

Reply to comments by Reviewers # 1 and #2 of Technical Note: A simple generalization of the Brutsaert and Nieber analysis

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Abstract. In our answer to the comments by two anonymous reviewers, we perform numerical simulations of the full nonlinear Boussinesq equation. Our proposed modification to the Brutsaert-Nieber recession analysis can then be compared with those solutions. We show that our analytical approximations provide a significant improvement in the estimation of the aquifer physical parameters n_e and k_0 .

1 Introduction

We give our answers in a single file. All equation and figure *arabic* numbers still refer to the previous version of our manuscript. Added equation and figure numbers are given as (i), (ii), etc., to avoid confusion. Renumbering will be performed if the manuscript is accepted for HESS.

10 2 Answer to Referee #1

We thank Referee #1 for the valuable comments. In our original manuscript, we believed that our analytical results might alone be enough to carry our argument. In retrospect, we agree that an independent verification can strengthen it.

Indeed, our previous version shows results from estimates that amend the original BN77 theory for $\phi_0 \neq 0$, but still rely on Boussinesq (1903)'s *linearized* solution. It is well worth comparing how our (still approximate) estimates in Eqns (10) and (11) perform against true values resulting from the more physically accurate Boussinesq nonlinear differential equation.

However, we do not believe that field data exist where independent values of n_e and k_0 can be obtained with enough confidence — *i.e.* small enough uncertainty. After the reviewer's comments,

20 it has come to our attention that some laboratory experiments have been performed that might be
 useful as validation sets (Hewlett and Hibbert, 1963; Sanford et al., 1993; Mizumura, (2002),but
 we haven't yet looked at those data in detail. Moreover, and importantly, it is not clear that a wide
 enough range of carefully controlled values of ϕ_0 would be available to validate our results.

Therefore, we have decided to run an extensive set of numerical simulations of the full nonlinear
 25 Boussinesq equation. We would like to argue that this is a valid alternative, that has been used in
 important research related to the theme (see Szilagyi et al., 1998; Rupp and Selker, 2006)

We describe our simulations here; the description is also relevant to our answers to Referee #
 2. With an implicit finite-difference method, we solved the fully nonlinear Boussinesq equation in
 dimensionless form, *i. e.*

$$30 \quad \frac{\partial \phi}{\partial \tau} = \frac{\partial}{\partial \eta} \left(\phi \frac{\partial \phi}{\partial \eta} \right); \quad \phi(\eta, 0) = 1, \quad \phi(0, \tau) = \phi_0, \quad \frac{\partial \phi}{\partial \eta}(1, \tau) = 0, \quad (i)$$

with

$$\phi \equiv \frac{h}{H}, \quad (ii)$$

$$\eta \equiv \frac{x}{B}, \quad (iii)$$

$$35 \quad \tau \equiv \frac{k_0 H}{n_e B^2} t. \quad (iv)$$

Before anything else, we checked (with excellent results) the numerical solution against the only
 known analytical solution of the nonlinear PDE, which is valid for $\phi_0 = 0$, as was obtained by
 Boussinesq (1904). We do not show the comparison here, but it is available upon request. Then we
 varied ϕ_0 from 0 to 0.95 in increments of 0.05. For each ϕ_0 , a Brutsaert-Nieber recession analysis
 40 was performed and k_0 and n_e were estimated against their true values \bar{k}_0 and \bar{n}_e using Equations (10)
 and (11). The evident advantage of the dimensionless form is economy: one need not “vary” k_0 and
 n_e (which are kept at nominal unity values, as well as H and B), but only ϕ_0 : all that matters are the
 ratios of the estimates, namely n_e/\bar{n}_e and k_0/\bar{k}_0 . In order to be consistent with the estimates given
 by equations (10) and (11), the slopes of the recession analyses were fixed at $\beta_1 = 3$ (early time) and
 45 $\beta_2 = 1$ (late time), and only α_1 and α_2 were estimated using a nonlinear Levenberg-Marquardt least
 squares method.

We now re-plot our Figure 2 in the manuscript as two new figures, one for n_e and the other for
 k_0 , in Figures i and ii. Notice the change in the choice of (for instance) k_0/\bar{k}_0 instead of the former
 $k_0(\text{BN})/k_0$, which avoids the log scale of Figure 2 and allows a more clear picture to emerge.
 50 Moreover, while our Figure 2 was actually the ratio of two *estimates*, we are now able (after our
 numerical simulations) to plot the results against true known values \bar{n}_e and \bar{k}_0 .

As can be seen, the k_0 estimate using the original equations remains “robust” up to $\phi_0 = 0.4$
 approximately. On the other hand, there is a more or less linear trend in n_e estimated with the
 original equations all the way from $\phi_0 = 0$. Our modified equations (10) and (11) give estimated

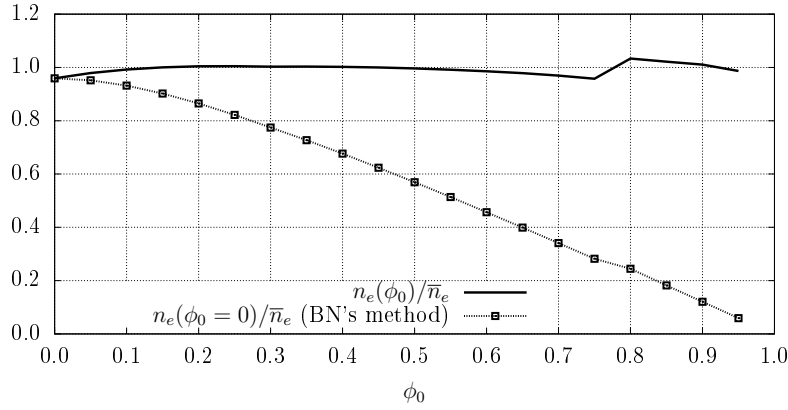


Figure i. Performance of n_e/\bar{n}_e using the proposed equation (10) (solid line), and error incurred by the traditional BN77 analysis (dotted line and symbols).

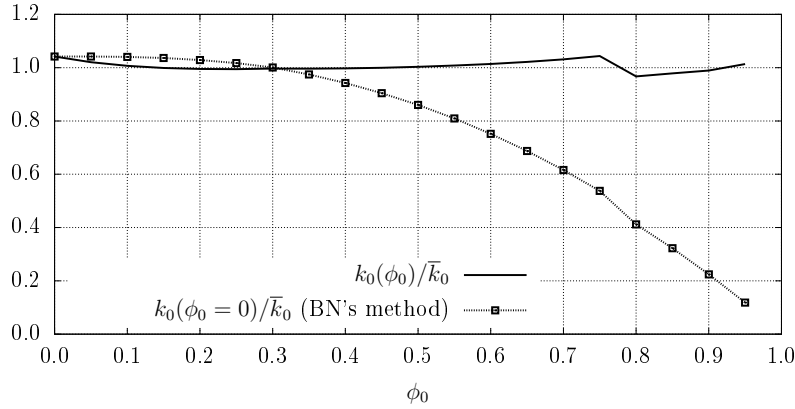


Figure ii. Performance of k_0/\bar{k}_0 using the proposed equation (11) (solid line), and error incurred by the traditional BN77 analysis (dotted line and symbols).

55 values of k_0 and n_e that differ very little from the true ones for the whole ϕ_0 range, and as such represent a considerable improvement over the original equations.

The small kinks between $\phi_0 = 0.7$ and $\phi_0 = 0.8$ are an artifact of the choice of the range of the streamflow Q for fitting α_1 and α_2 used in the recession analysis. This (to the best of our knowledge) is still a subjective part of the BN77 analysis: the ranges were chosen to fit the recession plots
60 $dQ/dt \times Q$ reasonably well, but they were not “fudged” to “optimize”, in any way, the estimated k_0 and n_e . Our recession data are also available so that these results can be verified independently.

We now address specific comments by Reviewer 1.

Section 1: “... can be compared to the predictions from analytical solutions...” The authors may want to extend this paragraph by showing explicitly which analytical solutions they have in
65 mind and how they are used.

We suggest to rewrite this as:

can be compared to the predictions from the above-mentioned analytical solutions by Polubarinova-Kochina (1962), Boussinesq (1903) and Boussinesq (1904), among many others (see Rupp and Selker, 2006). In this work, the first two are used, and they are detailed in the sequence.

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Section 1: "... does not account for that case." and "... is not strictly true ...". The two statements seem to contradict each other and need clarification.

We were being tactful! We suggest the alternate text:

If one wishes to estimate only the soil hydraulic conductivity k_0 and the drainable porosity n_e , two of the three aforementioned solutions can be used. However, the solution by Polubarinova-Kochina (1962) is only valid for the case $H_0 = 0$: it is therefore important to assess how much this assumption affects the estimate of k_0 and of n_e for cases where it does not hold.

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Section 2: As the solutions of Chor (2013) and Dias (2014) are essential in this paragraph it may be worth noting the equations together with one or two sentences of explanation. This will give the reader the possibility to focus on the text rather than getting distracted by consulting the references to understand what follows.

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We suggest to extend the text right after our Eqn (3) with

where $h(x, t)$ is the water table height, x is the horizontal distance from the water stream and t is the time. Under the above change of variables, the Boussinesq equation is reduced to the dimensionless ordinary differential equation

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$$\frac{d}{d\xi} \left(\phi \frac{d\phi}{d\xi} \right) + 2\xi \frac{d\phi}{d\xi} = 0 \quad (\text{v})$$

together with the boundary conditions $\phi(0) = \phi_0$ and $\phi(\infty) = 1$. Due to the second boundary condition, the solution is only valid for the initial phase of aquifer draw-down. For $\phi_0 = 0$, as already noted, the solution by Polubarinova-Kochina (1962) suffices for the BN77 analysis; for $\phi_0 \neq 0$, a series solution of the form

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$$\phi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \quad (\text{vi})$$

has been proposed by Dias et al. (2014), with a recursion relation for the a_n 's. An important result in that work is an empirical equation, fitted to numerically obtained values of a_1 in the series above, for the value of ψ_0 , defined below. This is given as equation (12) in the present work.

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After eq. 12: Where do the numerical values for eq 12 come from? Please clarify.

After the values have been given, we propose to extend the text with:

100 As explained in Dias et al. (2014), even after a general recursion relation for the a_n 's in (vi) has been obtained, the values of the a_n 's still cannot be obtained analytically, essentially because the series' radius of convergence is limited so that the boundary condition $\phi(\infty) = 1$ cannot be imposed analytically. Instead, they must be obtained numerically with the aid of numerical solutions of (v). The coefficients above have been
105 obtained in Dias et al. (2014) by curve fitting with a large number of numerical solutions.

3 Answer to Referee # 2

We thank the referee for his comments. It appears to us that the Referee believes that the Brutsaert-Nieber analysis is now somewhat outdated, or rendered inapplicable, due to recent findings.

110 We believe that most of the criticism by the Referee can be traced back to the paper by Rupp and Selker (2006): because it is well known by us, and because it is already in our list of references, we would like, in the following, to argue on the basis of this reference.

115 First of all, we realize that our choice of words may lead to the optimistic impression that our results are all that is needed to "fix" the BN77 recession analysis. We know better than that, and are ready to admit that the issue of $\phi_0 \neq 0$ that we address in this manuscript does not, by any means, exhaust the subject.

As Rupp and Selker (2006) argue convincingly, there are at least two other issues that can compound the difficulty of BN77 recession analyses considerably: steep slopes and the k_0 dependency on aquifer depth h . The latter leads to an even more general non-linear equation than (i).

120 We also note, however, that exactly as we do here, Rupp and Selker (2006) resorted to numerical simulations. This may well have been chosen wisely, as (sadly!) real field data are bound to complicate the picture even more with measurement error, the existence of many more flow components contributing to the measured streamflow, complicated geometry, etc..

125 At any rate, Rupp and Selker (2006) results do not by any means sound a death knell on BN77. In particular, we call attention to their conclusions in paragraphs 60 and 62, reproduced in part below:

§ 60 "A definition for the recession parameter a was also derived for late time, meaning that in theory the Brutsaert and Nieber method can be used to determine the hydraulic properties of a sloping aquifer."

130 § 62 "In the case of a horizontal or very mildly sloping aquifer, Szilagyi et al. [1998] found the assumption of a representative single rectangular aquifer to be robust, based on numerical

solutions of the 2-D Boussinesq equation in a synthetic catchment. The general shape of the recession slope curve for catchment discharge was similar to that for discharge from a 1-D rectangular aquifer, though with a smoother transition between the early and late time domains. Furthermore, the basin-scale hydraulic and geometric aquifer parameters were reasonably estimated by recession slope analysis using (2), including cases where the saturated hydraulic conductivity varied across the catchment. As of yet, however, we are not aware of numerical experiments similar to that of Szilagyi et al. [1998] for catchments composed of hillslopes of moderate to steep gradient.”

In short, it seems to us that these remarks: (i) do not by any means consider BN77’s idea to be discarded, but only call our attention, very correctly, to possible complicating factors and (ii) also call our attention to the fact that there is good evidence supporting the approach for mild slopes.

None of the aforementioned complications dishearten us, and on a fundamental level we believe that they should not be used as arguments against conducting research using simplifying assumptions. In our manuscript, we chose a very simple approach: we studied a zero-slope, constant k_0 aquifer. These assumptions may, to varying degrees, not correspond to real watersheds, and we are more than ready to acknowledge this. Of course, such caveats will be incorporated in the manuscript if it is accepted. On the other hand, these simplifying assumptions allow us to concentrate on the issue of how $\phi_0 \neq 0$ affects the estimates of the physically-based parameters k_0 and n_e . Again, we would like to argue strongly, but cordially, that the simplifications are valid if one is — for the first time — investigating a new issue.

4 Conclusions

With the numerical results reported here we believe that we have a valid point that should be considered in recession analyses for recovering physical parameters of watersheds — a possibility which we think is still open, and potentially valid.

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