. Overcome non-locality of head equation for non-uniform flow
- 4. Effective Radial Coarse Graining transmissivity
- 5. Derivation of effective well flow head

The first three steps will be discussed shortly in the following. Step 4 and 5 will be explained in detail in Sec.2.3 and Sec. 2.4. 165

The Coarse Graining approach for uniform flow (step 1) was introduced by Attinger (2003), including derivation, mathematical proof and numerical simulations. The author started at a spatially variable transmissivity field T(x) and derived a filtered version $T_{\lambda}^{CG}(x)$, where fluctuation smaller than a cut-off length λ are filtered out. The resulting upscaled Coarse Graining transmissivity field

 $T_{\lambda}^{\text{CG}}(\boldsymbol{x})$ represents a log-normal distributed field with a smaller variance $\langle \sigma^2 \rangle_{\lambda}$, but larger correla-170 tion length $\langle \ell \rangle_{\lambda}$. Attinger (2003) showed, that the statistical parameters relate to the parameter of the unfiltered field by $\langle \sigma^2 \rangle_{\lambda} \equiv \sigma^2 \frac{\ell^2}{\ell^2 + \lambda^2/4}$ and $\langle \ell \rangle_{\lambda} \equiv \left(\ell^2 + \lambda^2/4\right)^{1/2}$.

The concept of Coarse Graining can similarly be applied to non-uniform flow (step 2). The critical point when transferring the results of Attinger (2003) to radial flow is the Fourier back-

175 transformation of the filtered head equation after localization. For uniform flow, this can be done due to the reasonable assumption of constant head gradient. For non-uniform flow, this assumption is not valid and thus, localization is not possible straight ahead.

A heuristic approach is taken to overcome the limitation of non-locality for well flow (step 3). Conditions of a quasi-constant head gradient are constructed by adapting the size of the volume

- elements over which flow takes place. The head gradient in well flow is proportional to the reciprocal 180 of the distance to the well: $\nabla h(r) = \frac{h(r) - h(r + \Delta r)}{\Delta r} \propto \frac{1}{r}$. The gradient is constant for volumes of size proportional to r. The step can be understood as a change from an equidistant Cartesian coordinate system to a polar coordinate system with cell sizes increasing with distance to the center, where the pumping well is located. Under this adaption, localization can be performed and so the following steps of the Coarse Graining procedure.
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The changed coordinate system impacts on the scaling procedure and the parameter λ . For uniform flow, λ is constant. Adapted to well flow, the scaling parameter needs to be proportional to r, because the filter width increases with distance to the well, thus $\lambda/2 = \zeta r$. The result is an upscaled lognormal distributed field $T_r^{\text{RCG}}(\boldsymbol{x})$ with an arithmetic mean $T_A^{\text{RCG}}(r)$ and a filtered fluctuation term.

The step was presented by Schneider and Attinger (2008). It is not performed in a mathematically 190 straight way, but problem adapted to well flow conditions.

2.3 Radial Coarse Graining Transmissivity

Spatial heterogeneity is still resolved in $T_r^{\text{RCG}}(\boldsymbol{x})$, although reduced to the amount relevant to the pumping test. A further step of averaging is necessary to derive an effective description of the trans-

missivity for well flow conditions. Thereby, two different aspects are of interest: (i) an effective 195 transmissivity for an ensemble and (ii) effective transmissivity for an individual field.

Schneider and Attinger (2008) presented two forms of $T_{RCG}(r)$, one for an ensemble of pumping tests and an adapted version for drawdowns of individual pumping tests, which are different from the ensemble behaviour due to a lack of ergodicity at the well. The ensemble version of $T_{RCG}(r)$ is given as

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A result for an effective ensemble description is derived by averaging $T_r^{
m RCG}({m x})$ appropriate to well flow condition. The averaging rule is determined by the boundary condition at the well, which is the harmonic mean for two-dimensional well flow (Dagan, 1989). Thus, the effective mean transmissivity, noticed by $T_{\rm RCG}(r)$, is calculated via the theoretical description of the harmonic

mean for log-normal distributed fields making use of the variance of the coarsened transmissivity 205 $\langle \sigma^2 \rangle_r = \frac{\sigma^2}{1 + \zeta^2 r^2 / \ell^2}$:

$$T_{\rm RCG}(r) = T_{\rm G} \exp\left(-\langle \sigma^2 \rangle_r / 2\right) = T_{\rm G} \exp\left(-\frac{1}{2} \frac{\sigma^2}{(1+\zeta^2 r^2/\ell^2)}\right) , \qquad (4)$$

where r is the radial distance to the well, $T_{\rm G}$ is the geometric mean, σ^2 is the variance and ℓ is the correlation length of the log-normally distributed transmissivity $T(\mathbf{x})$. ζ is a factor of proportionality, which was determined to be $\zeta = 1.6$, as discussed in detail by Zech et al. (2012).

 $T_{\rm RCG}(r)$ allows a transition from near field transmissivity can be interpreted as interpolating function between the representative transmissivity at the well $T_{\rm H} = T_{\rm G} \exp\left(-\frac{1}{2}\sigma^2\right)$ to the far field value $T_{\rm G}$ depending on the radial distance r, controlled by the correlation length ℓ (Fig. 1a).

- The adapted radial coarse graining transmissivity accounts for local effects of individual pumping **215** tests and is given by An effective description of well flow transmissivity for an individual field is derived from Eq (4). The behavior of individual fields is different especially at the well due to a lack of ergodicity there. The local transmissivity at the well $T_{\rm well}$ is not identical to the harmonic mean $T_{\rm H}$ as expected for the ensemble, but refers to the specific value of transmissivity at the well location. An adapted radial coarse graining transmissivity accounts for local effects by replacing the harmonic mean $T_{\rm H} = T_{\rm G} \exp\left(-\frac{1}{2}\sigma^2\right)$ by $T_{\rm well}$. In Eq. (4) this refers to substituting the variance by 220

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 $-\frac{1}{2}\sigma^2 = \ln T_{\rm well} - \ln T_{\rm G}$ and thus,

$$T_{\rm RCG}^{\rm local}(r) = T_{\rm G} \exp\left(\frac{\ln T_{\rm well} - \ln T_{\rm G}}{1 + \zeta^2 r^2/\ell^2}\right) = T_{\rm well}^{\frac{1}{1+\zeta^2 r^2/\ell^2}} T_{\rm G}^{\frac{\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}} .$$
(5)

where T_{well} is the local transmissivity at the well of an individual transmissivity field . $T_{\text{RCG}}^{\text{local}}(r)$ was derived from $T_{\text{RCG}}(r)$ by substituting the variance by $\sigma^2 = -2 \ln \frac{T_{\text{well}}}{T_{\text{G}}}$. The substitution is derived

225 from the relation of the variance to the harmonic mean, as asymptotic representative value at the well, $T_{\rm H} = T_{\rm G} \exp\left(\frac{1}{2}\sigma^2\right)$, thus $\sigma^2 = -2\ln\frac{T_{\rm H}}{T_{\rm G}}$. The theoretical value of $T_{\rm H}$ is than replaced by the local transmissivity at the well Twell. Due to the lack of ergodicity at the well for a single realization, T_{well} generally does not equal the harmonic mean transmissivity T_{H} of the entire field T(x).

 $T_{\rm RCG}^{\rm local}(r)$ interpolates between the specific transmissivity at the well $T_{\rm well}$ and the far field value $T_{\rm G}$ 230 depending on the radial distance r and the correlation length ℓ .

2.4 Effective well flow head

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Explicit results for the hydraulic head drawdown in steady state pumping test with a radially depending transmissivity can be are achieved by solving the integral in Eq. (3) making use of $T_{\text{RCG}}(r)$ (Eq. 4). Details on the derivation of $h_{\text{efw}}(r)$ can be found in the Appendix. A fully analytical solution, called the The result is the effective well flow head $h_{\text{efw}}(r)$ is given by

$$h_{\rm efw}(r) = -\frac{Q_{\rm w}}{4\pi T_{\rm G}} \exp\left(\frac{\sigma^2}{2}\right) \left(\Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 r^2/\ell^2}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{-\zeta^2 R^2/\ell^2}{1+\zeta^2 R^2/\ell^2}\right)\right) + \frac{Q_{\rm w}}{4\pi T_{\rm G}} \left(\Gamma\left(\frac{\sigma^2}{2} \frac{1}{1+\zeta^2 r^2/\ell^2}\right) - \Gamma\left(\frac{\sigma^2}{2} \frac{1}{1+\zeta^2 R^2/\ell^2}\right)\right) + h_R,$$
(6)

where r is the radial distance from the well, Q_w is the pumping rate, T_G is the geometric mean, σ^2 is the log-transmissivity variance, and ℓ is the correlation length. Again, ζ is the factor of proportionality determined to be 1.6 and R is an arbitrary distance from the well, where the hydraulic head $h(R) = h_R$ is known. $\Pr(x) = \int_{-\infty}^x \frac{\exp(z)}{z} dz$ is the exponential integral function with-

$$\Gamma(x) = \int_{-\infty}^{x} \frac{\exp(z)}{z} \, \mathrm{d}z.$$

Details on the derivation of $h_{efw}(r)$ can be found in the Appendix.

An approximate solution $h_{efw}^{approx}(r)$ can be is derived from Eq. (6) by making use of an approximation of the exponential integral function $\Gamma\Gamma(x)$. Details are given in the Appendix.

$$h_{\text{efw}}^{\text{approx}}(r) = -\frac{Q_{\text{w}}}{2\pi T_{\text{H}}} \ln \frac{r}{R} - \frac{Q_{\text{w}}}{4\pi T_{\text{G}}} \left(e^{\frac{\sigma^2}{2}} - 1 \right) \\ \cdot \left(\ln \frac{1 + \zeta^2 R^2 / \ell^2}{1 + \zeta^2 r^2 / \ell^2} + \frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 r^2 / \ell^2)} - \frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 R^2 / \ell^2)} \right) + h_R,$$
(7)

h_{efw}(r) is constructed to describe the mean drawdown of a pumping test in two dimensional heterogeneous media effectively. The drawdown curve of h_{efw}(r) for a specific choice of parameters
(Ensemble A of Table 1) is given in Fig. 1b in comparison to the equivalent drawdown h_{eq}(r), as the solution of the radial flow equation using the equivalent transmissivity T_{eq}, based on the same statistical parameters.

The effective well flow solution $h_{\text{efw}}(r)$ can be adapted to analyze individual pumping tests by using $T_{\text{RCG}}^{\text{local}}(r)$ (Eq. 5) instead of $T_{\text{RCG}}(r)$ (Eq. 4). The appropriate local effective well flow solutions $h_{\text{efw}}^{\text{local}}(r)$ are than given similarly to Eqsis than given by Eq. (6) and (7) with $\frac{\sigma^2}{2}$ substituted by $-\ln \frac{T_{\text{well}}}{T_{C}}$ and T_{H} substituted by T_{well} .

The local effective well flow solution $h_{\text{efw}}^{\text{local}}(r)$ can be used to analyze drawdowns of single pumping tests in heterogeneous media as encountered in practice. The solution is adapted to the lack of

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ergodicity at the well, by using transformed parameters T_{well} , T_{G} and ℓ . The geometric mean T_{G} and the correlation length ℓ for a single realization should also be interpreted as local values, not 260 necessarily representing the mean values of the entire field, but those of the pumping well vicinity. Owing to the nature of the pumping test, the drawdown signal does not sample the heterogeneity in transmissivity in a symmetric way, but the shape of the drawdown is mainly determined by the local heterogeneity close to well.

265 2.5 Impact of parameters

The analytical form of $h_{\rm efw}(r)$ allows to analyze the impact of the statistical parameters $T_{
m G}$, σ^2 and ℓ on the drawdown. The drawdown behaviour behavior for different choices of parameters can be seen in Fig. 2, which is discussed in detail later on.

Every parameter impacts on the drawdown in a different region. The geometric mean $T_{\rm G}$ as rep-270 resentative value for mean uniform flow determines the far field behaviour behavior. The variance σ^2 determines the drawdown at the well due to the dependence of the near-well asymptotic value $T_{\rm H} = T_{\rm G} \exp\left(-\frac{1}{2}\sigma^2\right)$. The larger the variance the larger are the differences between $T_{\rm G}$ and $T_{\rm H}$ and the steeper is the drawdown at the well. Whereas, the correlation length ℓ determines the transition from near to far field behavior.

275 The asymptotic behaviour behavior of $h_{efw}(r)$ can easily be analyzed using approximate functional description in Eq. (7): for distances close to the well, thus $r \ll \ell$, $h_{\rm efw}(r)$ converges to Thiem's solution with $T_{\rm H}$ as homogeneous substitute value. All terms, except the first one in Eq. (7), tend to zero or become constant. Thus, they are negligible compared to logarithmic first term for very small r,

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$$h_{\text{efw}}^{\text{approx}}(r \ll \ell) \approx -\frac{Q_{\text{w}}}{2\pi T_{\text{H}}} \ln \frac{r}{R} - \frac{Q_{\text{w}}}{4\pi T_{\text{G}}} \left(e^{\frac{\sigma^2}{2}} - 1\right) \cdot \left(\ln\left(1 + \zeta^2 R^2/\ell^2\right) + \frac{\sigma^2}{2}\right) + h_R$$

 $\approx -\frac{Q_{\text{w}}}{2\pi T_{\text{H}}} \ln \frac{r}{R} + h_R.$

For large distances from the well, i.e. $r \gg \ell$, the solution converges to Thiem's solution with $T_{\rm G}$ as homogeneous substitute value. The third and fourth term in Eq. (7) tend to zero and cancel out. The ones in the second term can be neglected, thus

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$$h_{\text{efw}}^{\text{approx}}(r \gg \ell) \approx -\frac{Q_{\text{w}}}{2\pi T_{\text{H}}} \ln \frac{r}{R} - \left(\frac{Q_{\text{w}}}{4\pi T_{\text{H}}} - \frac{Q_{\text{w}}}{4\pi T_{\text{G}}}\right) \cdot \ln \left(\frac{\zeta^2 R^2/\ell^2}{\zeta^2 r^2/\ell^2}\right) + h_R \approx -\frac{Q_{\text{w}}}{2\pi T_{\text{G}}} \ln \frac{r}{R} + h_R$$

The larger the correlation length ℓ the longer takes the transition of the drawdown from near well to far field behaviour behavior. The influence of ℓ on $h_{efw}(r)$ vanishes quickly with increasing distance to the well. The drawdown reaches the far field behavior after approximately two correlation lengths $h^{\text{efw}}(r > 2\ell) = h_{\text{Thiem}}(r > 2\ell)$ with T_{G} as homogeneous substitute value (Fig. 1). b). These findings are in line with the results of Neuman et al. (2004).

3 Robust estimation of statistical parameters

3.1 Numerical pumping tests

Numerical pumping tests in heterogeneous porous media were generated as artificial measurements. They serve-were used to test the capability of $h_{efw}(r)$ in reproducing the mean drawdown and in estimating the underlying parameters of heterogeneity. Pumping tests were simulated using the finite element software OpenGeoSys. The software was successfully tested against a wide range of benchmarks (Kolditz et al., 2012). Results of a steady state simulation with homogeneous transmissivity are-were in perfect agreement with Thiem's analytical solution Eq. (1).

The numerical grid is-was constructed as a square of 256×256 elements with a uniform grid cell size of 1 m except for cells in the vicinity of the pumping well. The mesh was refined in the range of 4 m around the well, which ensures a fine resolution of the steep head gradients at the well. The well in the center of the mesh is was included as a hollow cylinder with radius $r_w = 0.01$ m. The constant pumping rate of $Q_w = -10^{-4}$ m³ s⁻¹ is was distributed equally to all elements at the well. At the radial distance R = 128 m a constant head of h(R) = 0 m is was applied giving a circular outer head

305 boundary condition.

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Log-normally distributed, Gaussian correlated transmissivity fields were generated using a statistical field generator based on the randomization method (Heße et al., 2014). Multiple ensembles with different statistical parameter values were generated, including high variances up to $\sigma^2 = 4$ (Table 1). Ensemble A with $T_{\rm G} = 10^{-4} \,\mathrm{m}^2 \,\mathrm{s}^{-1}$, $\sigma^2 = 1$ and $\ell = 10 \,\mathrm{m}$ will serve served as reference ensemble for specific cases. Every ensemble consist consists of N = 5000 realizations, which was

tested as sufficiently large to ensure ensemble convergence.

Pumping test simulations are-were post-processed by performing an angular and an ensemble average. For every realization *i*, the simulated drawdown $\langle h_i(r,\phi) \rangle$ at the radial and angular location (r,ϕ) in polar coordinates is-was averaged over the four axial directions: $\langle h_i(r) \rangle = \sum_{\phi_j} \langle h_i(r,\phi_j) \rangle$. The ensemble mean is-was the sum over the angular mean of all individual realizations: $\overline{\langle h_i(r) \rangle} = \langle h_i(r) \rangle$

315 The ensemble mean is was the sum over the angular mean of all individual realizations: $\overline{\langle h(r) \rangle} = \sum_{i=1}^{N} \langle h_i(r) \rangle$.

Non-linear regression is-was used to find the best fitting values for the statistical parameters, denoted by $\hat{T}_{\rm G}$, $\hat{\sigma}^2$, and $\hat{\ell}$. The best fit estimates are fitting estimates were determined by minimizing the mean square error of the difference between the analytical solution $h_{\rm efw}(r)$ and the measured drawdown samples h(r): $\min_{T_{\rm G},\sigma^2,\ell} \sum_r (h(r) - h_{\rm efw}(r))^2$ making use of the Levenberg–Marquardt

algorithm. The reliability of the estimated parameters is was evaluated using 95%-confidence intervals.

The estimation procedure is-was applied to the head measurements at every meter distance starting at the well up to a distance of 80 m. The range beyond 80 m is-was not taken into account to avoid boundary effects. The range of 80 m includes at least 4 correlation lengths for all tested ensembles,

which is sufficient to ensure convergence to the far field behaviourbehavior. The question of the

applicability of $h_{efw}(r)$ on limited head data is of quite complex nature. For a detailed discussion on that issue the reader is referred to Zech et al. (2015).

3.2 Ensemble pumping test interpretation

- 330 We first analyze First, the simulated ensemble means were analyzed making use of the ensemble version of $T_{\text{RCG}}(r)$ and $h_{\text{efw}}(r)$ (Eqs. 4 and 6). Simulated ensemble means $\overline{\langle h(r) \rangle}$ for multiple choices of statistical parameters $T_{\rm G}$, σ^2 and ℓ are visualized in Fig. 2 in combination with $h_{\rm efw}(r)$ for the best fit fitting parameter estimates $\hat{T}_{\rm G}$, $\hat{\sigma}^2$, and $\hat{\ell}$. Input parameters as well as inverse estimation results for all tested ensembles are listed in Table 1.
- 335 The best fit fitting estimates show, that all three parameters can could be inferred from the ensemble mean with a high degree of accuracy. The deviation of the geometric mean from the input value is in general less than 10%, only for high variances the deviations are up to 30%. Variances deviate in a range of 20% and estimated correlation lengths are accurate within 10% of the initial input parameter.
- The confidence intervals of the estimates \hat{T}_{G} and $\hat{\sigma}^{2}$ are very small, showing a high sensitivity 340 of the effective well flow solution $h_{efw}(r)$ towards geometric mean and variance. The confidence intervals of the correlation length are larger due to the dependence of the estimate of $\hat{\ell}$ on the estimates $\hat{T}_{\rm G}$ and $\hat{\sigma}^2$. This is due to the fact, that the correlation length determines the transition from $\hat{T}_{\text{well}} = \hat{T}_{\text{G}} \exp\left(-\frac{1}{2}\hat{\sigma}^2\right)$ to \hat{T}_{G} , which results in larger uncertainties in the estimates of $\hat{\ell}$.

345 3.3 Individual pumping test interpretation

In the following, pumping test drawdowns of individual transmissivity fields are interpreted based on the adaption version $h_{efw}^{local}(r)$ as discussed in Sect. 2.4. The drawdowns along the four axial directions as well as the radial mean for two realizations from Ensemble A ($T_{\rm G} = 10^{-4} \,\mathrm{m^2 \, s^{-1}}$, $\sigma^2 = 1, \ell = 10 \,\mathrm{m}$) are visualized in Fig. 3a and b.

- Both realizations from Fig. 3a and b differ significantly in the value of the local transmissiv-350 ity at the well. The analysis of the transmissivity fields at the well gave "measured" values of $T_{\rm well}^{\rm (a)} = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \text{ and } T_{\rm well}^{\rm (b)} = 1.11 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \text{ sampled values of } < T_{\rm well}^{\rm (a)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (a)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (a)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \times 10^{-4} \, {\rm m}^2 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > = 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm (b)} > 0.204 \, {\rm s}^{-1} \, {\rm sampled values of } < T_{\rm well}^{\rm$ $10^{-4} \mathrm{m}^2 \mathrm{s}^{-1}$ and $\langle T_{\text{well}}^{(b)} \rangle = 1.11 \times 10^{-4} \mathrm{m}^2 \mathrm{s}^{-1}$, which is in both cases far from the theoretical harmonic mean value $T_{\rm H} = 0.61 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$ as being the representative value for the near well 355
- behaviourbehavior.

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Inverse estimation results for the realization in Fig. 3a differ for the drawdowns along the four axial directions $\langle h(r,\phi_i) \rangle$ and the radial mean $\langle h(r) \rangle$: the estimated geometric mean ranges between 1.03×10^{-4} and $1.45 \times 10^{-4} \, \mathrm{m^2 \, s^{-1}}$ for the four axial directions, with an average value of $\hat{T}_{\rm G} = 1.17 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$. The estimates for the local transmissivity at the well are between 0.195×10^{-4} and $0.212 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$, with an average value of $\hat{T}_{\mathrm{well}} = 0.204 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$, which

is exactly the "measured" local transmissivity $T_{well}^{(a)}$ sampled local transmissivity $< T_{well}^{(a)} >$. The value

of $\hat{T}_{well} = 0.204 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$ is equivalent to a local variance of $\hat{\sigma}^2 = 3.49$. The estimated correlation length ranges between 7.95 and 18.15 m, with an average of $\hat{\ell} = 12.77 \,\mathrm{m}$. The differences in the estimates for the drawdowns in different direction for the same realization of transmissivity shows

365 that the parameter estimates reflect local heterogeneity in the vicinity of the well rather than the global statistical parameters of the transmissivity field. This was studied and discussed in detail for pumping tests in three dimensional heterogeneous media by Zech et al. (2015).

Realizations The realization in Fig. 3b does not allow to infer the parameters of variance and correlation length, due to the similarity of T_{well} and T_{G} . Near and far field representative transmis-

370 sivities are nearly identical, thus the pumping test appears to behave like in a homogeneous medium (Fig. 3b). However, the behaviour behavior is not representative but a result of the coincidental choice of the location of the pumping well.

A statistical analysis of the estimation results is presented in Fig. 4 for 100all 5000 realizations of Ensemble A. Estimation results Histogram on the best fitting estimates in normalized form are

- shownas standard box plots, where normalization of results means that they are were divided by the input parameters. It can be inferred that the estimate of the geometric mean \hat{T}_{G} is in general close to the input value T_{G} . A few exceptionally small and large values of \hat{T}_{G} show that the mean transmissivity observed by a pumping test in the vicinity of the well is not necessary close to the mean value of the entire transmissivity field. The estimate of the local transmissivity at the well \hat{T}_{well}
- 380 from the drawdown data-is very close to the "measured" values T_{well} sampled values $\langle T_{well} \rangle$ for nearly all realizations. Thus, the method reproduces reproduced very well the local transmissivity at the well. However, the local value T_{well} of every realization can be far from the theoretical value of T_{H} , where both realizations in Fig. 3 gave example. The estimates of the correlation length show a very large scatter. Exceptionally large and small value for $\hat{\ell}$ refer to realizations, where it is-was
- nearly impossible to infer it due to the similarity of T_{well} and T_G, as for the realization of Fig. 3b. The large range of estimated correlation lengths also point towards the fact that *l* of a single drawdown needs to be interpreted as a highly-local value, which is determined by the transmissivity distribution in the vicinity of the well rather than the distribution of the entire field. However, the median of the normalized estimated correlation lengths is close to one, pointing to the fact that representative values can be inferred by taking the mean from multiple pumping tests.

Representative values of $T_{\rm G}$, σ^2 and ℓ for individual transmissivity fields can be inferred by performing pumping tests at multiple location of the field. Thus, the sampled area increases and the effect of local heterogeneity at the well is reduced. Parameters should be estimated for every test separately and statistically analyzed. Mean values and the range of estimation results can than be

- 395 interpreted as representative for the underlying transmissivity field.
 - 4 Application Example: Single Aquifer Analysis

Pumping test campaigns in the field often include the performance of multiple pumping tests within one aquifer. Drawdown measurements at multiple test locations can be used to gain representative parameters of the heterogeneous transmissivity field. The sampled area increases and the effect of local heterogeneity reduces. In the following, it is shown, how mean $T_{\rm G}$, variance and $\sigma^2 \sigma^2$ and

400 local heterogeneity reduces. In the following, it is shown, how mean $T_{\rm G}$, variance and $\sigma^2 \sigma^2$ and the correlation length ℓ for of an individual transmissivity fields can be inferred making use of a sampling strategy in combination with $h_{\rm efw}^{\rm local}(r)$.

4.1 Sampling Strategy

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The sampling strategy was constructed as pumping test campaign in a virtual aquifer with heterogeneous transmissivity. A series of steady state pumping tests was performed at n different wells. For each test, drawdowns were measured at all n wells and at m additional observation wells. A similar sampling strategy to infer the aquifer statistics from drawdown measurements have been pursued by e.g. Neuman et al. (2004); Copty and Findikakis (2004); Firmani et al. (2006).

The used sampling strategy includes n = 8 pumping wells and m = 4 observation wells. The specific location of all wells are indicated in Fig. 5. All 8 pumping wells are located within a distance of 18 m. The observation wells are located at larger distances and in all four directions. The well locations were designed to gain numerous drawdown measurements in the vicinity of each pumping well to allow a reliable estimation of T_{well} (or σ_{local}^2 , respectively) and ℓ . The additional observation wells provide head observations in the far field to gain a representative value for T_G . The choice of the well locations does not interfere with the refinement of the numerical grid at the pumping well.

Each of the 8 pumping tests was analyzed with $h_{\text{efw}}^{\text{local}}(r)$ (Sect. 2.4). The best fitting estimates \hat{T}_{G} , \hat{T}_{well} , and $\hat{\ell}$ for all tests were inferred by minimizing the difference between the analytical solution and the 12 measurements. Additionally, parameter estimates were inferred by analyzing the drawdown measurements of all tests jointly.

420 4.2 Aquifer Analysis

The sampling strategy was applied to fields of all ensembles A-G (Table 1). Results are presented for two fields: D1 out of Ensemble D ($\sigma^2 = 2.25$, $\ell = 20 \text{ m}$) and E1 out of Ensemble E ($\sigma^2 = 4.0$, $\ell = 10 \text{ m}$). Each field was generated according to the theoretical values defined for the particular ensemble and afterwards analyzed geostatistically to determine the sampled values. The fields D1

425 and E1 are visualized in Fig. 5. The drawdown measurements for all 8 pumping tests at both fields are given in Fig. 6. The inverse estimates as well as theoretical input and sampling values for the statistical parameters are summarized in Table 2.

Analyzing the data from all 8 pumping tests at field D1 jointly yields very close estimates of all parameters $\hat{T}_{\rm G}$, $\hat{T}_{\rm well}$ (corresponding to $\hat{\sigma}^2 = 2.255$), and $\hat{\ell}$ to the theoretical and sampled values.

430 The geometric mean estimate is similar for all of the 8 individual pumping tests. In contrast, the values of \hat{T}_{well} vary within one order of magnitude. This behavior was expected, since \hat{T}_{well} repre-

sents the local transmissivity value at the pumping well. The wide range of estimates is a results of the high variance of the transmissivity field. The estimates of the correlation length $\hat{\ell}$ differ between the individual tests within a reasonable range of a few meters. The only exception is the estimate

- 435 for pumping at PW₅. For this specific pumping test is highly uncertain due to the coincidence of the values of \hat{T}_{well} and \hat{T}_{G} , similar to the realizations in Fig. 3b, as discussed in section 3.3. However, the mean value over the individual tests as well as the estimate from the joint analysis of all measurements gave reliable estimates for the correlation length.
- The analysis of the sampling strategy at field E1 yields similar results as for D1. The geometric 440 mean values $\hat{T}_{\rm G}$ differ little among the 8 individual pumping tests and for the joint analysis. The mean value is double the value as the theoretical one, but close to the sampled geometric mean (Table 2). The local transmissivities $\hat{T}_{\rm well}$ again vary within one order of magnitude, reflecting the high variance of the field. The mean and jointly estimated values are higher than theoretical one, which is in correspondence to the difference in the geometric mean. The estimates of the correlation
- 445 length $\hat{\ell}$ deviate in a reasonable range of a few meters, which reflects the impact of the location of the pumping well with regard to the shape of the correlation structure around the well.

Finally, the analysis shows that representative values of the statistical parameters can be determined by performing pumping test at multiple locations of an individual transmissivity field. It was shown, that $h_{efw}(r)$ is feasible to interpret steady state pumping tests in highly heterogeneous fields.

5 Conclusions

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The analytical effective well flow solution h_{efw}(r) is presented, which can be interpreted as extension of Thiem's equation to heterogeneous media. h_{efw}(r) depends on the statistical parameters of lognormal distributed transmissivity: geometric mean T_G, variance σ² and correlation length l. h_{efw}(r)
455 was derived based on the Radial Coarse Graining transmissivity T_{RCG}(r) introduced by Schneider and Attinger (2008), which interpolates between the near well and far field representative transmissivities for well flow. Simulation of pumping tests were performed in log-normally distributed transmissivity fields and compared with h_{efw}(r). Based on the results, the following conclusions can be drawn:

- 460 1. $h_{\text{efw}}(r)$ describes the mean drawdown of a pumping test in two dimensional heterogeneous isotropic media effectively. It is not limited to small variance, but is tested to reproduce ensemble means for highly heterogeneous media with variances up to $\sigma^2 = 4$.
 - 2. The analytical character of $h_{\text{efw}}(r)$ allows to perform inverse estimation of the statistical parameters of the transmissivity fields from measured drawdowns. Geometric mean T_{G} , variances σ^2 and correlation length ℓ can be estimated for a wide range of parameters with a high accuracy and certainty from ensemble mean drawdowns.

- 3. Parameter estimates from individual drawdowns reflect local heterogeneity at the well rather than the global statistical parameters of the transmissivity field.
- 4. Representative values of geometric mean, variance and correlation length for an individual field of transmissivity can be determined by performing pumping test at multiple locations of that field, estimating the parameters for every test separately and than performing a statistical analysis of the results.

 $h_{\text{efw}}(r)$ is a promising tool to interpret steady state pumping tests in order to infer the statistical parameters of the underlying transmissivity field without time- and cost-intensive laboratory inves-

475 tigations. Future steps will include the expansion of the method to interpret transient pumping test data.

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480 Appendix

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The effective well flow head $h_{efw}(r)$ as solution of the well flow equation (2) is derived by solving the integral (Eq. 3) with the analytical expression of $T_{RCG}(r)$ from Eq. (4).

$$h(r_2) - h(r_1) = \frac{C_1}{T_G} \int_{r_1}^{r_2} \frac{1}{r} \exp\left(\frac{\sigma^2}{2} \frac{1}{(1 + \zeta^2 r^2/\ell^2)}\right) dr.$$
 (A1)

The integral is evaluated analytically by making use of the exponential integral function

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$$\Gamma(x) - \Gamma(X) = \int_{X}^{x} \frac{\exp(z)}{z} dz = \ln \frac{x}{X} + \sum_{k=1}^{\infty} \frac{x^k - X^k}{k!k}.$$
 (A2)

The argument in the exponent in Eq. (A1) is substituted by $z(r) = \frac{\sigma^2}{2} \frac{1}{(1+\zeta^2 r^2/\ell^2)}$ with integrator $dr = -\frac{\ell\sigma^2}{4\zeta z^2} \left(\frac{\sigma^2}{2z} - 1\right)^{-\frac{1}{2}} dz$, furthermore partial fraction decomposition is used, resulting in

$$h(r_{2}) - h(r_{1}) = \frac{C_{1}}{T_{G}} \frac{\sigma^{2}}{4} \int_{z(r_{1})}^{z(r_{2})} \frac{\exp(z)}{z(z - \frac{\sigma^{2}}{2})} dz$$

$$= \frac{C_{1}}{2T_{G}} \int_{z(r_{1}) - \frac{\sigma^{2}}{2}}^{z(r_{2}) - \frac{\sigma^{2}}{2}} \frac{\exp\left(z + \frac{\sigma^{2}}{2}\right)}{z} dz - \frac{C_{1}}{2T_{G}} \int_{z(r_{1})}^{z(r_{2})} \frac{\exp(z)}{z} dz$$

$$= \frac{C_{1}}{2T_{G}} e^{\frac{\sigma^{2}}{2}} \left(\Gamma\left(z(r_{2}) - \frac{\sigma^{2}}{2}\right) - \Gamma\left(z(r_{1}) - \frac{\sigma^{2}}{2}\right) \right)$$

$$- \frac{C_{1}}{2T_{G}} \left(\Gamma(z(r_{2})) - \Gamma(z(r_{1}))\right).$$
(A3)

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The final solution for the effective well flow head as given in Eq. (6) results by re-substituting the abbreviation $z(r) = \frac{\sigma^2}{2} \frac{1}{(1+\zeta^2 r^2/\ell^2)}$ with $r_2 = r$ and $r_1 = R$ and inserting $C_1 = -\frac{Q_w}{2\pi}$ as derived from the constant flux boundary condition (Sect. 2.1).

An approximate formulation of Eq. (A3) can be derived by using the definition of the exponential integral function as infinite sum, given in Eq. (A2) in combination with the relationship $z(r) - \frac{\sigma^2}{2} = \frac{\sigma^2}{2} \frac{-\zeta^2 r^2/\ell^2}{1-\zeta^2 r^2/\ell^2} = z(r) \left(-\frac{\zeta^2 r^2}{\ell^2}\right)$:

$$\begin{split} h(r_{2}) - h(r_{1}) &= \frac{C_{1}}{2T_{G}} e^{\frac{\sigma^{2}}{2}} \left(\ln \frac{z(r_{2}) - \frac{\sigma^{2}}{2}}{z(r_{1}) - \frac{\sigma^{2}}{2}} + \sum_{k=1}^{\infty} \frac{\left(z(r_{2}) - \frac{\sigma^{2}}{2}\right)^{k} - \left(z(r_{1}) - \frac{\sigma^{2}}{2}\right)^{k}}{k!k} \right) \\ &- \frac{C_{1}}{2T_{G}} \left(\ln \frac{z(r_{2})}{z(r_{1})} + \sum_{k=1}^{\infty} \frac{z(r_{2})^{k} - z(r_{1})^{k}}{k!k} \right) \\ &= \frac{C_{1}}{T_{G}} e^{\frac{\sigma^{2}}{2}} \ln \frac{r_{2}}{r_{1}} + \frac{C_{1}}{2T_{G}} \left(e^{\frac{\sigma^{2}}{2}} - 1 \right) \ln \frac{z(r_{2})}{z(r_{1})} \\ &+ \frac{C_{1}}{2T_{G}} \sum_{k=1}^{\infty} \frac{z(r_{2})^{k} \left(e^{\frac{\sigma^{2}}{2}} (-\zeta^{2}r_{2}^{2}/\ell^{2})^{k} - 1 \right) - z(r_{1})^{k} \left(e^{\frac{\sigma^{2}}{2}} (-\zeta^{2}r_{1}^{2}/\ell^{2})^{k} - 1 \right)}{k!k} \\ &\approx \frac{C_{1}}{T_{G}} e^{\frac{\sigma^{2}}{2}} \ln \frac{r_{2}}{r_{1}} + \frac{C_{1}}{2T_{G}} \left(e^{\frac{\sigma^{2}}{2}} - 1 \right) \ln \frac{z(r_{2})}{z(r_{1})} + \frac{C_{1}}{2T_{G}} \left(e^{\frac{\sigma^{2}}{2}} - 1 \right) (z(r_{2}) - z(r_{1})). \end{split}$$
(A4)

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The final approximate solution as given in Eq. (7) results by re-substituting z(r) with $r_2 = r$ and $r_1 = R$ and inserting $C_1 = -\frac{Q_w}{2\pi}$.

505 References

- Attinger, S.: Generalized coarse graining procedures for flow in porous media, Comput. Geosci., 7, 253–273, doi:10.1023/B:COMG.0000005243.73381.e3, 2003.
- Copty, N. K., and A. N. Findikakis, Stochastic analysis of pumping test drawdown data in heterogeneous geologic formations, *J. Hydraul. Res.*, *42, EXTRA ISSUE*, 59–67, doi:10.1080/00221680409500048, 2004.
- 510 Dagan, G., Flow and Transport on Porous Formations, Springer, New York, 1989.
 - Dagan, G. and Lessoff, S. C.: Transmissivity upscaling in numerical aquifer models of steady well flow: unconditional statistics, Water Resour. Res., 43, W05431, doi:10.1029/2006WR005235, 2007.
 - Desbarats, A.: Spatial averaging of transmissivity in heterogeneous fields with flow toward a well, Water Resour. Res., 28, 757–767, doi:10.1029/91WR03099, 1992.
- 515 Firmani, G., A. Fiori, and A. Bellin, Three-dimensional numerical analysis of steady state pumping tests in heterogeneous confined aquifers, *Water Resour. Res.*, 42, W03422, doi:10.1029/2005WR004382, 2006. Gelhar, L.: Stochastic Subsurface Hydrology, Prentice Hall, Englewood Cliffs, New York, 1993.
 - Heße, F., Prykhodko, V., Schlüter, S., and Attinger, S.: Generating random fields with a truncated powerlaw variogram: a comparison of several numerical methods, Environ. Modell. Softw., 55, 32–48,
- 520 doi:10.1016/j.envsoft.2014.01.013, 2014.
 - Indelman, P., Fiori, A., and Dagan, G.: Steady flow toward wells in heterogeneous formations: mean head and equivalent conductivity, Water Resour. Res., 32, 1975–1984, doi:10.1029/96WR00990, 1996.
 - Kolditz, O., Görke, U., Shao, H., and Wang, W. (Eds.): Thermo-Hydro-Mechanical-Chemical Processes in Porous Media, Springer Verlag, Berlin Heidelberg, 2012b.
- Matheron, G.: Elements Pour Une Theorie des Milieux Poreux, Maisson et Cie, Paris, 1967.
 Neuman, S. P., A. Guadagnini, and M. Riva, Type-curve estimation of statistical heterogeneity, *Water Resour*. *Res.*, 40, W04201, doi:10.1029/2003WR002405, 2004.
 - Neuman, S. P., Blattstein, A., Riva, M., Tartakovsky, D., Guadagnini, A., and Ptak, T.: Type curve interpretation of late-time pumping test data in randomly heterogeneous aquifers, Water Resour. Res., 43, W10421,
- 530 doi:10.1029/2007WR005871, 2007.
 - Sánchez-Vila, X., Axness, C. L., and Carrera, J.: Upscaling transmissivity under radially convergent flow in heterogeneous media, Water Resour. Res., 35, 613–621, doi:10.1029/1998WR900056, 1999.
 - Sánchez-Vila, X., Guadagnini, A., and Carrera, J.: Representative hydraulic conductivities in saturated groundwater flow, Rev. Geophys., 44, RG3002, doi:10.1029/2005RG000169, 2006.
- 535 Schneider, C. L. and Attinger, S.: Beyond Thiem a new method for interpreting large scale pumping tests in heterogeneous aquifers, Water Resour. Res., 44, W04427, doi:10.1029/2007WR005898, 2008.
 - Thiem, G.: Hydrologische Methoden, J. M. Gebhardt, Leipzig, 1906.
 - Zech, A., Schneider, C. L., and Attinger, S.: The extended Thiem's solution including the impact of heterogeneity, Water Resour. Res., 48, W10535, doi:10.1029/2012WR011852, 2012.
- 540 Zech, A., Arnold, S., Schneider, C., and Attinger, S.: Estimating Parameters of Aquifer Heterogeneity Using Pumping Tests – Implications for Field Applications, Adv. Water Resour., 83, 137–147, doi:10.1016/j.advwatres.2015.05.021, 2015.

| | $T_{\rm G}$ | $\hat{T}_{\rm G} [10^{-4} {\rm m}^2 {\rm s}^{-1}]$ | σ^2 | $\hat{\sigma}^2 \left[- ight]$ | l | $\hat{\ell}$ [m] |
|---|-------------------|---|----------------|---|---------------|----------------------|
| А | 1.0 | $1.03 \ (\pm 0.0011)$ | 1.0 | 1.04 (±0.0022) | 10 | 9.80 (±0.086) |
| В | 1.0 | $1.08 \ (\pm 0.0013)$ | 1.0 | $1.19 (\pm 0.0022)$ | 20 | $21.6 (\pm 0.127)$ |
| С | 1.0 | $1.08 \ (\pm 0.0021)$ | 2.25 | $2.49 \ (\pm 0.0038)$ | 10 | $10.1 \ (\pm 0.050)$ |
| D | 1.0 | $1.19 (\pm 0.0024)$ | 2.25 | $2.67 (\pm 0.0039)$ | 20 | $22.2 (\pm 0.077)$ |
| Е | 1.0 | 1.16 (±0.0046) | 4.0 | $4.34~(\pm 0.0078)$ | 10 | $11.0 \ (\pm 0.042)$ |
| F | 1.0 | $1.31 \ (\pm 0.0088)$ | 4.0 | $4.27~(\pm 0.0131)$ | 20 | $22.2 (\pm 0.120)$ |
| G | 1.5 | $1.55 \ (\pm 0.0012)$ | 1.0 | $1.03 \ (\pm 0.0016)$ | 10 | $10.1 \ (\pm 0.066)$ |
| Ħ | $\frac{1.5}{1.5}$ | $\frac{1.62}{\pm 0.0024}$ | 1.0 | $\frac{1.19(\pm 0.0028)}{1.19(\pm 0.0028)}$ | 20 | $21.2(\pm 0.158)$ |

Table 1. Ensemble input parameters $T_{\rm G}$, σ^2 and ℓ and best fit fitting inverse estimation results $\hat{T}_{\rm G}$, $\hat{\sigma}^2$ and $\hat{\ell}$ with 95% confidence intervals (in brackets) for ensemble mean $\overline{\langle h(r) \rangle}$ for all generated ensembles.

Table 2. Parameter estimates of geometric mean transmissivity $\hat{T}_{\rm G}$ [10⁻⁴ m²/s], local transmissivity at the well $\hat{T}_{\rm well}$ [10⁻⁴ m²/s] and correlation length $\hat{\ell}$ [m] for the 8 pumping tests at fields D1 (from Ensemble D, $\sigma^2 = 2.25$) and E1 (from Ensemble E, $\sigma^2 = 4.0$). Additionally, the theoretical and the sampled values ($T_{\rm well} \equiv T_{\rm H}$) are given.

| | D1 | | | E1 | | | |
|-----------------|------------------|------------------------|--------------|------------------|------------------------|--------------|--|
| | $\hat{T}_{ m G}$ | $\hat{T}_{	ext{well}}$ | $\hat{\ell}$ | $\hat{T}_{ m G}$ | $\hat{T}_{	ext{well}}$ | $\hat{\ell}$ | |
| \mathbf{PW}_0 | 1.025 | 0.434 | 29.51 | 1.945 | 0.313 | 9.56 | |
| PW_1 | 1.023 | 0.362 | 27.23 | 2.202 | 0.445 | 11.55 | |
| \mathbf{PW}_2 | 1.076 | 0.220 | 23.68 | 2.093 | 0.437 | 10.03 | |
| PW_3 | 0.898 | 1.057 | 9.51 | 2.052 | 0.520 | 15.34 | |
| PW_4 | 1.001 | 0.147 | 20.53 | 2.174 | 1.847 | 12.30 | |
| PW_5 | 0.889 | 1.071 | 5.33 | 1.980 | 1.117 | 5.43 | |
| PW_6 | 1.038 | 0.177 | 20.39 | 1.840 | 0.148 | 8.78 | |
| PW ₇ | 0.901 | 1.700 | 16.48 | 1.969 | 0.476 | 17.04 | |
| Mean of 8 | 0.981 | 0.646 | 19.08 | 2.032 | 0.663 | 9.90 | |
| Jointly | 1.013 | 0.328 | 22.38 | 2.010 | 0.409 | 9.97 | |
| Theory | 1.0 | 0.325 | 20.0 | 1.0 | 0.135 | 10.0 | |
| Sampled | 0.985 | 0.333 | 23.43 | 1.999 | 0.491 | 12.66 | |



Figure 1. Comparison of equivalent and Radial Coarse Graining approach: (a) radially depending transmissivities interpolating between harmonic mean $T_{\rm H}$ and geometric mean $T_{\rm G}$: $T_{\rm RCG}(r)$ from Eq. (4) and $T_{\rm eq}(r)$ calculated based on Thiem's formula Eq. (1) with $h(r) = \overline{\langle h(r) \rangle}$, which is the ensemble mean for Ensemble A (Table 1), (b) hydraulic head drawdowns after pumping with: $h_{\rm efw}(r)$ from Eq. (6) as solution of the well flow equation using $T_{\rm RCG}(r)$, $h_{\rm eq}(r)$ as solution of the well flow equation using $T_{\rm eq}(r)$, Thiem's solution with homogeneous substitute values $T_{\rm G}$ and $T_{\rm H}$ as well as mean ensemble drawdown $\overline{\langle h(r) \rangle}$.



Figure 2. Simulated ensemble means $\overline{\langle h(r) \rangle}$ (dots) and $h_{\text{efw}}(r)$ with best fitting estimates (lines) for multiple Ensembles: A (blue), B (green), E (red), F (orange), G (purple). Parameter values are listed in Table 1. Black line shows $h_{\text{Thiem}}(r)$ with $T_{\text{G}} = 10^{-4} \,\text{m}^2 \,\text{s}^{-1}$.



Figure 3. Drawdowns simulated for two individual transmissivity field realizations of Ensemble A ($T_{\rm G} = 10^{-4} \,\mathrm{m^2 \, s^{-1}}$, $\sigma^2 = 1$, $\ell = 10 \,\mathrm{m}$): (a) realization with $T_{\rm well} = 0.204 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$ and (b) realization with $T_{\rm well} = 1.11 \times 10^{-4} \,\mathrm{m^2 \, s^{-1}}$. $\langle h(r) \rangle$ (dark color) is the radial mean, $\langle h(r, \phi) \rangle$ (light color) denotes the draw-downs along the four axes ($\phi = 0^{\circ}, 90^{\circ}, 180^{\circ}, 270^{\circ}$), as well as in black Thiem's solution for homogeneous substitute values.



Figure 4. Histogram on the best fitting estimates $(\hat{T}_{\rm G}, \hat{T}_{\rm well}, \hat{\ell})$ versus the theoretical input values $(T_{\rm G}, T_{\rm H}, \ell)$ and the sampled transmissivity at the pumping well ($\langle T_{\rm well} \rangle$) for the N = 5000 realizations of Ensemble A.



Figure 5. Spatial distribution of log-transmissivity for fields (a) D1 and (b) E1 and locations of the eight pumping wells ($PW_0, ..., PW_7$ in black) and the four observation wells ($OW_0, ..., OW_3$ in gray).



Figure 6. Simulated drawdown measurements (dots) and fitted effective well flow solution $h_{efw}(r)$ (lines) for eight pumping tests within the heterogeneous transmissivity fields (a) D1 and (b) E1. Colours indicate the results for the individual pumping tests at PW₀,..., PW₇ (from light to dark). The black line denotes the effective well flow solution $h_{efw}(r)$ fitted to all measurements jointly. Gray lines denote Thiem's solution for \hat{T}_{G} (solid) and for \hat{T}_{well} (dashed). Statistical parameters are given in Table 2.