Interactive comment on "Technical Note: Approximate solution of transient drawdown for constant-flux pumping at a partially penetrating well in a radial two-zone confined aquifer" by C.-S. Huang et al.

Anonymous Referee #1
Received and published: 31 March 2015

General Comments

C.-S. Huang, S.-Y. Yang, H.-D. Yeh present in the technical note a newly developed approximate solution for the drawdown of a pumping test at a partially penetrating well in a radial two-zone confined aquifer under constant-flux pumping conditions. The analytical solution for steady state and the approximate solution for transient pumping test is something new and interesting to the hydrological community. In general the publication is well-written. The readability could be improved by language check by a native speaker and restructuring several subsections, as described later. Figures and tables are in a good shape, minor improvements are suggested later on.

Response: Thanks for the comment. The manuscript has been revised on the basis of the comments below and edited by a colleague who is good at English writing.

Specific Comments

Abstract

Clarify in the abstract the type of pumping test solution you derive: Is it for homogeneous media (or heterogeneous media)? It is limited to 2D or valid for 3D aquifer description?

Response: To address the problem, we added the statement: "This study develops a new approximate solution for the problem based on a mathematical model describing steady-state radial and vertical flows in a two-zone aquifer. Hydraulic parameters in these two zones can be different but are assumed homogeneous in each zone." (lines $22 - 25$).

Introduction

What are potential applications of the derived approximate solution?

Response: Two sentences shown below are added in the revised manuscript to state its potential applications:

"The transient solution is in term of simple series with advantages of fast convergence, simplicity, and good accuracy from practical viewpoint. It can be used as a convenient tool to estimate temporal and spatial drawdown distributions for the constant-flux pumping and explore physical insight into the flow behavior affected by hydrogeological properties and aquifer configuration." (lines 115 – 119)

Mathematical Model

Specify the aim of the section at the beginning (p 2745, line 22).

Response: We added a sentence: "This section introduces a new mathematical model for steady-state flow due to the CFP at a finite-radius partially penetrating well in a radial two-zone confined aquifer." (lines 125 – 126).

The ordering of the content of the section could be improved: first specify the process of interest (pumping test), including boundary conditions, assumptions and characteristics (e.g. points mentioned in Table 1) first in words, than refer to figure and than in equations.

Response: The section is rewritten as suggested. The new one is listed below:

"This section introduces a new mathematical model for steady-state flow due to the CFP at a finite-radius partially penetrating well in a radial two-zone confined aquifer. The symbols representing variables and parameters for the model are listed in Table 2. The hydraulic parameters in the two zones are different but in each zone are assumed homogeneous. The outer boundary is considered to be under the Dirichlet condition of $\bar{s}_2 = 0$ at $\bar{r} = \bar{R}$. The top and bottom confining beds are under the no-flow conditions of $\partial \bar{s}_i / \partial \bar{z} = 0$ where $i \in (1, 2)$. The effect of wellbore storage on aquifer drawdown is assumed ignorable. Note that this effect diminishes when $t > 2.5 \times 10^2 r_c^2 / T_2$ mentioned in Papadopulos and Cooper (1967). In addition, Yeh and Chang (2013) also mentioned that this effect can be neglected for a well with $r_c \leq 0.25$ m. A schematic diagram for the CFP problem is illustrated in Figure 1.

The governing equations describing steady-state dimensionless drawdown distributions in the skin and formation zones are expressed, respectively, as

$$\frac{\partial^2 \bar{s}_1}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_1}{\partial \bar{r}} + \alpha_1 \frac{\partial^2 \bar{s}_1}{\partial \bar{z}^2} = 0 \quad \text{for} \quad 1 \leq \bar{r} \leq \bar{r}_s \tag{1}$$

and

$$\frac{\partial^2 \bar{s}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_2}{\partial \bar{r}} + \alpha_2 \frac{\partial^2 \bar{s}_2}{\partial \bar{z}^2} = 0 \quad \text{for} \quad \bar{r}_s \leq \bar{r} \leq \bar{R} \tag{2}$$

where α_1 and α_2 reflect the effect of aquifer anisotropy on dimensionless aquifer drawdown. The inner boundary designated at the rim of the wellbore is under the Neumann condition as

$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = -\frac{\gamma}{\phi} \left(U(\bar{z} - \bar{z}_1) - U(\bar{z} - \bar{z}_2) \right) \quad \text{at} \quad \bar{r} = 1 \quad \text{and} \quad 0 \leq \bar{z} \leq 1 \tag{3}$$

where $U(\,\cdot\,)$ is the unit step function. Equation (3) indicates that the flux is uniformly distributed over the screen. Two continuity conditions required at $\bar{r} = \bar{r}_s$ are

$$\bar{s}_1 = \bar{s}_2 \quad \text{at} \quad \bar{r} = \bar{r}_s \tag{4}$$

and

$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = \gamma \frac{\partial \bar{s}_2}{\partial \bar{r}} \quad \text{at} \quad \bar{r} = \bar{r}_s \tag{5}''$$

(lines 125 – 148)

A table containing the symbols of variables and parameters would improve the readability of the work significantly. Refer to that table in caption of fig. 1 and in the text.

Response: Thanks for the comment. We added Table 2 in which the symbols are defined. Two sentences given below related to the table are added in the revised manuscript. "The symbols representing variables and parameters for the model are listed in Table 2." (lines 126 – 127) and "The symbols of the variables are defined in Table 2" (in the caption of Figure 1).

Steady-State Solution

Is the derived solution for steady state already published before? If not, specify that these are new results. If yes give a reference.

Response: To our knowledge, the steady-state solution has not been published elsewhere, which is stated in the sentence of "A new solution derived by the application of the finite Fourier cosine transform to the model can be written as" (lines 150 – 151).

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Approximate Solution

The ordering of the content of the section could be improved: First state the aim of the approach (why), than the idea of the approach (as given in line 9-11, p 2749), than how it is done (line 20, p 2748 – line 6 p 2749) and than the result (line 17, p 2748). Finally elaborate in more detail on the way how R(t) was found (line 15, p. 2748): how was the trail and error procedure performed, what was the tested range of parameters, to what was the approximated solution compared to and how?

Response: Thanks for the comment. The section is rewritten on the basis of the comment. The new one is shown below:

"The inverse Laplace transform to Chiu et al. (2007) semi-analytical solution of drawdown leads to a time-domain result for the CFP in a two-zone aquifer system; however, the resultant solution involves laborious calculations. We therefore develop an approximate transient solution of drawdown for the CFP problem. The idea originated from the concept of a time-dependent diffusion layer for the solution of the diffusion equation in the field of electrochemistry (Fang et al., 2009). The approximate transient solution is obtained by replacing the $\overline{R}$ in the steady-state solution (i.e., Eqs. (6) − (15)) with a dimensionless time-dependent radius of influence $\overline{R}(\bar{t})$. The result is in terms of dimensionless time denoted as

$$s_1(\bar{r},\bar{z},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}_s) + \gamma \ln(\bar{r}_s/\bar{r}) + \frac{2\gamma}{\bar{z}_2 - \bar{z}_1} \sum_{n=1}^{\infty} F_1(\bar{r},n,\bar{t}) \cos(n\pi\bar{z}) \quad \text{for} \quad 1 \le \bar{r} \le \bar{r}_s \quad (16)$$

$$s_2(\bar{r},\bar{z},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}) + \frac{2\gamma}{\bar{r}_s(\bar{z}_2 - \bar{z}_1)} \sum_{n=1}^{\infty} F_2(\bar{r},n,\bar{t}) \cos(n\pi\bar{z}) \quad \text{for} \quad \bar{r}_s \le \bar{r} \le \overline{R} \quad (17)$$

and

$$\overline{R}(\bar{t}) = 1 + \sqrt{\pi\bar{t}/1.4} \quad (18)$$

where $F_1(\bar{r},n,\bar{t})$ and $F_2(\bar{r},n,\bar{t})$ obtained from Eqs. (8) and (9), respectively, with coefficients ψ, ζ, ξ, and $G(\mu,c)$ defined in Eqs. (10) − (13), respectively, are functions of dimensionless time due to substitution of Eq. (18). The time-dependent radius of influence $\overline{R}(\bar{t})$ was first assumed as $\overline{R}(\bar{t}) = 1 + \sqrt{\pi\bar{t}/c}$ where c is a constant. By trial and error, we found that the drawdowns predicted by the approximate solution and Chiu et

al. (2007) Laplace-domain solution with the Crump method agree well when c approaches 1.4. Detailed discussion is shown in section 3.1. Notice that Eq. (18) is similar to an equation given in Yang et al. (2014, Eq. (25)) but has a different coefficient value."
(lines 169 − 189)

Special Case

Give a link to the relation of the special case solution to previously derived results as given in the introduction. (Similar to the sentence in line 6, p. 2750 for the special case in 2.5.)
Response: To our knowledge, the special case of the present solution (i.e., eq. (19) and (20) in the revised manuscript) has never been published before.

Accuracy of approximate Solution

Specify the meaning of the parameters (e.g. line 12, p 2750 state what gamma is, etc.) for easier readability. Give a reason for the choice of parameters, e.g. the point in time t in Figure 2a. Did you test all choices of parameters? What are the ranges of tested parameters? For which choice of parameters did the solutions not match? I recommend to start a new paragraph in line 16, p 2750. The same questions concerning the choice and tested range of parameters as for Fig. 2a apply for Fig. 2b.
Response: To clarify the problem, a new text describing the choice of parameter values for plots in Figure 2 is added in the revised manuscript and also given below:
"On the basis of the comparison of predictions from the approximate solution and Chiu et al. (2007) Laplace-domain solution, we have concluded that the accuracy of the present solution depends only on dimensionless time $\bar{t}$ and radial distance $\bar{r}$ and does not relate to other dimensionless parameters and space variable. Consider representative parameters and variables as follows: $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\alpha_1 = \alpha_2 = 10^{-7}$, and $\gamma = 0.1$ for positive skins, 1 for no skin and 10 for negative skins." (lines 208 − 213)

Which discrepancies you mean in line 20, p 2750? I do not understand the message of the last sentence, especially what do you mean with " time during which the radius of influence arrives"?
Response: To clarify the problem, the associated sentence is rewritten as "The discrepancy

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in dimensionless drawdown at the early period of $0 \leq \bar{t} \leq 600$ can be attributed to the absence of the time derivative term in both Eqs. (1) and (2)." (lines 220 – 222)

The phrase "time during which the radius of influence arrives" has been deleted. In addition, the last sentence is rewritten as "It seems reasonable to conclude that the approximate transient solution gives good predicted drawdown in an observation well over the entire pumping period except at early time when the dynamic radius of influence reaches the well (i.e., $\bar{t} \cong 1.4(\bar{r}-1)^2/\pi$ derived by substituting $\overline{R}(\bar{t}) = \bar{r}$ into Eq. (18) and rearranging the result)." (lines 223 – 226)

Vertical Flow

Specify the meaning of the parameters (e.g. line 5, p 2751 alpha, etc.) to improve the readability. What is b in line 12, p 2751?

Response: Those parameters influence the vertical flow near the partially penetrating well. The original sentence is rewritten as "The vertical flow induced by well partial penetration is strongly dependent on both dimensionless lumped parameters $\alpha_1 \bar{r}^2$ and $\alpha_2 \bar{r}^2$ (i.e., $K_{z1} r^2/(K_{r1} b^2)$ and $K_{z2} r^2/(K_{r2} b^2)$, respectively)." (lines 228 – 230). The symbol b is aquifer thickness defined in Table 2 of the revised manuscript.

Give a reason for the choice of parameters (line 7, p 2751).

Response: Arbitrary choice of the parameter values won't affect the conclusion that the vertical flow induced by well partial penetration is ignorable when $\alpha_1 \bar{r}^2 \geq 1$ and $\alpha_2 \bar{r}^2 \geq 1$.

Are the results shown in Fig. 3 representative for other choices of parameters of r, z and gamma?

Response: Yes, they are.

Concluding remarks

I do not understand what is meant with " during which the time-dependent radius of influence just touches." (line 24, p 2751) State in words (not in formulas) what you mean

in line 1-4, p 2752. The conclusion should be understandable without searching for the meaning of the parameters.

Response: The phrase "during which the time-dependent radius of influence just touches" has been deleted. The conclusion is rewritten as:

"The analysis of the temporal drawdowns predicted by Eqs. (17) and (20) indicates that the vertical flow due to a partially penetrating well prevails under the conditions of thick aquifers, vicinity to the well, and/or small conductivity ratios (i.e., $\alpha_1 \bar{r}^2 < 1$ or $\alpha_2 \bar{r}^2 < 1$). Accordingly, conventional models neglecting the vertical flow will underestimate drawdown under those conditions." (lines 248 – 252)

Figures and Tables

Table 1: Avoid the abbreviation CFP in the caption.

Response: It is replaced by the constant-flux pumping.

Figure 1: Is the abbreviation CFT a typo or was it introduced before? Recommendation of not using abbreviation in caption in general. Variables and parameters used in the Figure are not explained in the caption. This would be OK, if it is given a link to a Table, where they are listed separately.

Response: We appreciate reviewer's eye for detail. The CFT is replaced by the constant-flux pumping.

Figure 2: The different lines in the plots are difficult to distinguish; probably use thicker lines and marker in combination with lines. Give a link to the equation in the text for the "approximate solution".

Response: Figure 2 is redrawn and also shown below.

Figure 3: The different lines in the plots are difficult to distinguish; probably use thicker lines. The color and line scheme appears somewhat arbitrary, this could be improved. List the choice of parameters (as done in caption of Fig. 2).

Response: Figure 3 is redrawn and its caption is rewritten. They are shown at the end of this response.

Technical Corrections

Language could be improvements by native speaker.

Response: The manuscript has been edited by a colleague who is good at English writing.

The usage of the abbreviation CFP is OK, but I would recommend to avoid it in the abstract and figure/table captions.

Response: The CFP in the abstract and figure/table captions are replaced by the constant-flux pumping.

Interactive comment on Hydrol. Earth Syst. Sci. Discuss., 12, 2741, 2015.

Table 2. Summary of symbols used in the text and their definitions

| Symbols	Definitions
(s_1, s_2)	Drawdowns in skin and formation zones, respectively
r	Radial distance from the center of the well
r_s	Radius of skin zone
R	Radius of cylinder aquifer domain or the radius of influence
(r_w, r_c)	Outer and inner radiuses of well, respectively
z	Elevation from the aquifer bottom
(z_1, z_2)	Lower and upper elevations of well screen, respectively
t	Time since pumping
b	Aquifer thickness
Q	Pumping rate of well
(K_{r1}, K_{r2})	Radial hydraulic conductivities of skin and formation zones, respectively
(K_{v1}, K_{v2})	Vertical hydraulic conductivities of skin and formation zones, respectively
S_{s2}	Specific storage of formation zone
(T_1, T_2)	Transmissivities of skin and formation zones, respectively
$(\bar{s}_1, \bar{s}_2)$	$(2\pi T_2 s_1/Q,\ 2\pi T_2 s_2/Q)$
$\bar{t}$	$K_{r2}\, t/(S_{s2} r_w^2)$
$(\bar{r},\ \bar{r}_s,\ \bar{R})$	$(r/r_w,\ r_s/r_w,\ R/r_w)$
$(\bar{z},\ \bar{z}_1,\ \bar{z}_2)$	$(z/b,\ z_1/b,\ z_2/b)$
$(\phi,\ \gamma)$	$(\bar{z}_2 - \bar{z}_1,\ K_{r2}/K_{r1})$
(α_1, α_2)	$(K_{z1} r_w^2/(K_{r1} b^2),\ K_{z2} r_w^2/(K_{r2} b^2))$
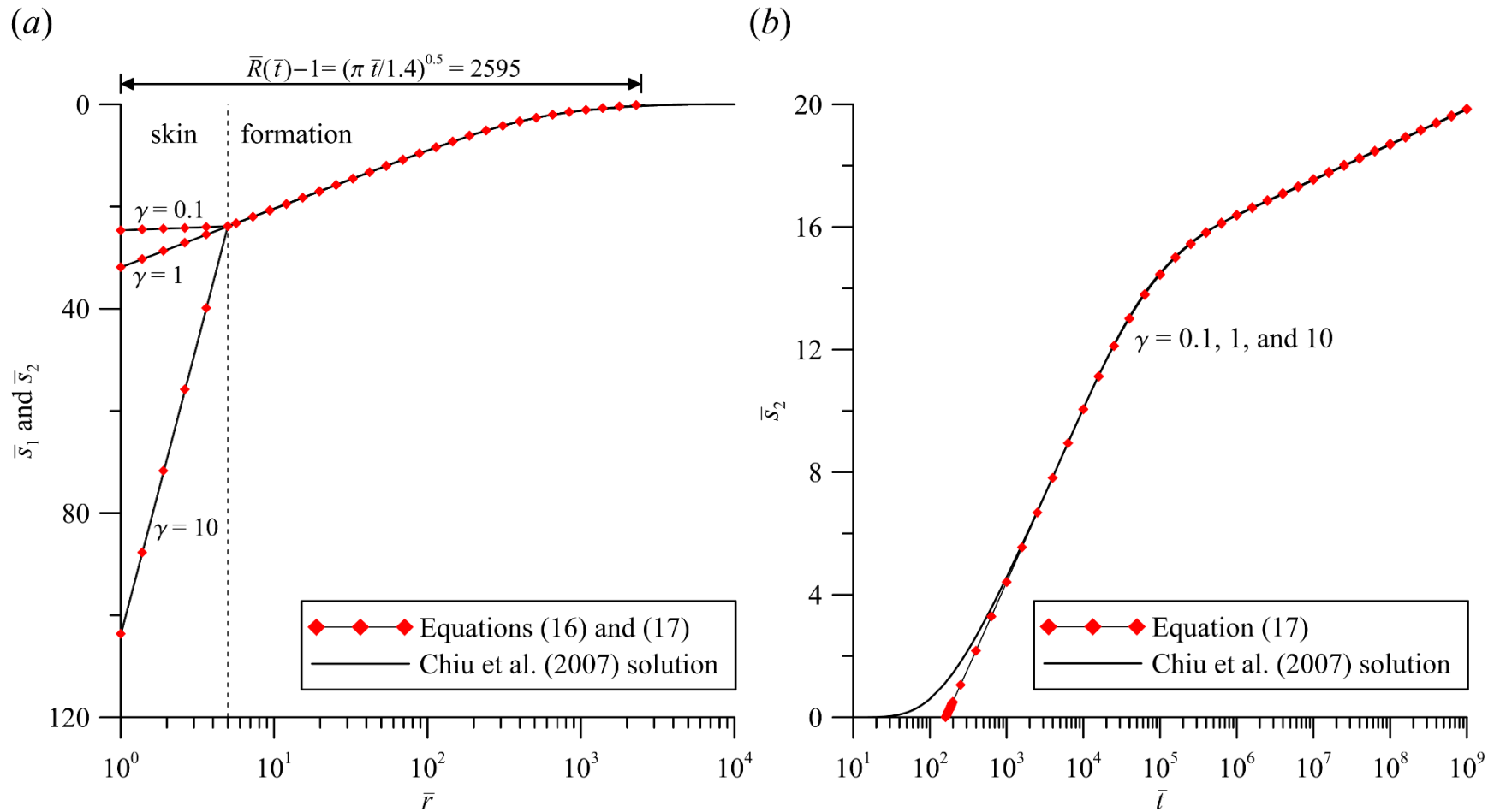
Figure 2. Predicted drawdowns by Chiu et al. (2007) solution and the approximate solution, Eqs. (16) and (17), with $\gamma = 0.1$, 1, and 10 for (a) spatial distributions at $\bar{t} = 3 \times 10^6$ and (b) temporal distributions at $\bar{r} = 20$ with $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$
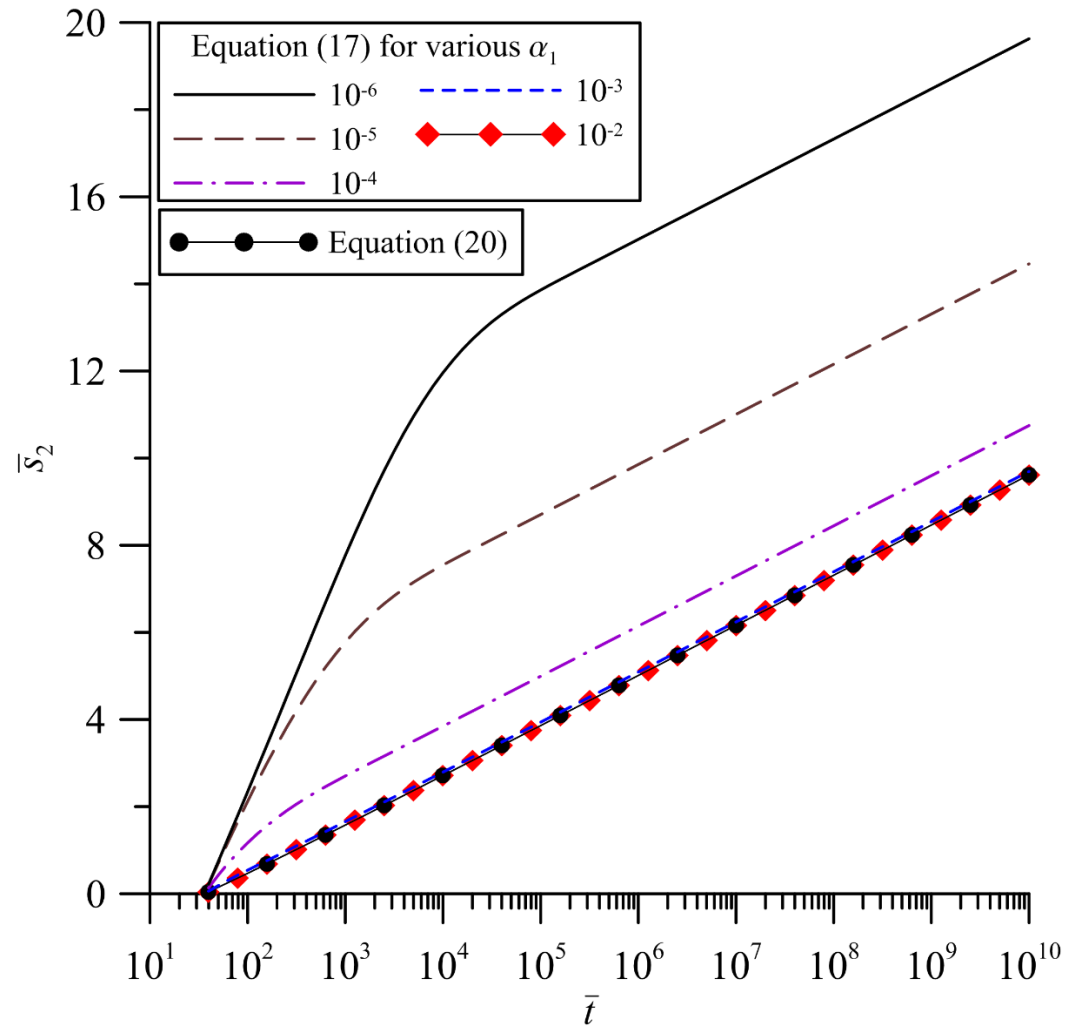
Figure 3. Temporal drawdown distributions predicted by the approximate solution, Eq. (17), with $\bar{r} = 10$, $\bar{z} = 0.5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\bar{r}_s = 5$, $\gamma = 0.1$ and various values of α_1 with $\alpha_1 = \alpha_2$

Interactive comment on "Technical Note: Approximate solution of transient drawdown for constant-flux pumping at a partially penetrating well in a radial two-zone confined aquifer" by C.-S. Huang et al.

Anonymous Referee #2
Received and published: 14 April 2015

General Comments

C.-S. Huang, S.-Y. Yang and H.-D. Yeh present in the technical note a new approximate solution for the drawdown induced by a constant rate pumping test in a radial two-zone confined aquifer. By considering partial penetration of the pumping well, their approximate solution for the transient drawdown is new and interesting to the hydrologic community. The publication is well-written. A more detailed description of the mathematical model and results in the discussion section could improve readability and understanding, as described later. Tables and figures are clear and comprehensible, minor improvements are suggested for captions.

Response: Thanks for the comment. The manuscript has been revised on the basis of the comments below.

Specific Comments

Abstract

Please clarify under which assumptions your solution is valid: 2D or 3D aquifer, heterogeneous or homogeneous media?

Response: To address the problem, we added the statement: "This study develops a new approximate solution for the problem based on a mathematical model describing steady-state radial and vertical flows in a two-zone aquifer. Hydraulic parameters in these two zones can be different but are assumed homogeneous in each zone." (lines $22 - 25$).

Introduction

The introduction might concentrate on studies directly related to the presented work, i.e. solutions which consider partial penetration or two-zone aquifers, respectively. A comprehensive overview on available solutions for constant rate pumping tests is already

given in table 1.

Response: Thanks.

What are potential applications for the presented solution?

Response: Two sentences shown below are added in the revised manuscript to state its potential applications:

"The transient solution is in term of simple series with advantages of fast convergence, simplicity, and good accuracy from practical viewpoint. It can be used as a convenient tool to estimate temporal and spatial drawdown distributions for the constant-flux pumping and explore physical insight into the flow behavior affected by hydrogeological properties and aquifer configuration." (lines $115 - 119$)

Mathematical Model

Please state the aim of the section a the beginning and explain model configurations and assumptions (2D or 3D, homogeneous or heterogeneous media, boundary conditions).

Response: Thanks for the comment. We added a new paragraph given below to state the aim and describe model configurations and assumptions.

"This section introduces a new mathematical model for steady-state flow due to the CFP at a finite-radius partially penetrating well in a radial two-zone confined aquifer. The symbols representing variables and parameters for the model are listed in Table 2. The hydraulic parameters in the two zones are different but in each zone are assumed homogeneous. The outer boundary is considered to be under the Dirichlet condition of $\bar{s}_2 = 0$ at $\bar{r} = \bar{R}$. The top and bottom confining beds are under the no-flow conditions of $\partial \bar{s}_i / \partial \bar{z} = 0$ where $i \in (1, 2)$. The effect of wellbore storage on aquifer drawdown is assumed ignorable. Note that this effect diminishes when $t > 2.5 \times 10^2 r_c^2 / T_2$ mentioned in Papadopulos and Cooper (1967). In addition, Yeh and Chang (2013) also mentioned that this effect can be neglected for a well with $r_c \leq 0.25$ m. A schematic diagram for the CFP problem is illustrated in Figure 1." (lines $125 - 134$).

Figure 1 shows the observation well to be screened over the entire aquifer. Is this a

prerequisite for your solution or is it also valid for partially penetrating observation wells?

Response: The present solution is applicable to a fully or partially penetrating observation well.

Refer to the meaning of α in the text.

Response: We added a new text shown below to describe it.

"The analysis of the temporal drawdowns predicted by Eqs. (17) and (20) indicates that the vertical flow due to a partially penetrating well prevails under the conditions of thick aquifers, vicinity to the well, and/or small conductivity ratio (i.e., $\alpha_1 \bar{r}^2 < 1$ or $\alpha_2 \bar{r}^2 < 1$)." (lines 248 – 251)

α includes the vertical hydraulic conductivity, K_z. Are you assuming the aquifer and the skin zone to be anisotropic? Please elaborate this in more detail.

Response: Yes, we added the phrase "α_1 and α_2 reflect the effect of aquifer anisotropy on dimensionless aquifer drawdown" in lines 140 – 141.

Approximate Solution

Please elaborate in more detail on the procedure how equation 18 was found to allow for reproducibility. What was the range of tested parameters. How is the accuracy of equation 18 for early, intermediate and late pumping times?

Response: The dynamic radius of influence, $\bar{R}(\bar{t}) = 1 + \sqrt{\pi \bar{t}/1.4}$, defined in the equation is applicable to any value of time t, well radius r_w, radial hydraulic conductivity K_{r2}, and specific storage S_{s2}. In addition, we added a new text in lines 184 – 187, also shown below, to explain how to obtain the equation:

"The time-dependent radius of influence $\bar{R}(\bar{t})$ was first assumed as $\bar{R}(\bar{t}) = 1 + \sqrt{\pi \bar{t}/c}$ where c is a constant. By trial and error, we found that the drawdowns predicted by the approximate solution and Chiu et al. (2007) Laplace-domain solution with the Crump method agree well when c approaches 1.4."

Why is the coefficient different to the one obtained by Yang et al. (2014)?

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Response: The coefficient equals unity found by Yang et al. (2014) for constant-head pumping tests and 1.4 found in this work for constant-flux pumping tests.

Accuracy of approximate Solution

Please specify "the approximate solution" in line 10, p 2750 by giving the equations you refer to.

Response: The equation numbers are provided as suggested.

Presented results generally assume that $\alpha_1 = \alpha_2$. What does this assumption imply?

Response: The assumption $\alpha_1 = \alpha_2$ (i.e., $K_{z1} r_w^2 /(K_{r1} b^2) = K_{z2} r_w^2 /(K_{r2} b^2)$) means that the conductivity ratios for both formation and skin zones are the same because of constant well radius r_w and aquifer thickness b.

Figures 2a and 2b show examples of the solution at specific values of dimensionless time and distance. Were these values chosen randomly or by some criterion? Did you test other choices?

Response: Yes, we did. The agreement on dimensionless drawdown in the figures stands for any chosen values of dimensionless parameters and variables.

I assume that by "discrepancy" in line 20, p 2750, you mean the deviation of your solution from Chiu et al. (2007) during early pumping times. As I understand, equation 18 should compensate for neglecting the temporal derivative in equations (2) and (3). Might the deviation between the two solutions possibly come from the definition of $\bar{R}(\bar{t})$ in equation 18? This could be answered by a more detailed description on the trail and error procedure regarding $\bar{R}(\bar{t})$ in section 2.3.

Response: Yes, you are right. The present solution was developed using equation 18 to compensate the neglect of the temporal derivative term in the transient groundwater flow equation. To clarify the problem, the associated sentence is rewritten as "The discrepancy in dimensionless drawdown at the early period of $0 \leq \bar{t} \leq 600$ can be attributed to the absence of the time derivative term in both Eqs. (1) and (2)." (lines 220 – 222)

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The phrase "except at early time during which the radius of influence arrives" (line 2, p 2751) is somehow unclear to me.

Response: The phrase "time during which the radius of influence arrives" has been deleted. The associated sentence is rewritten as "It seems reasonable to conclude that the approximate transient solution gives good predicted drawdown in an observation well over the entire pumping period except at early time when the dynamic radius of influence reaches the well (i.e., $\bar{t} \cong 1.4(\bar{r}-1)^2/\pi$ derived by substituting $\overline{R}(\bar{t}) = \bar{r}$ into Eq. (18) and rearranging the result)." (lines $223 - 226$)

Vertical Flow

What does the assumption $\alpha_1 = \alpha_2$ imply? Did you test for $\alpha_1 \neq \alpha_2$?

Response: Yes, we did. The assumption $\alpha_1 = \alpha_2$ indicates that the conductivity ratios for the formation zone and skin zone are the same.

How/why did you choose the set of parameters ($\bar{r}$, $\bar{z}$,...)? Did you test for other choices?

Response: Yes, we did. The criterion that the vertical flow effect on the aquifer drawdown vanishes is valid (or applicable) for any parameter values. We take a representative one of them for example.

In line 10, p 2751, "vertical flow vanishes": I guess the effect of vertical flow on the drawdown at an observation vanishes, but vertical flow itself does continue.

Response: Thanks for the comment. The original sentence is rewritten as "We may, therefore, reasonably conclude that the vertical flow effect on the aquifer drawdown at an observation well vanishes when $\alpha_1 \bar{r}^2 \geq 1$ and $\alpha_2 \bar{r}^2 \geq 1$, i.e., b is small, r is large, and/or the values of K_{z1}/K_{r1} and K_{z2}/K_{r2} are large." (lines $235 - 237$)

You state a criterion for the presented model by which vertical flow can be neglected. Do comparable criteria exist for other models (e.g. models mentioned in the introduction)? If so, please relate to those. Such criteria could also be mentioned in the introduction.

Response: To our knowledge, our work is the first to provide such a criterion.

Concluding remarks

Line 23, p 2751: The meaning of the phrase "during which the time-dependent radius of influence just touches" is somehow unclear to me. Please consider revising.

Response: The phrase "during which the time-dependent radius of influence just touches" has been deleted. The associated sentence is rewritten as:

"The comparison with the Chiu et al. (2007) solution reveals that the approximate solution gives accurate temporal drawdown distributions in an observation well over the entire pumping period except at early time when the dynamic radius of influence reaches the well (i.e., $\bar{t} \cong 1.4(\bar{r}-1)^2/\pi$ derived by substituting $\bar{R}(\bar{t}) = \bar{r}$ into Eq. (18) and rearranging the result)." (lines $244 - 248$)

Figures

Figure 1: Specify abbreviations.

Response: The abbreviation of "CFT" in the figure caption is replaced by "the constant-flux pumping".

Figure 2: Specify the approximate solution by referring to equations.

Response: Figure 2 is redrawn as suggested.

Figure 3: List chosen parameters (r, z, ...) in the caption, as done for figure 2.

Response: The caption is rewritten as suggested.

Technical Corrections

Figure 1: Is CFT a typo? If so, please change to CFP.

Response: We appreciate reviewer's eye for detail. It has been changed to "constant-flux pumping".

Please consider revising the language by a native speaker.

Response: The manuscript has been edited by a colleague who is good at English writing.

Interactive comment on Hydrol. Earth Syst. Sci. Discuss., 12, 2741, 2015.

Table 2. Summary of symbols used in the text and their definitions

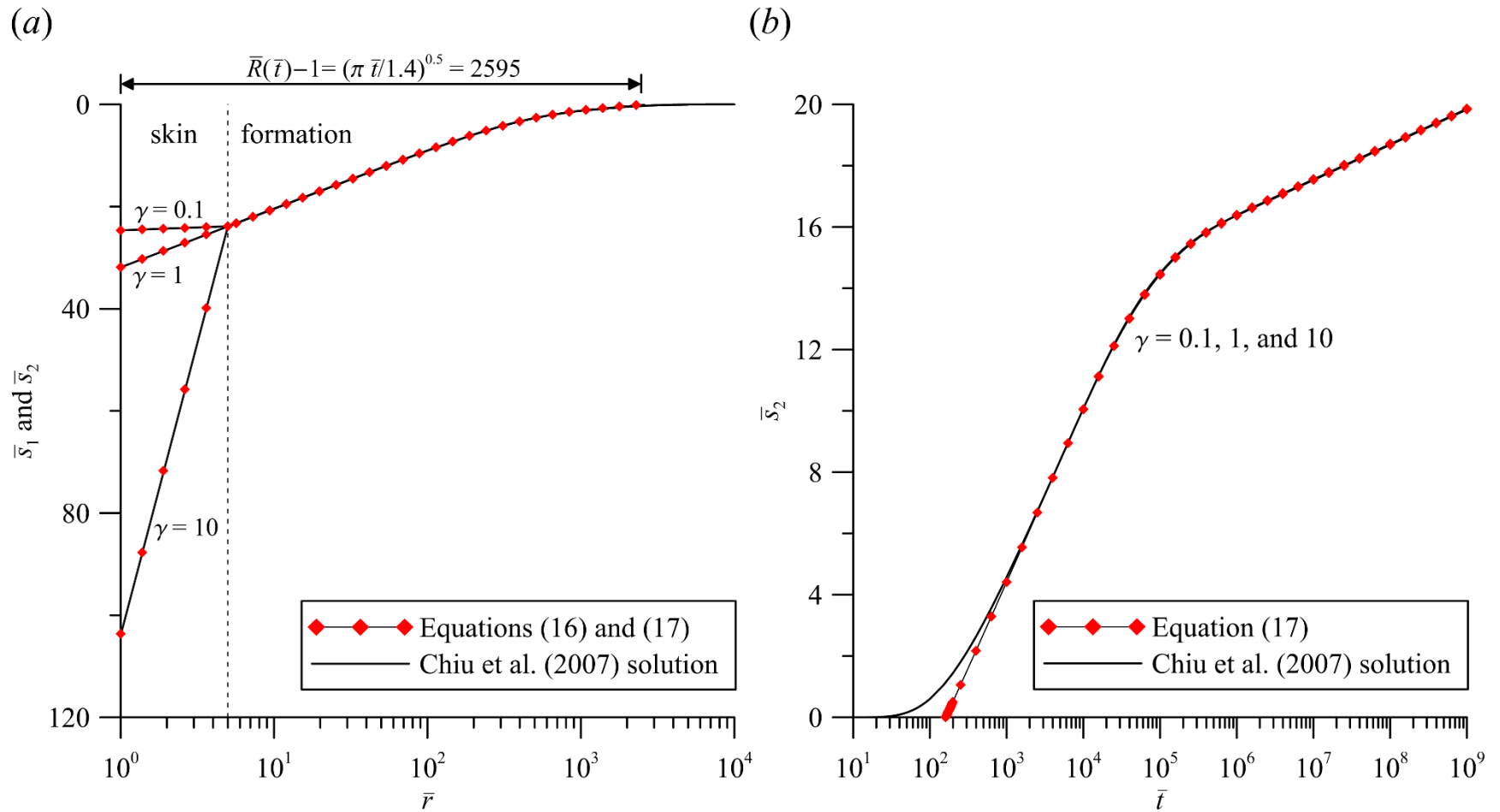| Symbols	Definitions
(s_1, s_2)	Drawdowns in skin and formation zones, respectively
r	Radial distance from the center of the well
r_s	Radius of skin zone
R	Radius of cylinder aquifer domain or the radius of influence
(r_w, r_c)	Outer and inner radiuses of well, respectively
z	Elevation from the aquifer bottom
(z_1, z_2)	Lower and upper elevations of well screen, respectively
t	Time since pumping
b	Aquifer thickness
Q	Pumping rate of well
(K_{r1}, K_{r2})	Radial hydraulic conductivities of skin and formation zones, respectively
(K_{v1}, K_{v2})	Vertical hydraulic conductivities of skin and formation zones, respectively
S_{s2}	Specific storage of formation zone
(T_1, T_2)	Transmissivities of skin and formation zones, respectively
$(\bar{s}_1, \bar{s}_2)$	$(2\pi T_2 s_1 / Q, 2\pi T_2 s_2 / Q)$
$\bar{t}$	$K_{r2} t / (S_{s2} r_w^2)$
$(\bar{r}, \bar{r}_s, \bar{R})$	$(r/r_w, r_s/r_w, R/r_w)$
$(\bar{z}, \bar{z}_1, \bar{z}_2)$	$(z/b, z_1/b, z_2/b)$
(ϕ, γ)	$(\bar{z}_2 - \bar{z}_1, K_{r2}/K_{r1})$
(α_1, α_2)	$(K_{z1} r_w^2 /(K_{r1} b^2), K_{z2} r_w^2 /(K_{r2} b^2))$

Figure 2. Predicted drawdowns by Chiu et al. (2007) solution and the approximate solution, Eqs. (16) and (17), with $\gamma = 0.1$, 1, and 10 for (a) spatial distributions at $\bar{t} = 3 \times 10^6$ and (b) temporal distributions at $\bar{r} = 20$ with $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$

1   **Technical Note: Approximate Solution of Transient Drawdown for**

2   **Constant-Flux Pumping at a Partially Penetrating Well in a Radial**

3   **Two-Zone Confined Aquifer**

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18

19 **Abstract**

20 An aquifer consisting of a skin zone and a formation zone is considered as a two-zone aquifer.

21 Existing solutions for the problem of constant-flux pumping in a two-zone confined aquifer

22 involve laborious calculation. This study develops a new approximate solution for the

23 problem based on a mathematical model describing steady-state radial and vertical flows in a

24 two-zone aquifer. Hydraulic parameters in these two zones can be different but are assumed

25 homogeneous in each zone. A partially penetrating well may be treated as the Neumann

26 condition with a known flux along the screened part and zero flux along the unscreened part.

27 The aquifer domain is finite with an outer circle boundary treated as the Dirichlet condition.

28 The steady-state drawdown solution of the model is derived by the finite Fourier cosine

29 transform. Then, an approximate transient solution is developed by replacing the radius of the

30 aquifer domain in the steady-state solution with an analytical expression for a dimensionless

31 time-dependent radius of influence. The approximate solution is capable of predicting good

32 temporal drawdown distributions over the whole pumping period except at the early stage. A

33 quantitative criterion for the validity of neglecting the vertical flow due to a partially

34 penetrating well is also provided. Conventional models considering radial flow without the

35 vertical component for the constant-flux pumping have good accuracy if satisfying the

36 criterion.

37 **Keywords:** skin zone, constant flux test, finite Fourier cosine transform, time-dependent

38   radius of influence

1. Introduction

The constant-flux pumping (CFP) test is a widely used well test for characterizing the aquifer properties such as transmissivity and storage coefficient. The test is performed with a constant pumping rate at a fully or partially penetration well in either a confined or unconfined aquifer. Existing analytical solutions for the CFP in a homogenous confined aquifer are briefly reviewed herein. Theis (1935) was the first article in the groundwater literature to present an analytical solution for aquifer drawdown due to pumping in a fully penetrating well with an infinitesimal radius. Carslaw and Jaeger (1959) presented analytical solutions for the three kinds of heat conduction problems which can be analogous to the CFP problems including the aquifers of the infinite domain with a finite-radius well, finite domain with a finite-radius well, and finite domain with an infinitesimal-radius well. Hantush (1962) developed an analytical solution of drawdown induced by a partially penetrating well for the CFP. Papadopulos and Cooper (1967) obtained an analytical solution of drawdown with considering the effects of well radius and wellbore storage. They provided a quantitative criterion of time for neglecting the effects. The criterion will be stated in the next section. Chen (1984) derived an analytical solution for drawdown in a circular aquifer with the Dirichlet boundary condition of zero drawdown and provided a quantitative criterion describing the beginning time of the boundary effect on the drawdown. Yang et al. (2006)

57   developed an analytical solution describing aquifer drawdown due to a partially penetrating

58   well with a finite radius. The effect of partial penetration on temporal drawdown distributions

59   was discussed. Wang and Yeh (2008) provided a quantitative criterion for the beginning time

60   of the boundary effect on drawdown induced by the CFP and constant-head pumping. Yeh and

61   Chang (2013) provided a comprehensive review on analytical solutions for the CFP in

62   unconfined and multilayered aquifer systems.

63   Drilling an aquifer to install a well may decrease or increase the permeability of the

64   formation around the wellbore. The perturbed formation, called as skin zone, extends from a

65   few millimeters to several meters. A positive skin zone means that its permeability is lower

66   than the original formation. On the other hand, a negative skin zone is of a higher

67   permeability than the original formation. Existing solutions accounting for the CFP in a

68   two-zone confined aquifer consisting of the skin zone and formation zone are reviewed.

69   Novakowski (1989) developed a semi-analytical solution of drawdown with the wellbore

70   storage effect and investigated the effect of an infinitesimally thin skin on temporal drawdown

71   curves. Hemker (1999) proposed an analytical-numerical solution describing pumping

72   drawdown in a multilayered aquifer system where the radial flow was analytically treated and

73   the vertical one was handled by a finite difference method. The flux along the well screen was

74   non-uniform through an infinitesimal thin skin, and the flow was subject to the wellbore

75   storage effect. Kabala and El-Sayegh (2002) presented a semi-analytical solution for the

76   transient flowmeter test in a multilayered aquifer system where the radial flow was considered

77   in each layer with assuming no vertical flow component and uniform flux along the well

78   screen. Predictions from the solution were compared with those from a numerical solution

79   which relaxes those two assumptions. Yeh et al. (2003) obtained an analytical solution for

80   pumping drawdown induced by a finite-radius well in a two-zone confined aquifer and

81   discussed the error caused by neglecting the well radius. Chen and Chang (2006) developed a

82   semi-analytical solution for the CFP on the basis of the Gram-Schmidt method to deal with

83   the non-uniform skin effect represented by an arbitrary piecewise function of elevation. They

84   indicated that flow near a pumping well is three dimensional due to the effect and away from

85   the well is radial. Perina and Lee (2006) proposed a general well function for transient flow

86   toward a partially penetrating well with considering the wellbore storage effect and

87   non-uniform flux between the screen and skin zone in a confined, unconfined, or leaky aquifer.

88   Chiu et al. (2007) developed a semi-analytical solution for the CFP at a partial penetrating

89   well in a two-zone confined aquifer. They indicated that the influence of the partial

90   penetration on drawdown is more significant for a negative skin zone than a positive one.

91   Wang et al. (2012a) provided an analytical solution of drawdown for the CFP in a two-zone

92   confined aquifer of finite extent with an outer boundary under the Dirichlet condition of zero

93 drawdown. They also derived a large-time drawdown solution which reduces to the Thiem

94 solution in the absence of the skin zone. Wang et al. (2012b) presented a finite layer method

95 (FLM) based on Galerkin's technique for simulating radial and vertical flows toward a

96 partially penetrating well in a multilayered aquifer system. The FLM was verified by an

97 analytical solution and finite difference solution.

98   It is informative to classify the above solutions into two groups, i.e., homogeneous

99 aquifer and two-zone aquifer systems in Table 1. The solutions in each group are categorized

100 according to the well penetration, well radius, and wellbore storage.

101   At the present, a time-domain analytical solution of drawdown for flow induced by the

102 CFP at a finite-radius partially penetrating well in a two-zone confined aquifer has not been

103 developed. The Laplace-domain result of the above-mentioned problem was presented by

104 Chiu et al. (2007) with resort to a numerical inversion scheme called the Crump method. The

105 application of their solution may therefore be inconvenient for those who are not familiar with

106 numerical approaches. The purpose of this note is to develop a new approximate transient

107 solution for the problem in a way similar to our previous work of Yang et al. (2014). A

108 mathematical model for steady-state flow due to a partially penetrating well in a finite-extent

109 two-zone confined aquifer is built. The flow equations describing spatial drawdowns in the

110 skin and formation zones are employed. The outer boundary of the aquifer is specified as the

111   Dirichlet condition of zero drawdown. The well is treated as the Neumann condition with a

112   constant flux for the screened part and zero flux for the unscreened part. The steady-state

113   solution of the model for drawdown is derived by the method of finite Fourier cosine

114   transform. The approximate transient solution of drawdown is then obtained on the basis of

115   the steady-state solution and a time-dependent radius of influence. The transient solution is in

116   term of simple series with advantages of fast convergence, simplicity, and good accuracy

117   from practical viewpoint. It can be used as a convenient tool to estimate temporal and spatial

118   drawdown distributions for the constant-flux pumping and explore physical insight into the

119   flow behavior affected by hydrogeological properties and aquifer configuration. The accuracy

120   of the solution is investigated in comparison with Chiu et al. (2007) solution. In addition, the

121   condition of neglecting the effect of the vertical flow on temporal drawdown distributions is

122   investigated.

123   **2.   Methodology**

124   **2.1. Mathematical Model**

125   This section introduces a new mathematical model for steady-state flow due to the CFP

126   at a finite-radius partially penetrating well in a radial two-zone confined aquifer. The symbols

127   representing variables and parameters for the model are listed in Table 2. The hydraulic

128   parameters in the two zones are different but in each zone are assumed homogeneous. The

outer boundary is considered to be under the Dirichlet condition of $\bar{s}_2 = 0$ at $\bar{r} = \overline{R}$. The top

and bottom confining beds are under the no-flow conditions of $\partial \bar{s}_i / \partial \bar{z} = 0$ where $i \in (1, 2)$.

The effect of wellbore storage on aquifer drawdown is assumed ignorable. Note that this

effect diminishes when $t > 2.5 \times 10^2 r_c^2 / T_2$ mentioned in Papadopulos and Cooper (1967). In

addition, Yeh and Chang (2013) also mentioned that this effect can be neglected for a well

with $r_c \leq 0.25$ m. A schematic diagram for the CFP problem is illustrated in Figure 1.

The governing equations describing steady-state dimensionless drawdown distributions

in the skin and formation zones are expressed, respectively, as

$$\frac{\partial^2 \bar{s}_1}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_1}{\partial \bar{r}} + \alpha_1 \frac{\partial^2 \bar{s}_1}{\partial \bar{z}^2} = 0 \quad \text{for} \quad 1 \leq \bar{r} \leq \bar{r}_s \tag{1}$$

and

$$\frac{\partial^2 \bar{s}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{s}_2}{\partial \bar{r}} + \alpha_2 \frac{\partial^2 \bar{s}_2}{\partial \bar{z}^2} = 0 \quad \text{for} \quad \bar{r}_s \leq \bar{r} \leq \overline{R} \tag{2}$$

where α_1 and α_2 reflect the effect of aquifer anisotropy on dimensionless aquifer

drawdown. The inner boundary designated at the rim of the wellbore is under the Neumann

condition as

$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = -\frac{\gamma}{\phi} \left(U(\bar{z} - \bar{z}_1) - U(\bar{z} - \bar{z}_2) \right) \quad \text{at} \quad \bar{r} = 1 \quad \text{and} \quad 0 \leq \bar{z} \leq 1 \tag{3}$$

where $U(\cdot)$ is the unit step function. Equation (3) indicates that the flux is uniformly

distributed over the screen. Two continuity conditions required at $\bar{r} = \bar{r}_s$ are

$$\bar{s}_1 = \bar{s}_2 \quad \text{at} \quad \bar{r} = \bar{r}_s \tag{4}$$

147 and

148
$$\frac{\partial \bar{s}_1}{\partial \bar{r}} = \gamma \frac{\partial \bar{s}_2}{\partial \bar{r}} \quad \text{at} \quad \bar{r} = \bar{r}_s \qquad (5)$$

149 **2.2. Steady-State Solution**

150   A new solution derived by the application of the finite Fourier cosine transform to the

151 model can be written as

152
$$s_1(\bar{r}, \bar{z}) = \ln(\bar{R}/\bar{r}_s) + \gamma \ln(\bar{r}_s/\bar{r}) + \frac{2\gamma}{\bar{z}_2 - \bar{z}_1} \sum_{n=1}^{\infty} F_1(\bar{r}, n) \cos(n\pi\bar{z}) \quad \text{for} \quad 1 \le \bar{r} \le \bar{r}_s \qquad (6)$$

153 and

154
$$s_2(\bar{r}, \bar{z}) = \ln(\bar{R}/\bar{r}) + \frac{2\gamma}{\bar{r}_s(\bar{z}_2 - \bar{z}_1)} \sum_{n=1}^{\infty} F_2(\bar{r}, n) \cos(n\pi\bar{z}) \quad \text{for} \quad \bar{r}_s \le \bar{r} \le \bar{R} \qquad (7)$$

155 with

156
$$F_1(\bar{r}, n) = \omega \left(\zeta\, I_0(\lambda_1 \bar{r}) + \xi\, K_0(\lambda_1 \bar{r}) \right)/(\lambda_1 \psi) \qquad (8)$$

157
$$F_2(\bar{r}, n) = \omega \left(K_0(\lambda_2 \bar{R}) I_0(\lambda_2 \bar{r}) - I_0(\lambda_2 \bar{R}) K_0(\lambda_2 \bar{r}) \right)/(\lambda_1 \psi) \qquad (9)$$

158
$$\psi = \lambda_1 G(0, -1) H(1, -1) - \gamma \lambda_2 G(1,1) H(0,1) \qquad (10)$$

159
$$\zeta = \lambda_1 K_1(\lambda_1 \bar{r}_s) G(0, -1) + \gamma \lambda_2 K_0(\lambda_1 \bar{r}_s) G(1,1) \qquad (11)$$

160
$$\xi = \lambda_1 I_1(\lambda_1 \bar{r}_s) G(0, -1) - \gamma \lambda_2 I_0(\lambda_1 \bar{r}_s) G(1,1) \qquad (12)$$

161
$$G(\mu, c) = I_\mu(\lambda_2 \bar{r}_s) K_0(\lambda_2 \bar{R}) + c K_\mu(\lambda_2 \bar{r}_s) I_0(\lambda_2 \bar{R}) \qquad (13)$$

162
$$H(\mu, c) = K_1(\lambda_1) I_\mu(\lambda_1 \bar{r}_s) + c I_1(\lambda_1) K_\mu(\lambda_1 \bar{r}_s) \qquad (14)$$

163 and

164
$$\omega = \left(\sin(\bar{z}_2 \pi n) - \sin(\bar{z}_1 \pi n) \right)/(\pi n) \qquad (15)$$

165 where $\lambda_i = \pi n \sqrt{\alpha_i}$, and $I_\mu(\,\cdot\,)$ and $K_\mu(\,\cdot\,)$ are the modified Bessel functions of the first and

166 second kinds with order μ, respectively. The detailed derivation of the solution is given in

167 Appendix A.

168 **2.3. Approximate Transient Solution**

169 The inverse Laplace transform to Chiu et al. (2007) semi-analytical solution of

170 drawdown leads to a time-domain result for the CFP in a two-zone aquifer system; however,

171 the resultant solution involves laborious calculations. We therefore develop an approximate

172 transient solution of drawdown for the CFP problem. The idea originated from the concept of

173 a time-dependent diffusion layer for the solution of the diffusion equation in the field of

174 electrochemistry (Fang et al., 2009). The approximate transient solution is obtained by

175 replacing the $\overline{R}$ in the steady-state solution (i.e., Eqs. (6) − (15)) with a dimensionless

176 time-dependent radius of influence $\overline{R}(\overline{t})$. The result is in terms of dimensionless time

177 denoted as

178 $$s_1(\overline{r},\overline{z},\overline{t}) = \ln(\overline{R}(\overline{t})/\overline{r}_s) + \gamma\ln(\overline{r}_s/\overline{r}) + \frac{2\gamma}{\overline{z}_2-\overline{z}_1}\sum_{n=1}^{\infty}F_1(\overline{r},n,\overline{t})\cos(n\pi\overline{z}) \quad \text{for} \ \ 1\leq\overline{r}\leq\overline{r}_s \qquad (16)$$

179 $$s_2(\overline{r},\overline{z},\overline{t}) = \ln(\overline{R}(\overline{t})/\overline{r}) + \frac{2\gamma}{\overline{r}_s(\overline{z}_2-\overline{z}_1)}\sum_{n=1}^{\infty}F_2(\overline{r},n,\overline{t})\cos(n\pi\overline{z}) \quad \text{for} \ \ \overline{r}_s\leq\overline{r}\leq\overline{R} \qquad (17)$$

180 and

181 $$\overline{R}(\overline{t}) = 1 + \sqrt{\pi\overline{t}/1.4} \qquad (18)$$

182 where $F_1(\overline{r},n,\overline{t})$ and $F_2(\overline{r},n,\overline{t})$ obtained from Eqs. (8) and (9), respectively, with

11

coefficients ψ, ζ, ξ, and $G(\mu,c)$ defined in Eqs. (10) − (13), respectively, are functions

of dimensionless time due to substitution of Eq. (18). The time-dependent radius of influence

$\overline{R}(\bar{t})$ was first assumed as $\overline{R}(\bar{t}) = 1 + \sqrt{\pi \bar{t}/c}$ where c is a constant. By trial and error, we

found that the drawdowns predicted by the approximate solution and Chiu et al. (2007)

Laplace-domain solution with the Crump method agree well when c approaches 1.4. Detailed

discussion is shown in section 3.1. Notice that Eq. (18) is similar to an equation given in Yang

et al. (2014, Eq. (25)) but has a different coefficient value.

2.4. Special Case 1: Solution for CFP at Fully Penetration Well in Two-Zone Aquifer

When $\bar{z}_1 = 0$ and $\bar{z}_2 = 1$ (i.e., $z_1 = 0$ and $z_2 = b$) for the case of well full penetration,

one can obtain $\omega = 0$ according to Eq. (15). The simple series in Eqs. (16) and (17) then

vanishes, and the solution for temporal drawdown distributions subject to the skin effect

reduces to

$$s_1(\bar{r},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}_s) + \gamma \ln(\bar{r}_s/\bar{r}) \text{ for } 1 \le \bar{r} \le \bar{r}_s \tag{19}$$

and

$$s_2(\bar{r},\bar{t}) = \ln(\overline{R}(\bar{t})/\bar{r}) \text{ for } \bar{r}_s \le \bar{r} \le \overline{R}(\bar{t}) \tag{20}$$

Note that Eqs. (19) and (20) are independent of $\bar{z}$, indicating that groundwater flow is only

horizontal.

2.5. Special Case 2: Solution for CFP at Fully Penetration Well in Homogeneous Aquifer

201   When $\bar{z}_1 = 0$, $\bar{z}_2 = 1$, and $\gamma = 1$ (i.e., $z_1 = 0$, $z_2 = b$, and $K_{r1} = K_{r2}$) for the case of a fully

202   penetrating well in a homogeneous aquifer, Eqs. (16) and (17) yield

203     $s(\bar{r},\bar{t}) = \ln(\bar{R}(\bar{t})/\bar{r})$ for $1 \leq \bar{r} \leq \bar{R}(\bar{t})$ (21)

204   which is indeed a dimensionless form of Thiem's equation. Note that Eq. (21) can also be

205   derived by substituting $\gamma = 1$ into Eq. (19).

206   **3. Results and Discussion**

207   **3.1. Accuracy of Approximate Solution**

208   On the basis of the comparison of predictions from the approximate solution and Chiu et

209   al. (2007) Laplace-domain solution, we have concluded that the accuracy of the present

210   solution depends only on dimensionless time $\bar{t}$ and radial distance $\bar{r}$ and does not relate to

211   other dimensionless parameters and space variable. Consider representative parameters and

212   variables as follows: $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\alpha_1 = \alpha_2 = 10^{-7}$, and $\gamma = 0.1$ for

213   positive skins, 1 for no skin and 10 for negative skins. Figure 2(*a*) shows the spatial

214   drawdown distributions predicted by both solutions when $\bar{t} = 3 \times 10^6$. The figure indicates

215   that both solutions agree very well on the drawdown within the time-dependent radius of

216   influence represented by $\bar{R}(\bar{t})$. The drawdown curves of $\gamma = 0.1$, 1 and 10 in the formation

217   zone merge together at and beyond the interface, i.e., $\bar{r}_s = 5$, because of $\alpha_1 = \alpha_2$. Figure 2(*b*)

218   displays the temporal drawdown distributions predicted by both solutions for an observation

13

219   well at $\bar{r} = 20$. This figure demonstrates that the drawdown curves also have good match over

220   the intermediate and late pumping periods. The discrepancy in dimensionless drawdown at the

221   early period of $0 \leq \bar{t} \leq 600$ can be attributed to the absence of the time derivative term in

222   both Eqs. (1) and (2). The drawdown dramatically increases at $\bar{t} = 160$ as soon as

223    $\bar{R}(\bar{t} = 160) = 20$. It seems reasonable to conclude that the approximate transient solution

224   gives good predicted drawdown in an observation well over the entire pumping period except

225   at early time when the dynamic radius of influence reaches the well (i.e., $\bar{t} \cong 1.4(\bar{r}-1)^2 / \pi$

226   derived by substituting $\bar{R}(\bar{t}) = \bar{r}$ into Eq. (18) and rearranging the result).

227   **3.2. Vertical Flow**

228   The vertical flow induced by well partial penetration is strongly dependent on both

229   dimensionless lumped parameters $\alpha_1 \bar{r}^2$ and $\alpha_2 \bar{r}^2$ (i.e., $K_{z1} r^2 /(K_{r1} b^2)$ and

230    $K_{z2} r^2 /(K_{r2} b^2)$, respectively). Figure 3 shows temporal drawdown distributions predicted by

231   the approximate solution, Eq. (17), for $\alpha_1 = \alpha_2$ ranging from 10^{-6} to 10^{-2} when $\bar{r} = 10$, $\bar{z} =$

232   0.5, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\bar{r}_s = 5$ and $\gamma = 0.1$. Equation (20) is the drawdown solution for the

233   CFP at a fully penetration well; therefore, the vertical flow is absent. When $\alpha_1 \bar{r}^2 = \alpha_2 \bar{r}^2 = 1$,

234   the drawdown distributions predicted by both equations agree well, indicating that the vertical

235   flow is negligible. We may, therefore, reasonably conclude that the vertical flow effect on the

236   aquifer drawdown at an observation well vanishes when $\alpha_1 \bar{r}^2 \geq 1$ and $\alpha_2 \bar{r}^2 \geq 1$, i.e., b is

14

237 small, r is large, and/or the values of K_{z1}/K_{r1} and K_{z2}/K_{r2} are large. On the other hand, Eq. (20)

238 underestimates the drawdown induced by the CFP at a partially penetration well because the

239 vertical flow prevails when $\alpha_1 \bar{r}^2 < 1$ or $\alpha_2 \bar{r}^2 < 1$.

4. Concluding Remarks

241   This study presents an approximate drawdown solution, Eqs. (16) and (17), in terms of a

242 simple series for the CFP at a partially penetrating well in a radial two-zone confined aquifer.

243 The solution is developed on the basis of the steady-state drawdown solution with an outer

244 boundary represented by the time-dependent radius of influence. The comparison with the

245 Chiu et al. (2007) solution reveals that the approximate solution gives accurate temporal

246 drawdown distributions in an observation well over the entire pumping period except at early

247 time when the dynamic radius of influence reaches the well (i.e., $\bar{t} \cong 1.4(\bar{r}-1)^2/\pi$ derived

248 by substituting $\bar{R}(\bar{t}) = \bar{r}$ into Eq. (18) and rearranging the result). The analysis of the

249 temporal drawdowns predicted by Eqs. (17) and (20) indicates that the vertical flow due to a

250 partially penetrating well prevails under the conditions of thick aquifers, vicinity to the well,

251 and/or small conductivity ratios (i.e., $\alpha_1 \bar{r}^2 < 1$ or $\alpha_2 \bar{r}^2 < 1$). Accordingly, conventional

252 models neglecting the vertical flow will underestimate drawdown under those conditions.

Appendix A: Derivation of Eqs. (6) and (7)

254   The finite Fourier cosine transform is defined, in our notation, as

15

255      $\hat{s}_i = \int_0^1 \bar{s}_i \cos(n\pi \bar{z}) \, d\bar{z}$ (A1)

256   where $i \in (1, 2)$. The formula for the inverse transform is expressed as

257      $\bar{s}_i = \hat{s}_i(0) + 2 \sum_{n=1}^{\infty} \hat{s}_i(n) \cos(n\pi\bar{z})$ (A2)

258   where $\hat{s}_i(n)$, a function of n, is the solution in the transform domain. Replacing $\bar{s}_i$ in Eq.

259   (A1) by $\partial^2 \bar{s}_i / \partial \bar{z}^2$ and applying integration by parts twice yields

260      $\int_0^1 \dfrac{\partial^2 \bar{s}_i}{\partial \bar{z}^2} \cos(n\pi \bar{z}) \, d\bar{z} = (-1)^n \left. \dfrac{\partial \bar{s}_i}{\partial \bar{z}} \right|_{\bar{z}=1} - \left. \dfrac{\partial \bar{s}_i}{\partial \bar{z}} \right|_{\bar{z}=0} - (n\pi)^2 \hat{s}_i$ (A3)

261   Applying the transform to Eqs. (1) − (5) on the basis of Eq. (A3) with $\partial \bar{s}_i / \partial \bar{z} = 0$

262   results in the following equations:

263      $\dfrac{\partial^2 \hat{s}_1}{\partial \bar{r}^2} + \dfrac{1}{\bar{r}} \dfrac{\partial \hat{s}_1}{\partial \bar{r}} - \lambda_1^2 \, \hat{s}_1 = 0$ for $1 \le \bar{r} \le \bar{r}_s$ (A4)

264      $\dfrac{\partial^2 \hat{s}_2}{\partial \bar{r}^2} + \dfrac{1}{\bar{r}} \dfrac{\partial \hat{s}_2}{\partial \bar{r}} - \lambda_2^2 \, \hat{s}_2 = 0$ for $\bar{r}_s \le \bar{r} \le \bar{R}$ (A5)

265      $\hat{s}_2 = 0$ at $\bar{r} = \bar{R}$ (A6)

266      $\dfrac{\partial \hat{s}_1}{\partial \bar{r}} = -\gamma \omega / \phi$ at $\bar{r} = 1$ (A7)

267      $\hat{s}_1 = \hat{s}_2$ at $\bar{r} = \bar{r}_s$ (A8)

268   and

269      $\dfrac{\partial \hat{s}_1}{\partial \bar{r}} = \gamma \dfrac{\partial \hat{s}_2}{\partial \bar{r}}$ at $\bar{r} = \bar{r}_s$ (A9)

270   The Fourier-domain solution of Eqs. (A4) and (A5) can be expressed as

271      $\hat{s}_1 = c_1 I_0(\lambda_1 \bar{r}) + c_2 K_0(\lambda_1 \bar{r})$ (A10)

272   and

273      $\hat{s}_2 = c_3 I_0(\lambda_2 \bar{r}) + c_4 K_0(\lambda_2 \bar{r})$ (A11)

274   where $I_0(\cdot)$ and $K_0(\cdot)$ are the modified Bessel functions of the first and second kinds of order

275   zero, respectively, and c_1, c_2, c_3 and c_4 are undetermined coefficients. Substituting Eqs. (A10)

276   and (A11) into Eqs. (A6) − (A9) and solving the four resultant equations leads to

277      $(c_1, c_2, c_3, c_4) = \left(\dfrac{\gamma \zeta \omega}{\phi \lambda_1 \psi}, \ \dfrac{\gamma \xi \omega}{\phi \lambda_1 \psi}, \ \dfrac{\gamma \omega K_0(\lambda_2 \bar{R})}{\phi r_s \lambda_1 \psi}, \ -\dfrac{\gamma \omega I_0(\lambda_2 \bar{R})}{\phi r_s \lambda_1 \psi} \right)$ (A12)

278   where ψ, ζ and ξ are defined in Eqs. (10), (11) and (12), respectively. According to Eq.

279   (A12), Eqs. (A10) and (A11) can be written, respectively, as

280      $\hat{s}_1(\bar{r}, n) = \gamma F_1(\bar{r}, n) / \phi$ (A13)

281   and

282      $\hat{s}_2(\bar{r}, n) = \gamma F_2(\bar{r}, n) / (\phi \bar{r}_s)$ (A14)

283   where $F_1(\bar{r}, n)$ and $F_2(\bar{r}, n)$ are defined in Eqs. (8) and (9), respectively. In the light of Eq.

284   (A2), the inverse transforms to Eqs. (A13) and (A14) lead to Eqs. (6) and (7), respectively.

285   Note that the first terms on the right-hand side of Eqs. (6) and (7) are derived via L'Hospital's

286   law.

287   **Acknowledgements**

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339 **Table 1.** Categorization of the solutions for the constant-flux pumping in confined aquifers

References	Well Penetration	Well Radius	Wellbore Storage	Remark
		Homogeneous Aquifer		
Theis (1935)[a]	Fully	Infinitesimal	None	Infinite aquifer
Carslaw and Jaeger (1959, p.328)[a]	Fully	Finite	None	Infinite aquifer
Carslaw and Jaeger (1959, p.332)[a]	Fully	Finite	None	Finite aquifer with Dirichlet boundary
Carslaw and Jaeger (1959, p.335)[a]	Fully	Infinitesimal	None	Finite aquifer with Dirichlet boundary
Hantush (1962)[a]	Partially	Infinitesimal	None	Infinite aquifer
Papadopulos and Cooper (1967)[a]	Fully	Finite	Considered	Infinite aquifer
Chen (1984)[a]	Fully	Infinitesimal	None	Finite aquifer with Dirichlet boundary
Yang et al. (2006)[a]	Partially	Finite	None	Infinite aquifer
		Two-Zone Aquifer		
Novakowski (1989)[b]	Fully	Finite	Considered	Infinite aquifer
Hemker (1999)[c, #]	Partially	Finite	Considered	Multilayered aquifer with radial and vertical flows
Kabala and El-Sayegh (2002)[b]	Fully	Finite	Considered	Multilayered aquifer with radial flow only
Yeh et al. (2003)[a]	Fully	Finite	None	Infinite aquifer
Chen and Chang (2006)[b, #]	Fully	Finite	Considered	Non-uniform skin effect
Perina and Lee (2006)[b]	Partially	Finite	Considered	General well functions for three-kinds of aquifers
Chiu et al. (2007)[b]	Partially	Finite	None	Infinite aquifer
Wang et al. (2012a)[a]	Fully	Finite	None	Finite aquifer with Dirichlet boundary
Wang et al. (2012b)[a]	Partially	Infinitesimal	None	Multilayered aquifer with radial and vertical flows

340   The superscripts *a*, *b* and *c* represent analytical, semi-analytical and analytical-numerical solutions, respectively.

341   The superscript # represents an infinitesimal thin skin zone.

Table 2. Summary of symbols used in the text and their definitions

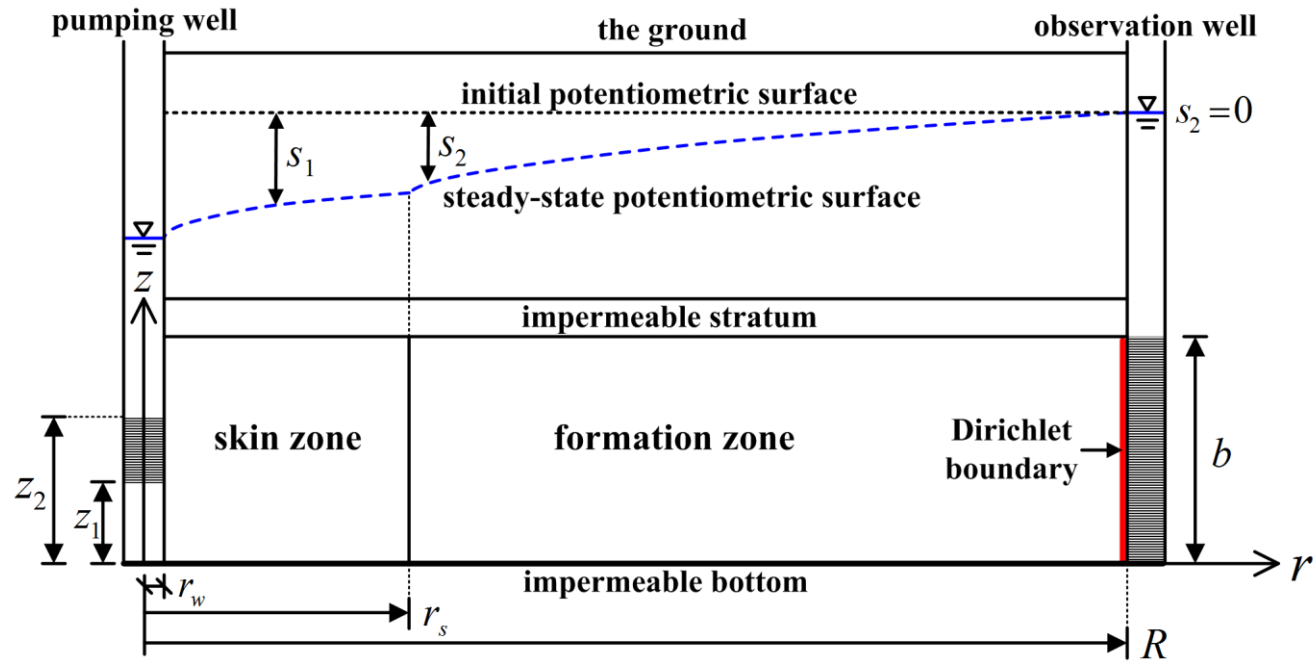| Symbols	Definitions
(s_1, s_2)	Drawdowns in skin and formation zones, respectively
r	Radial distance from the center of the well
r_s	Radius of skin zone
R	Radius of cylinder aquifer domain or the radius of influence
(r_w, r_c)	Outer and inner radiuses of well, respectively
z	Elevation from the aquifer bottom
(z_1, z_2)	Lower and upper elevations of well screen, respectively
t	Time since pumping
b	Aquifer thickness
Q	Pumping rate of well
(K_{r1}, K_{r2})	Radial hydraulic conductivities of skin and formation zones, respectively
(K_{v1}, K_{v2})	Vertical hydraulic conductivities of skin and formation zones, respectively
S_{s2}	Specific storage of formation zone
(T_1, T_2)	Transmissivities of skin and formation zones, respectively
$(\bar{s}_1, \bar{s}_2)$	$(2\pi T_2 s_1 / Q, 2\pi T_2 s_2 / Q)$
$\bar{t}$	$K_{r2} t / (S_{s2} r_w^2)$
$(\bar{r}, \bar{r}_s, \bar{R})$	$(r/r_w, r_s/r_w, R/r_w)$
$(\bar{z}, \bar{z}_1, \bar{z}_2)$	$(z/b, z_1/b, z_2/b)$
(ϕ, γ)	$(\bar{z}_2 - \bar{z}_1, K_{r2}/K_{r1})$
(α_1, α_2)	$(K_{z1} r_w^2 /(K_{r1} b^2), K_{z2} r_w^2 /(K_{r2} b^2))$

Figure 1. A schematic diagram of the constant-flux pumping at a partially penetrating well in a cylinder two-zone confined aquifer with the Dirichlet boundary (The symbols of the variables are defined in Table 2.)
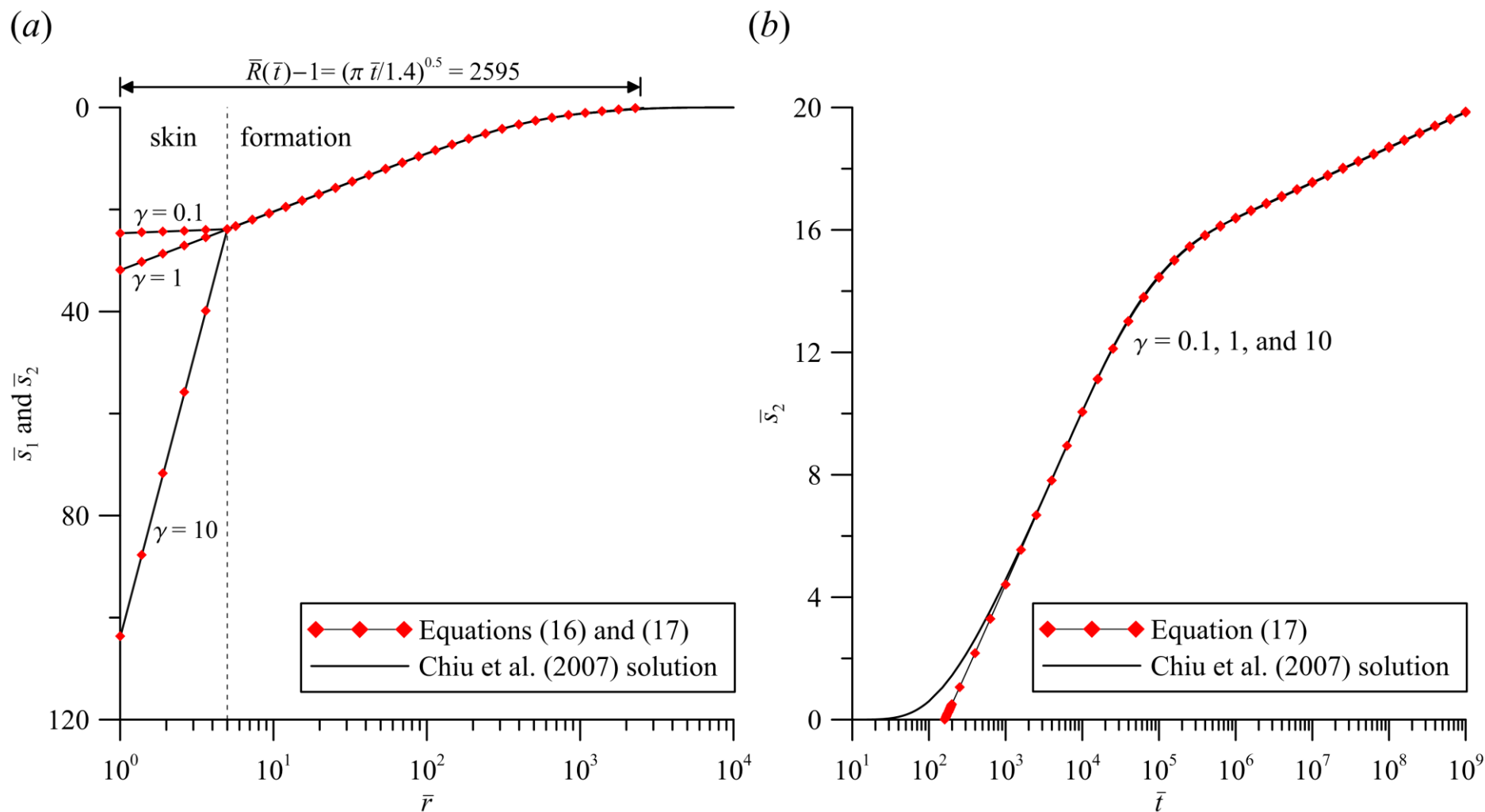
Figure 2. Predicted drawdowns by <mark>Chiu et al. (2007) solution and the approximate solution, Eqs. (16) and (17),</mark> with $\gamma = 0.1$, 1, and 10 for (a) spatial distributions at $\bar{t} = 3 \times 10^6$ and (b) temporal distributions at $\bar{r} = 20$ with $\bar{z} = 0.5$, $\bar{r}_s = 5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, and $\alpha_1 = \alpha_2 = 10^{-7}$
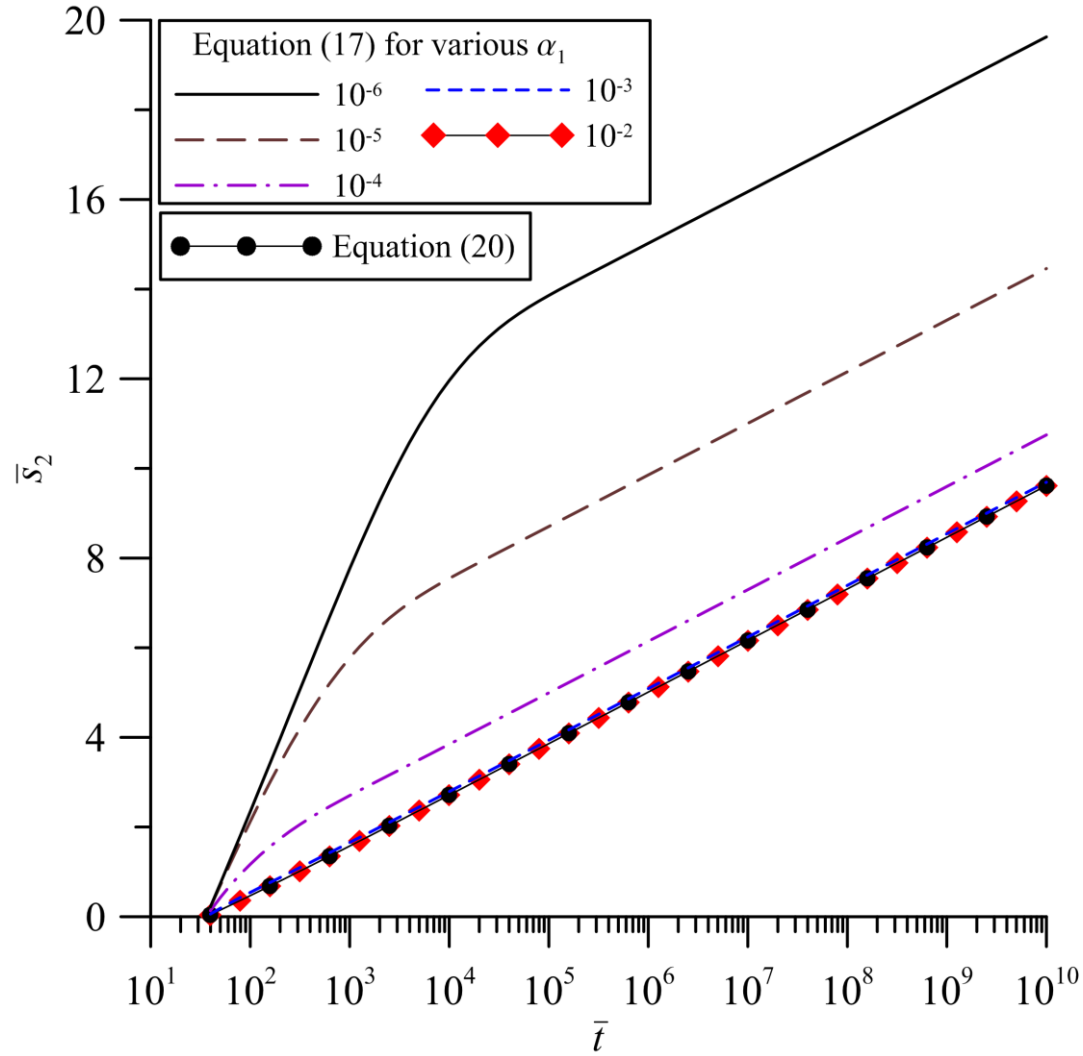
Figure 3. Temporal drawdown distributions predicted by the approximate solution, Eq. (17), with $\bar{r} = 10$, $\bar{z} = 0.5$, $\bar{z}_1 = 0.4$, $\bar{z}_2 = 0.6$, $\bar{r}_s = 5$, $\gamma = 0.1$ and various values of α_1 with $\alpha_1 = \alpha_2$