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Technical Note: Approximate solution of transient drawdown for constant-flux pumping at a partially penetrating well in a radial two-zone confined aquifer

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Abstract

An aquifer consisting of a skin zone and a formation zone is considered as a two-zone aquifer. Existing solutions for the problem of constant-flux pumping (CFP) in a two-zone confined aquifer involve laborious calculation. This study develops a new approximate solution for the problem based on a mathematical model including two steady-state flow equations with different hydraulic parameters for the skin and formation zones. A partially penetrating well may be treated as the Neumann condition with a known flux along the screened part and zero flux along the unscreened part. The aquifer domain is finite with an outer circle boundary treated as the Dirichlet condition. The steady-state drawdown solution of the model is derived by the finite Fourier cosine transform. Then, an approximate transient solution is developed by replacing the radius of the boundary in the steady-state solution with an analytical expression for a dimensionless time-dependent radius of influence. The approximate solution is capable of predicting good temporal drawdown distributions over the whole pumping period except at the early stage. A quantitative criterion for the validity of neglecting the vertical flow component due to a partially penetrating well is also provided. Conventional models considering radial flow without the vertical component for the CFP have good accuracy if satisfying the criterion.

1 Introduction

The constant-flux pumping (CFP) test is a widely used well test for characterizing the aquifer properties such as transmissivity and storage coefficient. The test is performed with a constant pumping rate at a fully or partially penetration well in either a confined or unconfined aquifer. Existing analytical solutions for the CFP in a homogenous confined aquifer are briefly reviewed herein. Theis (1935) was the first article in the groundwater literature to present an analytical solution for aquifer drawdown due to pumping in a fully penetrating well with an infinitesimal radius. Carslaw and Jaeger (1959) pre-

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sented analytical solutions for the three kinds of heat conduction problems which can be analogous to the CFP problems including the aquifers of the infinite domain with a finite-radius well, finite domain with a finite-radius well, and finite domain with an infinitesimal-radius well. Hantush (1962) developed an analytical solution of drawdown induced by a partially penetrating well for the CFP. Papadopulos and Cooper (1967) obtained an analytical solution of drawdown with considering the effects of well radius and wellbore storage. They provided a quantitative criterion of time for neglecting the effects. The criterion will be stated in the next section. Chen (1984) derived an analytical solution for drawdown in a circular aquifer with the Dirichlet boundary condition of zero drawdown and provided a quantitative criterion describing the beginning time of the boundary effect on the drawdown. Yang et al. (2006) developed an analytical solution describing aquifer drawdown due to a partially penetrating well with a finite radius. The effect of partial penetration on temporal drawdown distributions was discussed. Wang and Yeh (2008) also provided a quantitative criterion for the beginning time of the boundary effect on drawdown induced by the CFP and constant-head pumping. Yeh and Chang (2013) provided a comprehensive review on analytical solutions for the CFP in unconfined and multilayered aquifer systems.

Drilling an aquifer to install a well may decrease or increase the permeability of the formation around the wellbore. The perturbed formation, called as skin zone, extends from a few millimeters to several meters. A positive skin zone means that its permeability is lower than the original formation. On the other hand, a negative skin zone is of a higher permeability than the original formation. Existing solutions accounting for the CFP in a two-zone confined aquifer consisting of the skin zone and formation zone are reviewed. Novakowski (1989) developed a semi-analytical solution of drawdown with the wellbore storage effect and investigated the effect of an infinitesimally thin skin on temporal drawdown curves. Hemker (1999) proposed an analytical-numerical solution describing pumping drawdown in a multilayered aquifer system where the radial flow component was analytically treated and the vertical one was dealt with by a finite difference method. The flux along the well screen was non-uniform through an

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infinitesimal thin skin, and the flow was subject to the wellbore storage effect. Kabala and El-Sayegh (2002) presented a semi-analytical solution for the transient flowmeter test in a multilayered aquifer system where the radial flow was considered in each layer with assuming no vertical flow component and uniform flux along the well screen. Predictions from the solution were compared with those from a numerical solution which relaxes those two assumptions. Yeh et al. (2003) obtained an analytical solution for pumping drawdown induced by pumping at a finite-radius well in a two-zone confined aquifer and discussed the error caused by neglecting the well radius. Chen and Chang (2006) developed a semi-analytical solution for the CFP on the basis of the Gram–Schmidt method to deal with the non-uniform skin effect represented by an arbitrary piecewise function of the elevation. They indicated that flow near the pumping well is three dimensional due to the effect and away from the well is radial. Perina and Lee (2006) proposed a general well function for transient flow toward a partially penetrating well with considering the wellbore storage effect and non-uniform flux between the screen and skin zone in a confined, unconfined, or leaky aquifer. Chiu et al. (2007) developed a semi-analytical solution for the CFP at a partial penetrating well in a two-zone confined aquifer. They indicated that the influence of the partial penetration on drawdown is more significant for a negative skin zone than a positive one. C.-T. Wang et al. (2012) provided an analytical solution of drawdown for the CFP in a two-zone confined aquifer of finite extent with an outer boundary under the Dirichlet condition of zero drawdown. They also derived a large-time drawdown solution which reduces to the Thiem solution when the skin zone is absent. X. Wang et al. (2012) presented a finite layer method (FLM) based on Galerkin's technique for simulating radial and vertical flows toward a partially penetrating well in a multilayered aquifer system. The FLM was verified by an analytical solution and finite difference solution.

It is informative to classify the above solutions into two groups, i.e. homogeneous aquifer and two-zone aquifer systems in Table 1. The solutions in each group are categorized according to the well penetration, well radius, and wellbore storage.

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At the present, a time-domain analytical solution of drawdown for flow induced by the CFP at a finite-radius partially penetrating well in a two-zone confined aquifer has not been developed. The Laplace-domain result of the above-mentioned problem was presented by Chiu et al. (2007) with resort to a numerical inversion such as the Crump method. The application of their solution may therefore be inconvenient for those who are not familiar with numerical approaches. The purpose of this note is to develop a new approximate transient solution for the problem in a way similar to our previous work of Yang et al. (2014). A mathematical model for steady-state flow due to a partially penetrating well in a finite-extent two-zone confined aquifer is built. The flow equations describing spatial drawdowns in the skin and formation zones are employed. The outer boundary of the aquifer is specified as the Dirichlet condition of zero drawdown. The well is treated as the Neumann condition with a constant flux for the screened part and zero flux for the unscreened part. The steady-state solution of the model for drawdown is derived by the method of finite Fourier cosine transform. The approximate transient solution of drawdown is then obtained on the basis of the steady-state solution and a time-dependent radius of influence. The transient solution is in term of simple series with fast convergence. The accuracy of the solution is investigated in comparison with Chiu et al. (2007) solution. In addition, the condition of neglecting the effect of vertical flow on the temporal drawdown distribution is investigated.

2 Methodology

2.1 Mathematical model

Figure 1 shows a schematic diagram of the CFP performed at a partially penetrating well in a radial two-zone confined aquifer of finite extent. The symbols are defined as follows: r is a radial distance from the center of the well; r_w is the radius of the well; r_s is the outer radius of the skin zone; R is the radius of influence; z is an elevation from the impermeable bottom; z_1 and z_2 are the lower and upper elevations of the well

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screen, respectively; b is an aquifer thickness; s_1 and s_2 are drawdowns in the skin and formation zones, respectively. The dimensionless parameters and variables are defined as

$$\bar{s}_1 = 2\pi T_2 s_1/Q, \quad \bar{s}_2 = 2\pi T_2 s_2/Q, \quad \bar{r} = r/r_\mathrm{w}, \quad \bar{z} = z/b, \quad \bar{r}_\mathrm{s} = r_\mathrm{s}/r_\mathrm{w}, \quad \bar{R} = R/r_\mathrm{w},$$

$$\bar{z}_1 = z_1/b, \quad \bar{z}_2 = z_2/b, \quad \varphi = \bar{z}_2 - \bar{z}_1, \quad \gamma = K_{\mathrm{r}2}/K_{\mathrm{r}1}, \quad \alpha_1 = \frac{K_{z1} r_\mathrm{w}^2}{K_{\mathrm{r}1} b^2}, \quad \alpha_2 = \frac{K_{z2} r_\mathrm{w}^2}{K_{\mathrm{r}2} b^2}, \tag{1}$$

where the bar represents a dimensionless symbol, the subscripts 1 and 2 represent the skin and formation zones, respectively, Q is the pumping rate of the well, K_r and K_z are the radial and vertical hydraulic conductivities, respectively, and $T_1 = K_{\mathrm{r}1} b$ and $T_2 = K_{\mathrm{r}2} b$ are the transmissivities of the skin and formation zones, respectively. The φ is the penetration ratio of the screen length over the aquifer thickness.

The governing equations describing steady-state dimensionless drawdown distributions in the skin and formation zones are expressed, respectively, as

$$\frac{\partial^2 \bar{s}_1}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_1}{\partial \bar{r}} + \alpha_1 \frac{\partial^2 \bar{s}_1}{\partial \bar{z}^2} = 0 \text{ for } 1 \le \bar{r} \le \bar{r}_\mathrm{s} \tag{2}$$

and

$$\frac{\partial^2 \bar{s}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_2}{\partial \bar{r}} + \alpha_2 \frac{\partial^2 \bar{s}_2}{\partial \bar{z}^2} = 0 \text{ for } \bar{r}_\mathrm{s} \le \bar{r} \le \bar{R}. \tag{3}$$

The outer boundary specified at $\bar{r} = \bar{R}$ is under the Dirichlet condition as

$$\bar{s}_2 = 0 \text{ at } \bar{r} = \bar{R} \tag{4}$$

while the inner boundary designated at the rim of the wellbore is under the Neumann condition as

$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = -\frac{\gamma}{\varphi} \left(U\left(\bar{z} - \bar{z}_1\right) - U\left(\bar{z} - \bar{z}_2\right) \right) \text{ at } \bar{r} = 1 \text{ and } 0 \le \bar{z} \le 1 \tag{5}$$

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where $U(\cdot)$ is the unit step function. Equation (5) indicates that the flux is uniformly distributed over the screen with neglecting the wellbore storage effect. Papadopulos and Cooper (1967) mentioned that the storage effect on a temporal drawdown distribution diminishes for $t > 2.5 \times 10^2 r_c^2 / T_2$ where t is dimensional time since pumping and r_c is the inner radius of the well. In addition, Yeh and Chang (2013) also mentioned that this effect is negligible for a well with a small radius within 0.25 m. Two continuity conditions required at $\bar{r} = \bar{r}_s$ are

$$\bar{s}_1 = \bar{s}_2 \text{ at } \bar{r} = \bar{r}_s \tag{6}$$

and

$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = \gamma \frac{\partial \bar{s}_2}{\partial \bar{r}} \text{ at } \bar{r} = \bar{r}_s. \tag{7}$$

The top and bottom confining beds are under the no-flow conditions of $\partial \bar{s}_i / \partial \bar{z} = 0$ where $i \in (1,\ 2)$.

2.2 Steady-state solution

The solution of the model, obtained by applying the finite Fourier cosine transform, can be written as

$$s_1(\bar{r},\bar{z}) = \ln\left(\bar{R}/\bar{r}_s\right) + \gamma \ln\left(\bar{r}_s/\bar{r}\right) + \frac{2\gamma}{\bar{z}_2 - \bar{z}_1} \sum_{n=1}^{\infty} F_1(\bar{r},n)\cos(n\pi\bar{z}) \text{ for } 1 \le \bar{r} \le \bar{r}_s \tag{8}$$

and

$$s_2(\bar{r},\bar{z}) = \ln(\bar{R}/\bar{r}) + \frac{2\gamma}{\bar{r}_s\left(\bar{z}_2 - \bar{z}_1\right)} \sum_{n=1}^{\infty} F_2(\bar{r},n)\cos(n\pi\bar{z}) \text{ for } \bar{r}_s \le \bar{r} \le \bar{R} \tag{9}$$

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with

$$F_1(\bar{r},n) = \varpi \left(\zeta I_0\left(\lambda_1\bar{r}\right) + \xi K_0\left(\lambda_1\bar{r}\right)\right)/(\lambda_1\psi) \tag{10}$$

$$F_2(\bar{r},n) = \varpi \left(K_0\left(\lambda_2\bar{R}\right) I_0\left(\lambda_2\bar{r}\right) - I_0\left(\lambda_2\bar{R}\right) K_0\left(\lambda_2\bar{r}\right)\right)/(\lambda_1\psi) \tag{11}$$

$$\psi = \lambda_1 G(0,\,-1)\,H(1,\,-1) - \gamma\lambda_2 G(1,\,1)\,H(0,\,1) \tag{12}$$

$$\zeta = \lambda_1 K_1\left(\lambda_1\bar{r}_\mathrm{s}\right) G(0,\,-1) + \gamma\lambda_2 K_0\left(\lambda_1\bar{r}_\mathrm{s}\right) G(1,\,1) \tag{13}$$

$$\xi = \lambda_1 I_1\left(\lambda_1\bar{r}_\mathrm{s}\right) G(0,\,-1) - \gamma\lambda_2 I_0\left(\lambda_1\bar{r}_\mathrm{s}\right) G(1,\,1) \tag{14}$$

$$G(\mu,c) = I_\mu\left(\lambda_2\bar{r}_\mathrm{s}\right) K_0\left(\lambda_2\bar{R}\right) + c\,K_\mu\left(\lambda_2\bar{r}_\mathrm{s}\right) I_0\left(\lambda_2\bar{R}\right) \tag{15}$$

$$H(\mu,c) = K_1(\lambda_1) I_\mu\left(\lambda_1\bar{r}_\mathrm{s}\right) + c\,I_1(\lambda_1) K_\mu\left(\lambda_1\bar{r}_\mathrm{s}\right) \tag{16}$$

and

$$\varpi = \left(\sin\left(\bar{z}_2\pi n\right) - \sin\left(\bar{z}_1\pi n\right)\right)/(\pi n) \tag{17}$$

where $\lambda_i = \pi n\sqrt{\alpha_i}$, and I_μ $(\cdot)$ and K_μ $(\cdot)$ are the modified Bessel functions of the first and second kinds with order μ, respectively. The detailed derivation of the solution is given in the Appendix.

2.3 Approximate transient solution

By trial and error, we find that a time-dependent radius of influence induced by the CFP can be approximated as

$$\bar{R}(\bar{t}) = 1 + \sqrt{\pi\bar{t}/1.4} \tag{18}$$

where $\bar{t} = K_{r2}\,t/(S_{s2}r_\mathrm{w}^2)$ is a dimensionless time, and S_{s2} is the specific storage of the formation zone. Notice that Eq. (18) is similar to an equation given in Yang et al. (2014, Eq. 25) but has a different coefficient value. The transient solution of drawdown for

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the CFP in a two-zone aquifer system can be obtained approximately by replacing the parameter $\overline{R}$ in the steady-state solution (i.e. Eqs. 8–17) with Eq. (18). The result is a function of dimensionless time denoted as

$$s_1(\bar{r},\bar{z},\bar{t}) = \ln\left(\overline{R}(\bar{t})/\bar{r}_s\right) + \gamma \ln\left(\bar{r}_s/\bar{r}\right) + \frac{2\gamma}{\bar{z}_2 - \bar{z}_1}\sum_{n=1}^{\infty} F_1(\bar{r},n,\bar{t})\cos(n\pi\bar{z}) \text{ for } 1 \le \bar{r} \le \bar{r}_s \tag{19}$$

and

$$s_2(\bar{r},\bar{z},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}) + \frac{2\gamma}{\bar{r}_s\left(\bar{z}_2 - \bar{z}_1\right)}\sum_{n=1}^{\infty} F_2(\bar{r},n,\bar{t})\cos(n\pi\bar{z}) \text{ for } \bar{r}_s \le \bar{r} \le \overline{R}. \tag{20}$$

Notice that $F_1(\bar{r},n,\bar{t})$ and $F_2(\bar{r},n,\bar{t})$ obtained from Eqs. (10) and (11), respectively, with coefficients ψ, ζ, ξ, and $G(\mu,c)$ defined in Eqs. (12)–(15), respectively, are functions of dimensionless time because of Eq. (18). The idea of deriving Eqs. (19) and (20) originated from the concept of a time-dependent diffusion layer for the solution of the diffusion equation in the field of electrochemistry (Fang et al., 2009).

2.4 Special case 1: solution for CFP at fully penetration well in two-zone aquifer

When $\bar{z}_1 = 0$ and $\bar{z}_2 = 1$ (i.e. $z_1 = 0$ and $z_2 = b$) for the case of well full penetration, one can obtain $\varpi = 0$ according to Eq. (17). The simple series in Eqs. (19) and (20) then vanishes, and the solution for temporal drawdown distributions subject to the skin effect reduces to

$$s_1(\bar{r},\bar{t}) = \ln\left(\overline{R}(\bar{t})/\bar{r}_s\right) + \gamma \ln\left(\bar{r}_s/\bar{r}\right) \text{ for } 1 \le \bar{r} \le \bar{r}_s \tag{21}$$

and

$$s_2(\bar{r},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}) \text{ for } \bar{r}_s \le \bar{r} \le \overline{R}(\bar{t}). \tag{22}$$

Note that Eqs. (21) and (22) are independent of $\bar{z}$, indicating that the groundwater flow is only horizontal.

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2.5 Special case 2: solution for CFP at fully penetration well in homogeneous aquifer

When $\bar{z}_1 = 0$, $\bar{z}_2 = 1$, and $\gamma = 1$ (i.e. $z_1 =$, $z_2 = b$, and $K_{r1} = K_{r2}$) for the case of a fully penetrating well in a homogeneous aquifer, Eqs. (19) and (20) yield

$$s(\bar{r},\bar{t}) = \ln(\bar{R}(\bar{t})/\bar{r}) \text{ for } 1 \le \bar{r} \le \bar{R}(\bar{t}) \tag{23}$$

which is indeed a dimensionless form of Thiem's equation. Note that Eq. (23) can also be derived by substituting $\gamma = 1$ into Eq. (21).

3 Results and discussion

3.1 Accuracy of approximate solution

The predictions from the approximate solution are compared with those from the Laplace-domain solution of Chiu et al. (2007). Figure 2a shows the spatial drawdown distributions predicted from both solutions with $\gamma = 0.1$, 1 and 10 at $\bar{t} = 3 \times 10^6$ for $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$. The figure indicates that both solutions agree very well on the drawdown within the time-dependent radius of influence represented by $\bar{R}(\bar{t})$. The drawdown curves of $\gamma = 0.1$, 1 and 10 in the formation zone merge together at and after the interface, i.e. $\bar{r}_s = 5$, because of $\alpha_1 = \alpha_2$. Figure 2b displays the temporal drawdown distributions predicted from both solutions at $\bar{r} = 20$ with $\gamma = 0.1$, 1 and 10 for $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$. This figure demonstrates that the drawdown curves predicted by both solutions also have good match over the intermediate and late pumping periods. The discrepancy can be attributed to the neglect of the time derivative term in Eqs. (2) and (3). The drawdown dramatically increases at $\bar{t} = 160$ as soon as the radius of influence arrives at $\bar{r} = 20$. It seems reasonable to conclude that the approximate transient solution gives

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good predicted drawdown over the entire pumping period except at early time during which the radius of influence arrives.

3.2 Vertical flow

A partially penetrating well induces vertical flow over a region depending on both magnitudes of $\alpha_1\bar{r}^2$ and $\alpha_2\bar{r}^2$ (i.e. $K_{z1}r^2/(K_{r1}b^2)$ and $K_{z2}r^2/(K_{r2}b^2)$, respectively). Figure 3 shows temporal drawdown distributions predicted by the approximate solution, Eq. (20), for $\alpha_1 = \alpha_2$ ranging from 10^{-6} to 10^{-2} when $\bar{r} = 10$, $\bar{z} = 0.5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\bar{r}_s = 5$ and $\gamma = 0.1$. Equation (22) is the drawdown solution for the CFP at a fully penetration well; therefore, the vertical flow is absent. When $\alpha_1\bar{r}^2 = \alpha_2\bar{r}^2 = 1$, the drawdown distributions predicted by both equations agree well, indicating that the vertical flow is negligible. We may, therefore, reasonably conclude that the vertical flow vanishes when $\alpha_1\bar{r}^2 \geq 1$ and $\alpha_2\bar{r}^2 \geq 1$, i.e. b is small, r is large, and/or the values of K_{z1}/K_{r1} and K_{z2}/K_{r2} are large. On the other hand, Eq. (22) underestimates the drawdown induced by the CFP at a partially penetration well because the vertical flow prevails when $\alpha_1\bar{r}^2 < 1$ or $\alpha_2\bar{r}^2 < 1$.

4 Concluding remarks

This study presents an approximate drawdown solution, Eqs. (19) and (20), in terms of a simple series for the CFP at a partially penetrating well in a radial two-zone confined aquifer. The solution is developed on the basis of the steady-state drawdown solution with an outer boundary represented by the time-dependent radius of influence. The comparison with the Chiu et al. (2007) solution reveals that the approximate solution gives accurate temporal drawdown distributions over the entire pumping period except at early time during which the time-dependent radius of influence just touches. The analysis of the temporal drawdowns predicted by Eqs. (20) and (22) indicates that the

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vertical flow component due to a partially penetrating well prevails when $\alpha_1\bar{r}^2 < 1$ or $\alpha_2\bar{r}^2 < 1$ and vanishes when $\alpha_1\bar{r}^2 \geq 1$ and $\alpha_2\bar{r}^2 \geq 1$. Conventional models neglecting the vertical flow underestimate the drawdown when $\alpha_1\bar{r}^2 < 1$ or $\alpha_2\bar{r}^2 < 1$ and predict accurate drawdown when $\alpha_1\bar{r}^2 \geq 1$ and $\alpha_2\bar{r}^2 \geq 1$.

Appendix: Derivation of Eqs. (8) and (9)

The finite Fourier cosine transform is defined, in our notation, as

$$\hat{s}_i = \int_0^1 \bar{s}_i \cos(n\pi\bar{z})\,\mathrm{d}\bar{z} \tag{A1}$$

where $i \in (1, 2)$. The formula for the inverse transform is expressed as

$$\bar{s}_i = \hat{s}_i(0) + 2\sum_{n=1}^{\infty} \hat{s}_i(n)\cos(n\pi\bar{z}) \tag{A2}$$

where $\hat{s}_i(n)$, a function of n, is the solution in the transform domain. Replacing $\bar{s}_i$ in Eq. (A1) by $\partial^2\bar{s}_i/\partial\bar{z}^2$ and applying integration by parts twice yields

$$\int_0^1 \frac{\partial^2\bar{s}_i}{\partial\bar{z}^2}\cos(n\pi\bar{z})\,\mathrm{d}\bar{z} = (-1)^n.\left.\frac{\partial\bar{s}_i}{\partial\bar{z}}\right|_{\bar{z}=1} - \left.\frac{\partial\bar{s}_i}{\partial\bar{z}}\right|_{\bar{z}=0} - (n\pi)^2\hat{s}_i. \tag{A3}$$

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Applying the transform to Eqs. (2)–(7) on the basis of Eq. (A3) with $\partial \bar{s}_i / \partial \bar{z} = 0$ results in the following equations:

$$\frac{\partial^2 \hat{s}_1}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{s}_1}{\partial \bar{r}} - \lambda_1^2 \hat{s}_1 = 0 \text{ for } 1 \le \bar{r} \le \bar{r}_\mathrm{s} \tag{A4}$$

$$\frac{\partial^2 \hat{s}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \hat{s}_2}{\partial \bar{r}} - \lambda_2^2 \hat{s}_2 = 0 \text{ for } \bar{r}_\mathrm{s} \le \bar{r} \le \bar{R} \tag{A5}$$

$$\hat{s}_2 = 0 \text{ at } \bar{r} = \bar{R} \tag{A6}$$

$$\frac{\partial \hat{s}_1}{\partial \bar{r}} = -\gamma \varpi / \varphi \text{ at } \bar{r} = 1 \tag{A7}$$

$$\hat{s}_1 = \hat{s}_2 \text{ at } \bar{r} = \bar{r}_\mathrm{s} \tag{A8}$$

and

$$\frac{\partial \hat{s}_1}{\partial \bar{r}} = \gamma \frac{\partial \hat{s}_2}{\partial \bar{r}} \text{ at } \bar{r} = \bar{r}_\mathrm{s}. \tag{A9}$$

The Fourier-domain solution of Eqs. (A4) and (A5) can be expressed as

$$\hat{s}_1 = c_1 I_0\left(\lambda_1 \bar{r}\right) + c_2 K_0\left(\lambda_1 \bar{r}\right) \tag{A10}$$

and

$$\hat{s}_2 = c_3 I_0\left(\lambda_2 \bar{r}\right) + c_4 K_0\left(\lambda_2 \bar{r}\right) \tag{A11}$$

where $I(\cdot)$ and $K(\cdot)$ are the modified Bessel functions of the first and second kinds of order zero, respectively, and c_1, c_2, c_3 and c_4 are undetermined coefficients. Substituting Eqs. (A10) and (A11) into Eqs. (A6)–(A9) and solving the four resultant equations leads to

$$(c_1, c_2, c_3, c_4) = \left(\frac{\gamma \zeta \varpi}{\varphi \lambda_1 \psi}, \frac{\gamma \xi \varpi}{\varphi \lambda_1 \psi}, \frac{\gamma \varpi K_0(\lambda_2 \bar{R})}{\varphi r_\mathrm{s} \lambda_1 \psi}, -\frac{\gamma \varpi I_0(\lambda_2 \bar{R})}{\varphi r_\mathrm{s} \lambda_1 \psi} \right) \tag{A12}$$

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where ψ, ζ and ξ are defined in Eqs. (12)–(14), respectively. According to Eq. (A12), Eqs. (A10) and (A11) can be written, respectively, as

$$\hat{s}_1(\bar{r},n) = \gamma F_1(\bar{r},n)/\varphi \tag{A13}$$

and

$$\hat{s}_2(\bar{r},n) = \gamma F_2(\bar{r},n)/(\varphi \bar{r}_s) \tag{A14}$$

where $F_1(\bar{r},n)$ and $F_2(\bar{r},n)$ are defined in Eqs. (10) and (11), respectively. In the light of Eq. (A2), the inverse transforms to Eqs. (A13) and (A14) are defined in Eqs. (8) and (9), respectively. Note that the first terms on the right-hand side of Eqs. (8) and (9) are derived via L'Hospital's law.

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Table 1. Categorization of the solutions for the CFP in confined aquifers.

References	Well Penetration	Well Radius	Wellbore Storage	Remark
Homogeneous Aquifer				
Theis (1935)[a]	Fully	Infinitesimal	None	Infinite aquifer
Carslaw and Jaeger (1959, p. 328)[a]	Fully	Finite	None	Infinite aquifer
Carslaw and Jaeger (1959, p. 332)[a]	Fully	Finite	None	Finite aquifer with Dirichlet boundary
Carslaw and Jaeger (1959, p. 335)[a]	Fully	Infinitesimal	None	Finite aquifer with Dirichlet boundary
Hantush (1962)[a]	Partially	Infinitesimal	None	Infinite aquifer
Papadopulos and Cooper (1967)[a]	Fully	Finite	Considered	Infinite aquifer
Chen (1984)[a]	Fully	Infinitesimal	None	Finite aquifer with Dirichlet boundary
Yang et al. (2006)[a]	Partially	Finite	None	Infinite aquifer
Two-Zone Aquifer				
Novakowski (1989)[b]	Fully	Finite	Considered	Infinite aquifer
Hemker (1999)[c, d]	Partially	Finite	Considered	Multilayered aquifer with radial and vertical flows
Kabala and El-Sayegh (2002)[b]	Fully	Finite	Considered	Multilayered aquifer with radial flow only
Yeh et al. (2003)[a]	Fully	Finite	None	Infinite aquifer
Chen and Chang (2006)[b, d]	Fully	Finite	Considered	Non-uniform skin effect
Perina and Lee (2006)[b]	Partially	Finite	Considered	General well functions for three-kinds of aquifers
Chiu et al. (2007)[b]	Partially	Finite	None	Infinite aquifer
C.-T. Wang et al. (2012)[a]	Fully	Finite	None	Finite aquifer with Dirichlet boundary
X. Wang et al. (2012)[a]	Partially	Infinitesimal	None	Multilayered aquifer with radial and vertical flows

[a], [b] and [c] represent analytical, semi-analytical and analytical-numerical solutions, respectively.
[d] represents an infinitesimal thin skin zone.

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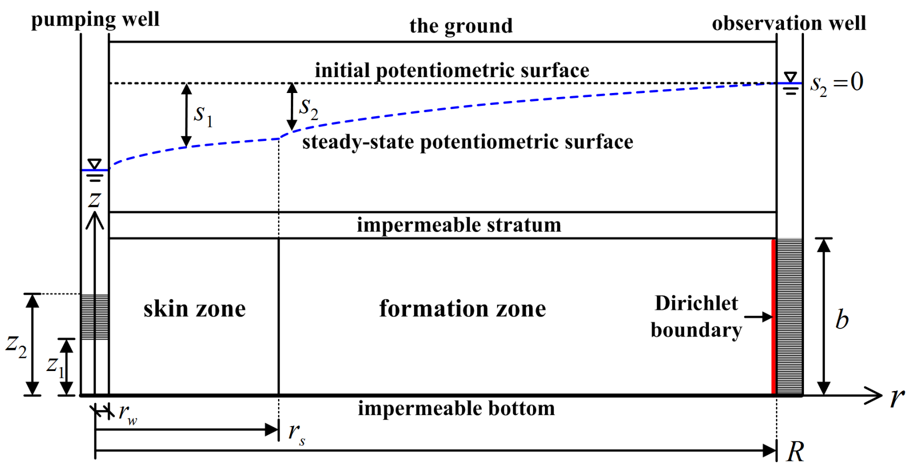

Figure 1. A schematic diagram of the CFT at a partially penetrating well in a radial two-zone confined aquifer with the Dirichlet boundary.

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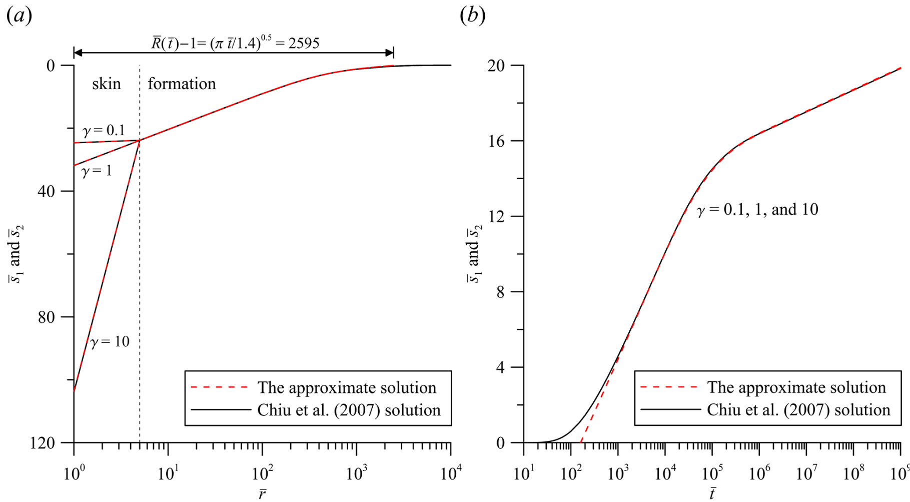

Figure 2. Predicted drawdowns by the approximate solution and Chiu et al. (2007) solution with $\gamma = 0.1$, 1, and 10 for **(a)** spatial distributions at $\bar{t} = 3 \times 10^6$ and **(b)** temporal distributions at $\bar{r} = 20$ with $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$.

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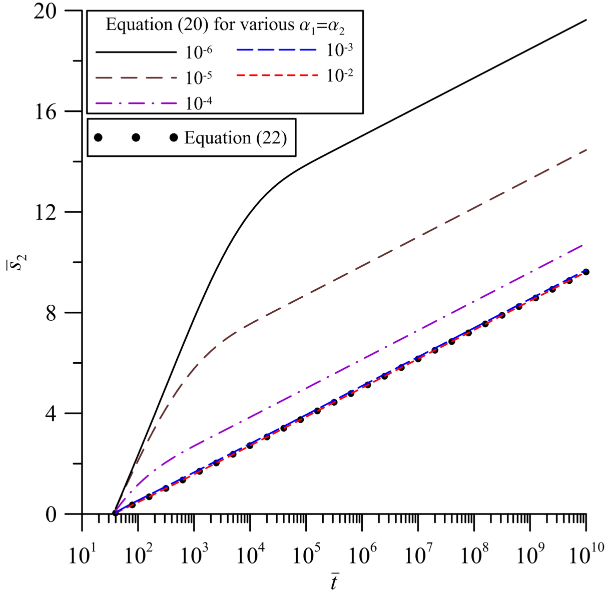

Figure 3. Temporal drawdown distributions predicted by the approximate solution, Eq. (20), for various values of α_1 and α_2.