Figures for „Short comment, 2nd response to referee #1“

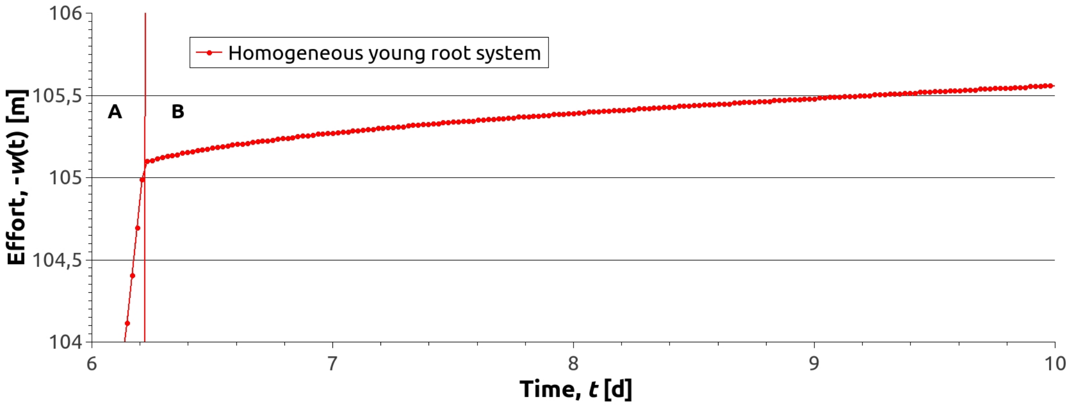

Fig. SC2.1: Temporal evolution of effort under a time constant transpiration rate before (A) and after (B) switching from constant flow to constant potential at the root collar (upon reaching the limiting collar potential of -150 m, see vertical line).

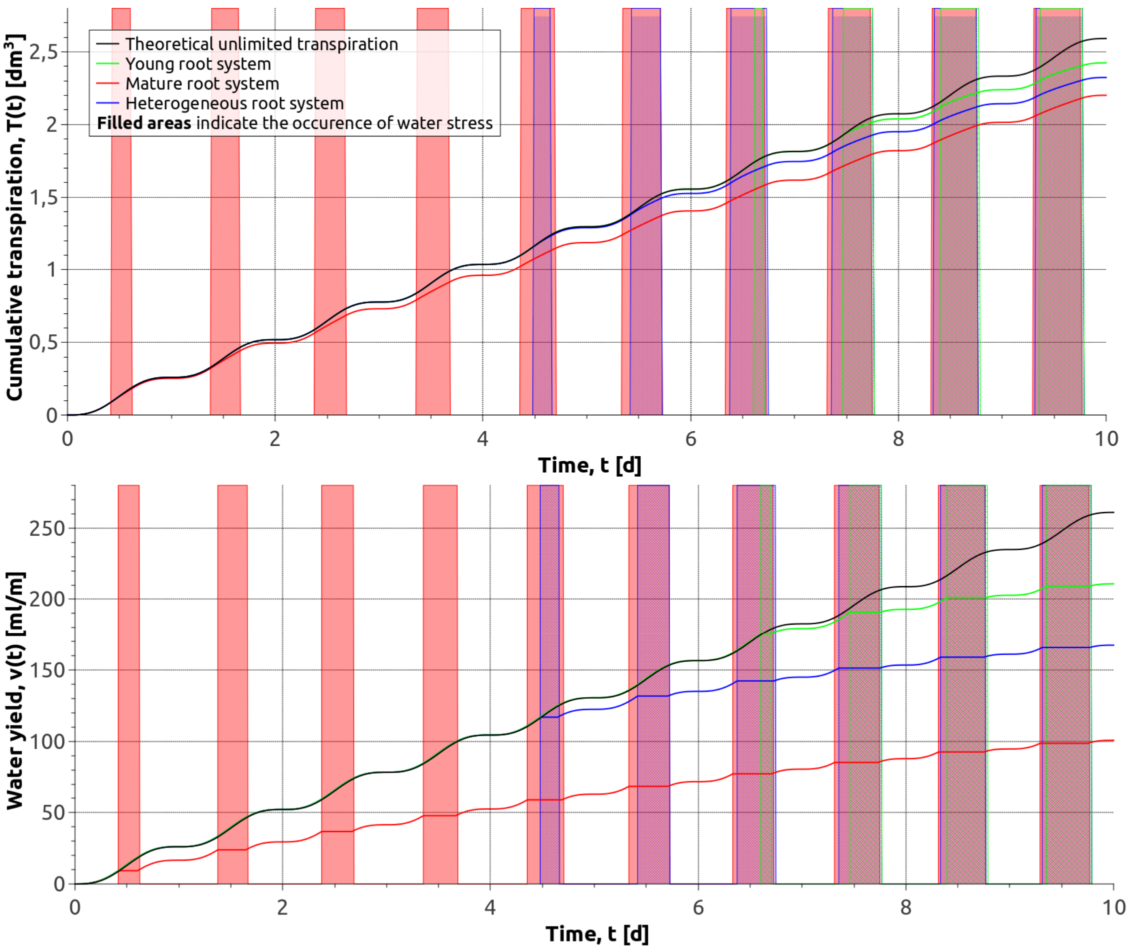

Fig. SC2.2: Cumulative transpiration (top) and water yield (bottom) for root systems with different hydraulic properties and under a sinusoidal flux boundary condition. Filled areas indicate the occurrence of water stress in homogeneous young (red), mature (blue) and a heterogeneous (green) root system. Note that the water yield does not increase during times of water stress

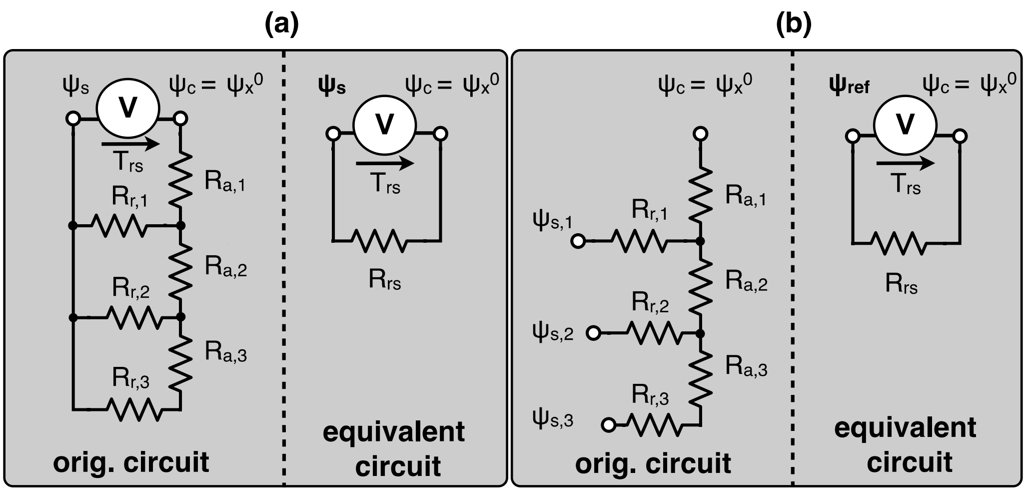

Fig. SC2.3: Electrical circuits and their equivalent circuits with a Thevenin resistance (R_{rs}) analogue for **(a)** a root network surrounded by homogenous soil water potential, ψ_s and **(b)** a root network surrounded by heterogenous soil water potentials $\psi_{s,i}$.

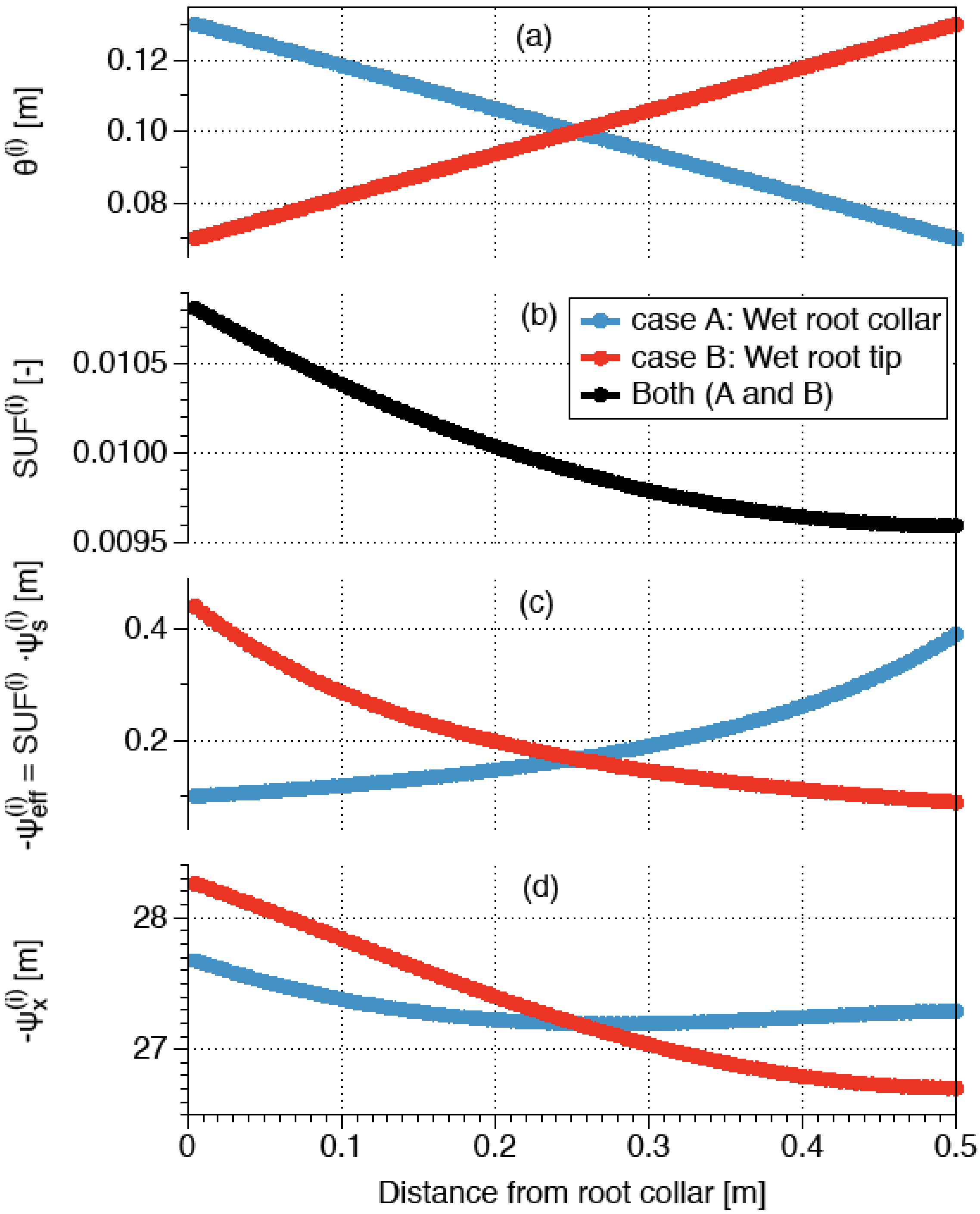

Fig. SC2.4: Boundary conditions and results for the same single un-branched root surrounded by two different heterogeneous soil water potential profiles. The parameters are the same as in the discussion paper for a mature root. Modeled are 100 segments. **(a)** Assumed soil water content surrounding the root, **(b)** SUF for this root, calculated according to Appendix 2, **(c)** Effective soil water potential, which is the product of SUF_i and $\psi_{s,i}$ at each segment, **(d)** resulting profiles in xylem water potential. The collar potential is the one at zero distance from the collar.

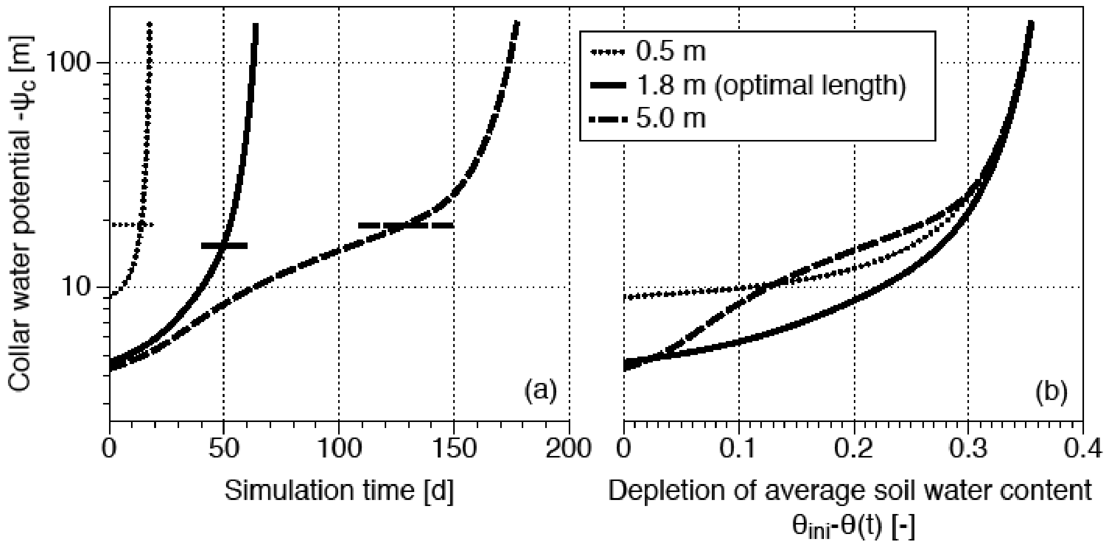

Fig. SC2.5: Plot of the evolution of collar water potential for three un-branched roots of the indicated length. Root (mature) and soil properties are the same as in the discussion paper. **(a)** Time evolution, the horizontal lines indicate the time average water potential between the start of the simulation and the onset of water stress (same as effort) **(b)** same simulations as (a) but plotted as a function of depletion of average volumetric water content around the root over the course of the simulation.

Second note to reviewer #1, referree comments
Appendix 1
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A.1 The temporal evolution of effort under water stress

In this section we will explain in a mathematical way why the effort increases slowly under water stress, although collar potential is high (see also Fig. 1 of this reply). Basically the slow increase of effort is caused by the decrease of overall root water uptake under water stress which shows that effort is not a temporal average but a flux weighted collar potential (see eq. (A.1-14)).
Like in the simulations used within our manuscript we consider a drying scenario and assume an initially time constant flux boundary condition $Q(\tau) = Q$ (m^3/s), which switches to a potential boundary condition $\psi(\tau) = \psi_{crit}$ (m) when collar potential drops below the critical value ψ_{crit} (m). Under our assumptions the occurence time of water stress is unique, we denote it with $\tilde{t}$ (s).
In order to calculate effort at a time $t > \tilde{t}$ (s), we use the general definition of effort and split the integrals in the enumerator and the denominator at $\tilde{t}$:

$$w(t) = \frac{\int_{\tau=0}^{t} \psi(\tau) \cdot Q(\tau) d\tau}{\int_{\tau=0}^{t} Q(\tau) d\tau} = \frac{\int_{\tau=0}^{\tilde{t}} \psi(\tau) \cdot Q(\tau) d\tau + \int_{\tau=\tilde{t}}^{t} \psi(\tau) \cdot Q(\tau) d\tau}{\int_{\tau=0}^{\tilde{t}} Q(\tau) d\tau + \int_{\tau=\tilde{t}}^{t} Q(\tau) d\tau} \quad \text{(A.1-1)}$$

We can now insert $Q(\tau) = Q$ for times $\tau = 0 \ldots \tilde{t}$ and $\psi(\tau) = \psi_{crit}$ for times $\tau = \tilde{t} \ldots t$. We thus obtain

$$w(t) = \frac{Q \cdot \int_{\tau=0}^{\tilde{t}} \psi(\tau) d\tau + \psi_{crit} \cdot \int_{\tau=\tilde{t}}^{t} Q(\tau) d\tau}{Q \cdot \tilde{t} + \int_{\tau=\tilde{t}}^{t} Q(\tau) d\tau} \quad \text{(A.1-2)}$$

We can transform the integrals in the stress periods by substituting $\tau = \tilde{t} \ldots t$ by $\tau = 0 \ldots \Delta t$ in which $\Delta t = t - \tilde{t}$ is the time since the occurence of water stress.

$$w(\tilde{t} + \Delta t) = \frac{Q \cdot \int_{\tau=0}^{\tilde{t}} \psi(\tau) d\tau + \psi_{crit} \cdot \int_{\tau=0}^{\Delta t} Q(\tilde{t} + \tau) d\tau}{Q \cdot \tilde{t} + \int_{\tau=0}^{\Delta t} Q(\tilde{t} + \tau) d\tau} \quad \text{(A.1-3)}$$

By defining

$$C := Q \cdot \int_{\tau=0}^{\tilde{t}} \psi(\tau) d\tau = const. \quad \text{(A.1-4)}$$

$$V_u := Q \cdot \tilde{t} = V(\tilde{t}) = const. \quad \text{(A.1-5)}$$

$$V_s(\Delta t) := \int_{\tau=0}^{\Delta t} Q(\tilde{t} + \tau) d\tau \quad \text{(A.1-6)}$$

the last equation reads as

$$w(t) = \frac{C + \psi_{crit} \cdot V_s}{V_u + V_s} = w(V_s(\Delta t)) \qquad \text{(A.1-7)}$$

Please note that C is the enumerator of effort used within our manuscript under a time constant flux boundary condition, that V_u equals the total amount of water that was extracted in unstressed conditions and thus is the denominator of effort in our manuscript and that V_s equals the cumulative amount of water that was taken up in stressed conditions (between $\tilde{t}$ and t). The variables C, V_u and ψ_{crit} are constant under the assumptions made here, only V_s depends on the duration of water stress.
In order to give a mathematical explanation of the slow increase of effort under water stress, we will use a first order taylor approximation of $w(V_s)$ around $V_s(\Delta t) = 0 \Leftrightarrow \Delta t = 0$. We will use the first terms

$$w\big|_{V_s=0} = C/V_u = \tilde{w} \qquad \text{(A.1-8)}$$

$$\left.\frac{\partial w}{\partial V_s}\right|_{V_s=0} = \frac{\psi_{crit}}{V_u} - \frac{C}{V_u^2} \qquad \text{(A.1-9)}$$

$$= \frac{\psi_{crit}}{V_u} - \frac{Q \cdot \int_{\tau=0}^{\tilde{t}} \psi(\tau) d\tau}{Q^2 \tilde{t}^2} \qquad \text{(A.1-10)}$$

$$= \frac{\psi_{crit}}{V_u} - \frac{1}{Q \cdot \tilde{t}} \cdot \frac{\int_{\tau=0}^{\tilde{t}} \psi(\tau) d\tau}{\tilde{t}} \qquad \text{(A.1-11)}$$

$$= \frac{\psi_{crit} - \tilde{w}}{V_u} \qquad \text{(A.1-12)}$$

in which $\tilde{w}$ denotes the effort as calculated in our manuscript. The first-order approximation is thus given by

$$w(\tilde{t} + \Delta t) = \tilde{w} + (\psi_{crit} - \tilde{w}) \cdot \frac{V_s(\Delta t)}{V_u} \qquad \text{(A.1-13)}$$

$$= \tilde{w} + (\psi_{crit} - \tilde{w}) \cdot \frac{\int_{\tau=0}^{\Delta t} Q(\tilde{t} + \tau) d\tau}{V_u} \qquad \text{(A.1-14)}$$

Equation (A.1-14) shows that in a first order approximation the effort $w(t)$ does further increase after water stress occurs, $w(t)$ would even reach ψ_{crit} as soon as the water uptake V_s under stressed conditions is equal to V_u. But as the water uptake rate $Q(t)$ decreases quickly under a constant potential boundary in the drying scenario, effort increases very slowly and an equilibration would take durations Δt of water stress that would be much longer than duration of unstressed water uptake $\tilde{t}$.

Second note to reviewer #1, referee comments
Appendix 2
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A.2 Dependence of (compensatory) root water uptake on soil water status and root hydraulic properties

Couvreur et al. (2012) presented an approach to quantify the redistribution of root water uptake with respect to heterogeneities in soil matric potential in an explicit root water uptake model. We apply this approach to a simple unbranched topology which allows to derive explicit solutions of the "standard uptake fraction" SUF [-] (distribution of root water uptake along roots within a soil with homogeneous matric potential) and the "compensatory root water uptake" φ [m^3/s] (deviation from these root water uptake patterns in homogeneous conditions). We will in particular show that to no surprise RWU rates and "compensatory root water uptake" along the root strand depend on the soil water status. However it will become clear that the local "compensatory root water uptake" at segment i does furthermore depend on the entire root hydraulic architecture. For that reason, K_{Rs} does not include the entire information about root hydraulic properties under heterogeneous water potentials.

Within the first section the algorithm is given that is used to calculate SUF and φ for root strands with arbitrary (heterogeneous) hydraulic properties and soil water statusses. In the second section we apply the algorithm to derive properties of unbranched roots with homogeneous root hydraulic properties. In particular we will show that the $SUF^{(i)}$ decrease from the root base towards the root tip (see eqs. (A.2-6-A.2-8)). The proof the correctness of this algorithm is given in the third section.

A.2.1 Algorithm to determine SUF and φ in an unbranched root strand

We consider an unbranched root strand (with generally heterogeneous root hydraulic properties), which is discretized into n segments. All segments have cylindrical shape, but may possess different length and radius. Root water uptake is driven by gradients in xylem water potentials h_{Xylem} and soil water potentials h_{Soil}. Actual values of roots radial and axial resistances can be derived from root geometry and root hydraulic resistivities as given in our manuscript. These resistivities are considered to be constant within each individual segment, but are allowed to differ between segments. The algorithm works as follows:

1. Collect/Calculate the roots radial and axial resistances $R_{Rad}^{(i)}$ [s/m^2] and $R_{Ax}^{(i)}$ [s/m^2] as well as the soil matric potentials $h_{Soil}^{(i)}$ [m].

2. Calculate the corresponding $\alpha^{(i)}$, $\beta^{(i)}$, $\delta^{(i)}$ and $\gamma^{(i)}$ defined for $i = 2 \ldots n$:

$$\alpha^{(i)} = \frac{R_{Rad}^{(i-1)} + R_{Ax}^{(i)}}{R_{Rad}^{(i)}} \qquad [-]$$

$$\beta^{(i)} = \frac{R_{Ax}^{(i)}}{R_{Rad}^{(i)}} \qquad [-]$$

$$\delta^{(i)} = h_{Soil}^{(i)} - h_{Soil}^{(i-1)} \qquad [m]$$

$$\gamma^{(i)} = \frac{\delta^{(i)}}{R_{Rad}^{(i)}} \qquad [m^3/s].$$

3. "Backward" calculate the auxiliary variables $C^{(i)}$ and $D^{(i)}$ which are recursively defined for $i = n-1, \ldots, 1$ by

$$C^{(n-1)} = \frac{1 + \beta^{(n)}}{1 + \alpha^{(n)}} \qquad [-]$$

$$C^{(n-i)} = \frac{C^{(n-i+1)} + \beta^{(n-i+1)}}{C^{(n-i+1)} + \alpha^{(n-i+1)}} \qquad [-]$$

$$D^{(n-1)} = \frac{\gamma^{(n)}}{1 + \alpha^{(n)}} \qquad [m^3/s]$$

$$D^{(n-i)} = \frac{D^{(n-i+1)} + \gamma^{(n-i+1)}}{C^{(n-i+1)} + \alpha^{(n-i+1)}} \qquad [m^3/s].$$

4. "Forward" calculate the soil water status independent "standard uptake fraction" $SUF^{(i)}$ as well as the soil water status dependent

auxiliary variable $\tilde{\varphi}^{(i)}$ for $i = 1, \ldots, n$ via

$$
\begin{aligned}
SUF^{(1)} &= C^{(1)} && [-] \\
SUF^{(2)} &= \alpha^{(2)} \cdot SUF^{(1)} - \beta^{(2)} && [-] \\
SUF^{(i)} &= \beta^{(i)} \cdot \sum_{j=1}^{i-2} SUF^{(j)} + \alpha^{(i)} \cdot SUF^{(i-1)} - \beta^{(i)} && [-]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\varphi}^{(1)} &= D^{(1)} && [m^3/s] \\
\tilde{\varphi}^{(2)} &= \alpha^{(2)} \cdot \tilde{\varphi}^{(1)} - \gamma^{(2)} && [m^3/s] \\
\tilde{\varphi}^{(i)} &= \beta^{(i)} \cdot \sum_{j=1}^{(n-2)} \tilde{\varphi}^{(j)} + \alpha^{(i)} \cdot \tilde{\varphi}^{(i-1)} - \gamma^{(i)} && [m^3/s].
\end{aligned}
$$

5. Calculate the soil water status dependent "compensatory root water uptake" $\varphi^{(i)}$ and the root water uptake rates $Q_{Rad}^{(i)}$ as follows:

$$
\varphi^{(i)} = \frac{\tilde{\varphi}^{(i)}}{SUF^{(i)}} \qquad [m^3/s]
$$

$$
\begin{aligned}
Q_{Rad}^{(i)} &= T \cdot SUF^{(i)} + \tilde{\varphi}^{(i)} \\
&= SUF^{(i)} \cdot (T + \varphi^{(i)}) && [m^3/s].
\end{aligned}
$$

A.2.2 Important properties of $SUF^{(i)}$ and $\varphi^{(i)}$ in unbranched homogeneous root strands

In a homogeneous unbranched root strand, the axial and radial resistivities do not change along the root. Thus, if the root is discretized into n equally long segments, the radial and axial resistances $R_{Ax}^{(i)} = R_{Ax}$ and $R_{Rad}^{(i)} = R_{Rad}$ are constant. If we substitute this into the above mentioned algorithm, we obtain:

$$
\begin{aligned}
\alpha := \alpha^{(i)} &= \frac{R_{Rad} + R_{Ax}}{R_{Rad}} = 1 + R_{Ax}/R_{Rad} && [-] \\
\beta := \beta^{(i)} &= R_{Ax}/R_{Rad} && [-]
\end{aligned}
$$

Since the resistances R_{Ax} and R_{Rad} are greater zero, we obtain

$$1 < \alpha = \beta + 1 \tag{A.2-1}$$
$$0 < \beta = \alpha - 1 \tag{A.2-2}$$

α and β both are dimensionless and greater than zero. They only depend on the time constant root hydraulic properties R_{Rad} and R_{Ax}. These information are also part to $D^{(i)}$ and thereof $\tilde{\varphi}^{(i)}$. However, since soil water status is arbitrary, it is impossible to derive any similar results for $\delta^{(i)}$, $\gamma^{(i)}$, $D^{(i)}$ or $\tilde{\varphi}^{(i)}$. Thus we are only able to deduce properties of $C^{(i)}$ and the thereof derived $SUF^{(i)}$. It follows from the recursive definition of $C^{(i)}$ that

$$0 < C^{(n-1)} = \frac{1+\beta}{1+\alpha} = \frac{\beta+1}{\beta+2} = \frac{\alpha}{1+\alpha} < 1 \tag{A.2-3}$$
$$0 < C^{(n-i)} = \frac{C^{(n-i+1)} + \beta^{(n-i+1)}}{C^{(n-i+1)} + \alpha^{(n-i+1)}} = \frac{C^{(n-i+1)} + \beta}{C^{(n-i+1)} + \alpha} = \frac{C^{(n-i+1)} + \alpha - 1}{C^{(n-i+1)} + \alpha} = 1 - \frac{1}{C^{(n-i+1)} + \alpha} < 1 \tag{A.2-4}$$

Please note that $C^{(k)}$ depends on the root hydraulic parameters of all successive segments $j > k$. In particular, $C^{(1)}$ contains the time independent information about root hydraulic properties of the entire root strand. Please note further that the recursive definitions of the $D^{(i)}$ do also contain this information, therefore the "compensatory root water uptake" $\tilde{\varphi}^{(i)}$ combines soil water status and the root hydraulic properties of the entire root system. However, we are not able to deduce results for $D^{(i)}$ or $\varphi^{(i)}$ which are valid under arbitrary soil water statusses.
Nevertheless we can proof that the $SUF^{(i)}$ have to decrease from the collar towards the tip in this specific topology by using the result

$$\sum_{i=1}^{n} SUF^{(i)} = 1 \tag{A.2-5}$$

(all SUFs sum to one, giving the total transpiration in homogeneous soils (Couvreur et al., 2012).

$$SUF^{(1)} - SUF^{(2)} = C^{(1)} - \alpha \cdot C^{(1)} + \beta = (1-\alpha) \cdot C^{(1)} + \beta = \beta \cdot (1 - C^{(1)}) > 0 \tag{A.2-6}$$

$$\begin{aligned} SUF^{(2)} - SUF^{(3)} &= SUF^{(2)} - \beta \cdot SUF^{(1)} - \alpha \cdot SUF^{(2)} + \beta = SUF^{(2)} \cdot (1-\alpha) + \beta \cdot (1 - SUF^{(1)}) \\ &= -\beta \cdot SUF^{(2)} + \beta \cdot (1 - SUF^{(1)}) = \beta \cdot (1 - SUF^{(1)} - SUF^{(2)}) > 0 \end{aligned} \tag{A.2-7}$$

$$\begin{aligned} SUF^{(i-1)} - SUF^{(i)} &= SUF^{(i-1)} - \left(\beta^{(i)} \cdot \sum_{j=1}^{i-2} SUF^{(j)} + \alpha^{(i)} \cdot SUF^{(i-1)} - \beta^{(i)} \right) \\ &= (1-\alpha) \cdot SUF^{(i-1)} + \beta \cdot \left(1 - \sum_{j=1}^{i-2} SUF^{(j)} \right) = \beta \cdot \left(1 - \sum_{j=1}^{i-1} SUF^{(j)} \right) > 0 \end{aligned} \tag{A.2-8}$$

A.2.3 Correctness of the algorithm

In this section we will give the scetch of a proof that shows the correctness of the above mentioned algorithm. Please remark that we consider an unbranched root strand which is composed of n segments. Within this proof we are following the approach presented in Couvreur et al., 2012.

Root water uptake is driven by gradients between $h^{(i)}_{Soil}$ [m] and the xylem water potential within segment i, denoted by $h^{(i)}_{Xylem}$ [m]. The actual value of root water uptake $Q^{(i)}_{Rad}$ [m^3/s] along this gradient is given by the segments radial resistance $R^{(i)}_{Rad}$ [s/m^2], according to Ohm's law. Water uptake is supposed to occur at the ends of each segment (thus water has to traverse the entire axial pathway and thus the entire axial resistance $R^{(i)}_{Ax}$ [s/m^2]). If the number of segments is sufficiently large, this assumption is not causing artifacts (see also Figure 2 of our first reply). Root water uptake from the successive segments confluences in the direction of the root collar, the axial rates of water transport within the segments are denoted with $Q^{(i)}_{Ax}$ [m^3/s].

Similar to Couvreur et. al 2012 there are $3n + 1$ unknown variables: The collor potential $h^{(0)}_{Xylem}$, the n xylem water potentials in each node $h^{(i)}_{Xylem}$, the n root water uptake rates $Q^{(i)}_{Rad}$ and the n axial rates of water transport $Q^{(i)}_{Ax}$. This set of unknown variables can be determined if $3n + 1$ linearly independent (linear) equations are given. These are given using the drops in water potential along the axial resistances, along radial resistances and in terms of mass conservation laws, the corresponding equations are the following:

$$
\begin{aligned}
R_{Ax}^{(i)} &= \frac{h_{Xylem}^{(i-1)} - h_{Xylem}^{(i)}}{Q_{Ax}^{(i)}} && i = 1, \ldots, n \\
Q_{Rad}^{(i)} &= \frac{h_{Xylem}^{(i)} - h_{Soil}^{(i)}}{R_{Rad}^{(i)}} && i = 1, \ldots, n \\
Q_{Ax}^{(i)} &= Q_{Rad}^{(i+1)} + Q_{Rad}^{(i)} && i = 1, \ldots, n-1 \\
Q_{Ax}^{(n)} &= Q_{Rad}^{(n)} && i = n
\end{aligned}
$$

As these are only $3n$ equations, one variable remains unknown. Closure of the system of equations can be achieved in terms of a boundary condition, which we decide to be of constant flux type:

$$
T_{act} = T = Q_{Ax}^{(1)}
$$

We eliminate the $Q_{Ax}^{(i)}$ and $h_{Xylem}^{(i)}$ by rearranging and recombining our equations. This results in a set of n equations for the $Q_{Rad}^{(i)}$. We do this as follows:

$$
\begin{aligned}
h_{Xylem}^{(1)} &= R_{Rad}^{(1)} \cdot Q_{Rad}^{(1)} + h_{Soil}^{(1)} && i = 1 \\
h_{Xylem}^{(i-1)} - h_{Xylem}^{(i)} &= (Q_{Rad}^{(i-1)} \cdot R_{Rad}^{(i-1)} - Q_{Rad}^{(i)} \cdot R_{Rad}^{(i)}) + (h_{Soil}^{(i-1)} - h_{Soil}^{(i)}) && i = 2, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
h_{Xylem}^{(0)} - h_{Xylem}^{(1)} &= R_{Ax}^{(1)} \cdot Q_{Ax}^{(1)} && i = 1 \\
h_{Xylem}^{(i-1)} - h_{Xylem}^{(i)} &= R_{Ax}^{(i)} \cdot Q_{Ax}^{(i)} && i = 2, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
Q_{Ax}^{(1)} &= T && i = 1 \\
Q_{Ax}^{(i)} &= \sum_{j=i}^{n} Q_{Rad}^{(i)} = T - \sum_{j=1}^{(i-1)} Q_{Rad}^{(i)} && i = 2, \ldots, n
\end{aligned}
$$

Combining the equations for $i = 2 \ldots n$, we obtain

$$(R_{Rad}^{(i-1)} \cdot Q_{Rad}^{(i-1)} - R_{Rad}^{(i)} \cdot Q_{Rad}^{(i)}) + (h_{Soil}^{(i-1)} - h_{Soil}^{(i)}) = h_{Xylem}^{(i-1)} - h_{Xylem}^{(i)} = R_{Ax}^{(i)} \cdot Q_{Ax}^{(i)} = R_{Ax}^{(i)} \cdot T - R_{Ax}^{(i)} \cdot \sum_{j=1}^{(i-1)} Q_{Rad}^{(j)}$$

By using the equations belonging to $i = 1$ we obtain

$$Q_{Rad}^{(1)} = T - \sum_{j=2}^{n} Q_{Rad}^{(j)}$$

We can bring these eqs. into a shape which is convenient to calculate $Q_{Rad}^{(i)}$ with the help of Gauss Algorithm. Firstly, we split

$$\sum_{j=1}^{(i-1)} Q_{Rad}^{(j)} = \sum_{j=1}^{i-2} Q_{Rad}^{(j)} + Q_{Rad}^{(i-1)} \quad \text{for } i > 2$$

and

$$\sum_{j=1}^{(i-1)} Q_{Rad}^{(j)} = Q_{Rad}^{(1)} \quad \text{for } i = 2.$$

Secondly, we collect all $Q_{Rad}^{(i)}$ on the left hand side. We thus obtain

$$\sum_{i=1}^{n} Q_{Rad}^{(i)} = T \tag{A.2-9}$$

$$(R^{(1)} + R_{Ax}^{(2)}) \cdot Q_{Rad}^{(1)} - R_{Rad}^{(2)} \cdot Q_{Rad}^{(2)} = R_{Ax}^{(2)} \cdot T + (h_{Soil}^{(2)} - h_{Soil}^{(1)}) \tag{A.2-10}$$

$$(R_{Rad}^{(i-1)} + R_{Ax}^{(i)}) \cdot Q_{Rad}^{(i-1)} - R_{Rad}^{(i)} \cdot Q_{Rad}^{(i)} + R_{Ax}^{(i)} \cdot \sum_{j}^{i-2} Q_{Rad}^{(j)} = R_{Ax}^{(i)} \cdot T + (h_{Soil}^{(i)} - h_{Soil}^{(i-1)}) \tag{A.2-11}$$

By defining $\delta^{(i)} := h_{Soil}^{(i)} - h_{Soil}^{(i-1)}$ [m] for $i >= 2$, we can rewrite this set of equations in matrix notation

$$M\vec{Q} = \vec{c} \tag{A.2-12}$$

in which M has the special structure which we will use to derive an explicit solution

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ R^{(1)} + R^{(2)}_{Ax} & -R^{(2)}_{Rad} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ R^{(3)}_{Ax} & R^{(2)}_{Rad} + R^{(3)}_{Ax} & -R^{(3)}_{Rad} & 0 & \dots & 0 & 0 & 0 & 0 \\ R^{(4)}_{Ax} & R^{(4)}_{Ax} & R^{(3)}_{Rad} + R^{(4)}_{Ax} & -R^{(4)}_{Rad} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ R^{(n-3)}_{Ax} & R^{(n-3)}_{Ax} & R^{(n-3)}_{Ax} & R^{(n-3)}_{Ax} & \dots & -R^{(n-3)}_{Rad} & 0 & 0 & 0 \\ R^{(n-2)}_{Ax} & R^{(n-2)}_{Ax} & R^{(n-2)}_{Ax} & R^{(n-2)}_{Ax} & \dots & R^{(n-3)}_{Rad} + R^{(n-2)}_{Ax} & -R^{(n-2)}_{Rad} & 0 & 0 \\ R^{(n-1)}_{Ax} & R^{(n-1)}_{Ax} & R^{(n-1)}_{Ax} & R^{(n-1)}_{Ax} & \dots & R^{(n-1)}_{Ax} & R^{(n-2)}_{Rad} + R^{(n-1)}_{Ax} & -R^{(n-1)}_{Rad} & 0 \\ R^{(n)}_{Ax} & R^{(n)}_{Ax} & R^{(n)}_{Ax} & R^{(n)}_{Ax} & \dots & R^{(n)}_{Ax} & R^{(n)}_{Ax} & R^{(n-1)}_{Rad} + R^{(n)}_{Ax} & -R^{(n)}_{Rad}, \end{pmatrix}$$

$\vec{Q}$ contains the n unknown variables $Q^{(i)}_{Rad}$

$$\vec{Q} = \begin{pmatrix} Q^{(1)}_{Rad} \\ Q^{(2)}_{Rad} \\ Q^{(3)}_{Rad} \\ Q^{(4)}_{Rad} \\ \vdots \\ Q^{(n-3)}_{Rad} \\ Q^{(n-2)}_{Rad} \\ Q^{(n-1)}_{Rad} \\ Q^{(n)}_{Rad}, \end{pmatrix}$$

and $\vec{c}$ contains the right hand sides of the equations

$$\vec{c} = \begin{pmatrix} T \\ R_{Ax}^{(2)} \cdot T + \delta^{(2)} \\ R_{Ax}^{(3)} \cdot T + \delta^{3} \\ R_{Ax}^{(4)} \cdot T + \delta^{4} \\ \vdots \\ R_{Ax}^{(n-3)} \cdot T + \delta^{(n-3)} \\ R_{Ax}^{(n-2)} \cdot T + \delta^{(n-2)} \\ R_{Ax}^{(n-1)} \cdot T + \delta^{(n-1)} \\ R_{Ax}^{(n)} \cdot T + \delta^{(n)} \end{pmatrix}$$

We now apply Gauss Algorithm to bring this set of equations into diagonal form and to derive an explicit solution.

$$\begin{array}{ccccccccccc|c}
(1) & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & T \\
(2) & R^{(1)} + R_{Ax}^{(2)} & -R_{Rad}^{(2)} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & R_{Ax}^{(2)} \cdot T + \delta^{(2)} \\
(3) & R_{Ax}^{(3)} & R_{Rad}^{(2)} + R_{Ax}^{(3)} & -R_{Rad}^{(3)} & 0 & \dots & 0 & 0 & 0 & 0 & R_{Ax}^{(3)} \cdot T + \delta^{(3)} \\
(4) & R_{Ax}^{(4)} & R_{Ax}^{(4)} & R_{Rad}^{(3)} + R_{Ax}^{(4)} & -R_{Rad}^{(4)} & \dots & 0 & 0 & 0 & 0 & R_{Ax}^{(4)} \cdot T + \delta^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & R_{Ax}^{(n-3)} & R_{Ax}^{(n-3)} & R_{Ax}^{(n-3)} & R_{Ax}^{(n-3)} & \dots & -R_{Rad}^{(n-3)} & 0 & 0 & 0 & R_{Ax}^{(n-3)} \cdot T + \delta^{(n-3)} \\
(n-2) & R_{Ax}^{(n-2)} & R_{Ax}^{(n-2)} & R_{Ax}^{(n-2)} & R_{Ax}^{(n-2)} & \dots & R_{Rad}^{(n-3)} + R_{Ax}^{(n-2)} & -R_{Rad}^{(n-2)} & 0 & 0 & R_{Ax}^{(n-2)} \cdot T + \delta^{(n-2)} \\
(n-1) & R_{Ax}^{(n-1)} & R_{Ax}^{(n-1)} & R_{Ax}^{(n-1)} & R_{Ax}^{(n-1)} & \dots & R_{Ax}^{(n-1)} & R_{Rad}^{(n-2)} + R_{Ax}^{(n-1)} & -R_{Rad}^{(n-1)} & 0 & R_{Ax}^{(n-1)} \cdot T + \delta^{(n-1)} \\
(n) & R_{Ax}^{(n)} & R_{Ax}^{(n)} & R_{Ax}^{(n)} & R_{Ax}^{(n)} & \dots & R_{Ax}^{(n)} & R_{Ax}^{(n)} & R_{Rad}^{(n-1)} + R_{Ax}^{(n)} & -R_{Rad}^{(n)} & R_{Ax}^{(n)} \cdot T + \delta^{(n)}
\end{array}$$

We will normalize the diagonals to in lines $2, \dots, n$ to -1 by dividing the respective lines by $R_{Rad}^{(i)}$. For convenience we introduce the following variables :

$$\alpha^{(i)} := \frac{R_{Rad}^{(i-1)} + R_{Ax}^{(i)}}{R_{Rad}^{(i)}} \qquad [-] \qquad \text{(A.2-13)}$$

$$\beta^{(i)} := \frac{R_{Ax}^{(i)}}{R_{Rad}^{(i)}} \qquad [-] \qquad \text{(A.2-14)}$$

$$\gamma^{(i)} := \frac{\delta^{(i)}}{R_{Rad}^{(i)}} \qquad [m^3/s] \qquad \text{(A.2-15)}$$

Please note that $\alpha^{(i)}$ and $\beta^{(i)}$ are dimensionless, strictly greater than zero and depend on the local, time invariant root hydraulic properties, whereas γ has units of a root water uptake rate and depends on soil water status as well as roots radial resistance. The system of equations now reads as

$$\begin{array}{cccccccccc|c}
(1) & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & T \\
(2) & \alpha^{(2)} & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(2)} \cdot T + \gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(3)} \cdot T + \gamma^{(3)} \\
(4) & \beta^{(4)} & \beta^{(4)} & \alpha^{(4)} & -1 & \dots & 0 & 0 & 0 & 0 & \beta^{(4)} \cdot T + \gamma^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \dots & -1 & 0 & 0 & 0 & \beta^{(n-3)} \cdot T + \gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \dots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)} \cdot T + \gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \dots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)} \cdot T + \gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \dots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)} \cdot T + \gamma^{(n)}
\end{array}$$

Please note that both sides of the equation are well defined, the left hand side is dimensionless and the solution Q_{Rad} indeed has units of a root water uptake rate. We will derive explicit solutions from this matrix by bringing it into a diagonal form. After adding line (n) to line (1), the system gets

$$
\begin{array}{c|ccccccccc|c}
(1) & 1+\beta^{(n)} & 1+\beta^{(n)} & 1+\beta^{(n)} & 1+\beta^{(n)} & \dots & 1+\beta^{(n)} & 1+\beta^{(n)} & 1+\alpha^{(n)} & 0 & (1+\beta^{(n)})\cdot T+\gamma^{(n)} \\
(2) & \alpha^{(2)} & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(2)}\cdot T+\gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(3)}\cdot T+\gamma^{(3)} \\
(4) & \beta^{(4)} & \beta^{(4)} & \alpha^{(4)} & -1 & \dots & 0 & 0 & 0 & 0 & \beta^{(4)}\cdot T+\gamma^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \dots & -1 & 0 & 0 & 0 & \beta^{(n-3)}\cdot T+\gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \dots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)}\cdot T+\gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \dots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)}\cdot T+\gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \dots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)}\cdot T+\gamma^{(n)}
\end{array}
$$

Please note that the last entry in the first line has vanished while all other lines remained unaltered. In order to continue the diagonalization of the matrix, we now divide the first line by $1+\alpha^{(n)}$ and define for convenience

$$C^{(n-1)} := \frac{1+\beta^{(n)}}{1+\alpha^{(n)}} \qquad [-] \qquad \text{(A.2-16)}$$

$$D^{(n-1)} := \frac{\gamma^{(n)}}{1+\alpha^{(n)}} \qquad [m^3/s] \qquad \text{(A.2-17)}$$

We obtain

$$
\begin{array}{c|ccccccccc|c}
(1) & C^{(n-1)} & C^{(n-1)} & C^{(n-1)} & C^{(n-1)} & \dots & C^{(n-1)} & C^{(n-1)} & 1 & 0 & C^{(n-1)}\cdot T+D^{(n-1)} \\
(2) & \alpha^{(2)} & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(2)}\cdot T+\gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(3)}\cdot T+\gamma^{(3)} \\
(4) & \beta^{(4)} & \beta^{(4)} & \alpha^{(4)} & -1 & \dots & 0 & 0 & 0 & 0 & \beta^{(4)}\cdot T+\gamma^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \dots & -1 & 0 & 0 & 0 & \beta^{(n-3)}\cdot T+\gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \dots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)}\cdot T+\gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \dots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)}\cdot T+\gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \dots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)}\cdot T+\gamma^{(n)}
\end{array}
$$

We now add line $(n-1)$ to line (1), the second entry in the first line vanishes:

$$
\begin{array}{cccccccc|c}
(1) & C^{(n-1)} + \beta^{(n-1)} & C^{(n-1)} + \beta^{(n-1)} & \cdots & C^{(n-1)} + \beta^{(n-1)} & C^{(n-1)} + \alpha^{(n-1)} & 0 & 0 & (C^{(n-1)} + \beta^{(n-1)}) \cdot T + D^{(n-1)} + \gamma^{(n-1)} \\
(2) & \alpha^{(2)} & -1 & \cdots & 0 & 0 & 0 & 0 & \beta^{(2)} \cdot T + \gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & \cdots & 0 & 0 & 0 & 0 & \beta^{(3)} \cdot T + \gamma^{(3)} \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \beta^{(4)} \cdot T + \gamma^{(4)} \\
 & & & & & & & & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \cdots & -1 & 0 & 0 & 0 & \beta^{(n-3)} \cdot T + \gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \cdots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)} \cdot T + \gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \cdots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)} \cdot T + \gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \cdots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)} \cdot T + \gamma^{(n)}
\end{array}
$$

The following recursive definitions and steps of calculations lead the way to an recursive construction of an diagonal matrix and thus the desired solution: We divide by $C^{(n-1)} + \alpha^{(n-1)}$, define for convenience

$$C^{(n-2)} := \frac{C^{(n-1)} + \beta^{(n-1)}}{C^{(n-1)} + \alpha^{(n-1)}} \qquad [-] \tag{A.2-18}$$

$$D^{(n-2)} := \frac{D^{(n-1)} + \gamma^{(n-1)}}{C^{(n-1)} + \alpha^{(n-1)}} \qquad [m^3/s] \tag{A.2-19}$$

and the system becomes

$$
\begin{array}{cccccccccc|c}
(1) & C^{(n-2)} & C^{(n-2)} & C^{(n-2)} & C^{(n-2)} & \cdots & C^{(n-2)} & 1 & 0 & 0 & C^{(n-2)} \cdot T + D^{(n-2)} \\
(2) & \alpha^{(2)} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \beta^{(2)} \cdot T + \gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \beta^{(3)} \cdot T + \gamma^{(3)} \\
(4) & \beta^{(4)} & \beta^{(4)} & \alpha^{(4)} & -1 & \cdots & 0 & 0 & 0 & 0 & \beta^{(4)} \cdot T + \gamma^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \cdots & -1 & 0 & 0 & 0 & \beta^{(n-3)} \cdot T + \gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \cdots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)} \cdot T + \gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \cdots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)} \cdot T + \gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \cdots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)} \cdot T + \gamma^{(n)}
\end{array}
$$

We can now add line $n-2$ to line one and continue with this procedure by using the recursive definitions

$$C^{(n-i)} := \frac{C^{(n-i+1)} + \beta^{(n-i+1)}}{C^{(n-i+1)} + \alpha^{(n-i+1)}} \qquad [-] \qquad \text{(A.2-20)}$$

$$D^{(n-i)} := \frac{D^{(n-i+1)} + \gamma^{(n-i+1)}}{C_{n-i+1} + \alpha^{(n-i+1)}} \qquad [m^3/s] \qquad \text{(A.2-21)}$$

Finally we obtain

$$\begin{array}{cccccccccc|c}
(1) & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & C^{(1)} \cdot T + D^{(1)} \\
(2) & \alpha^{(2)} & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(2)} \cdot T + \gamma^{(2)} \\
(3) & \beta^{(3)} & \alpha^{(3)} & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \beta^{(3)} \cdot T + \gamma^{(3)} \\
(4) & \beta^{(4)} & \beta^{(4)} & \alpha^{(4)} & -1 & \dots & 0 & 0 & 0 & 0 & \beta^{(4)} \cdot T + \gamma^{(4)} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-3) & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \beta^{(n-3)} & \dots & -1 & 0 & 0 & 0 & \beta^{(n-3)} \cdot T + \gamma^{(n-3)} \\
(n-2) & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \beta^{(n-2)} & \dots & \alpha^{(n-2)} & -1 & 0 & 0 & \beta^{(n-2)} \cdot T + \gamma^{(n-2)} \\
(n-1) & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \beta^{(n-1)} & \dots & \beta^{(n-1)} & \alpha^{(n-1)} & -1 & 0 & \beta^{(n-1)} \cdot T + \gamma^{(n-1)} \\
(n) & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \beta^{(n)} & \dots & \beta^{(n)} & \beta^{(n)} & \alpha^{(n)} & -1 & \beta^{(n)} \cdot T + \gamma^{(n)}
\end{array}$$

We can derive solutions $Q_{Rad}^{(i)}$ by forward substitution. In accordance with (Couvreur et al., 2012), we define

$$SUF^{(1)} := C^{(1)} \qquad [-] \qquad \text{(A.2-22)}$$

$$\tilde{\varphi}^{(1)} := D^{(1)} \qquad [m^3/s] \qquad \text{(A.2-23)}$$

and obtain for the first root water uptake rate

$$Q_{Rad}^{(1)} = SUF^{(1)} \cdot T + \tilde{\varphi}^{(1)}$$

For the second root water uptake we obtain

$$\begin{aligned}
Q_{Rad}^{(2)} &= \alpha^{(2)} \cdot Q_{Rad}^{(1)} - \beta^{(2)} \cdot T - \gamma^{(2)} \\
&= \alpha^{(2)} \cdot (SUF^{(1)} \cdot T + \tilde{\varphi}^{(1)}) - \beta^{(2)} \cdot T - \gamma^{(2)} \\
&= T \cdot (\alpha^{(2)} \cdot SUF^{(1)} - \beta^{(2)}) + \alpha^{(2)} \cdot \tilde{\varphi}^{(1)} - \gamma^{(2)}
\end{aligned}$$

We can express this in notation similar to (Couvreur et al., 2012) by defining

$$SUF^{(2)} := \alpha^{(2)} \cdot SUF^{(1)} - \beta^{(2)} \qquad [-] \tag{A.2-24}$$

$$\tilde{\varphi}^{(2)} := \alpha^{(2)} \cdot \tilde{\varphi}^{(1)} - \gamma^{(2)} \qquad [m^3/s] \tag{A.2-25}$$

and obtain

$$Q^{(2)}_{Rad} = SUF^{(2)} \cdot T + \tilde{\varphi}^{(2)}$$

The solutions for $i >= 3$ share a common structure. If we introduce

$$SUF^{(3)} := \beta^{(3)} \cdot SUF^{(1)} + \alpha^{(3)} SUF^{(2)} - \beta^{(3)} \tag{A.2-26}$$

$$\tilde{\varphi}^{(3)} := \beta^{(3)} \cdot \tilde{\varphi}^{(1)} + \alpha^{(3)} \cdot \tilde{\varphi}^{(2)} - \gamma^{(3)} \tag{A.2-27}$$

the solution for $Q^{(3)}_{Rad}$ is given by

$$\begin{aligned}
Q^{(3)}_{Rad} &= \beta^{(3)} \cdot Q^{(1)}_{Rad} + \alpha^{(3)} \cdot Q^{(2)}_{Rad} - \beta^{(3)} \cdot T - \gamma^{(3)} \\
&= \beta^{(3)} \cdot (SUF^{(1)} \cdot T + \tilde{\varphi}^{(1)}) + \alpha^{(3)} \cdot (SUF^{(2)} \cdot T + \tilde{\varphi}^{(2)}) - \beta^{(3)} \cdot T - \gamma^{(3)} \\
&= T \cdot (\beta^{(3)} \cdot SUF^{(1)} + \alpha^{(3)} SUF^{(2)} - \beta^{(3)}) + \beta^{(3)} \cdot \tilde{\varphi}^{(1)} + \alpha^{(3)} \cdot \tilde{\varphi}^{(2)} - \gamma^{(3)} \\
&= T \cdot SUF^{(3)} + \tilde{\varphi}^{(3)}
\end{aligned}$$

Observation of solution 4 unveils the general structure of the solutions for $Q_{Rad}^{(i)}$ and $i > 2$

$$
\begin{aligned}
Q_{Rad}^{(4)} &= \beta^{(4)} \cdot \sum_{j=1}^{2} Q_{Rad}^{(j)} + \alpha^{(4)} \cdot Q_{Rad}^{(3)} - \beta^{(4)} \cdot T - \gamma^{(4)} \\
&= \beta^{(4)} \cdot Q_{Rad}^{(1)} + \beta^{(4)} \cdot Q_{Rad}^{(2)} + \alpha^{(4)} \cdot Q_{Rad}^{(3)} - \beta^{(4)} \cdot T - \gamma^{(4)} \\
&= \beta^{(4)} \cdot (SUF^{(1)} \cdot T + \tilde{\varphi}^{(1)}) + \beta^{(4)} \cdot (SUF^{(2)} \cdot T + \tilde{\varphi}^{(2)}) + \alpha^{(4)} \cdot (SUF^{(3)} \cdot T + \tilde{\varphi}^{(3)}) - \beta^{(4)} \cdot T - \gamma^{(4)} \\
&= T \cdot (\beta^{(4)} \cdot SUF^{(1)} + \beta^{(4)} \cdot SUF^{(2)} + \alpha^{(4)} \cdot SUF^{(3)} - \beta^{(4)}) + \beta^{(4)} \cdot \tilde{\varphi}^{(1)} + \beta^{(4)} \cdot \tilde{\varphi}^{(2)} + \alpha^{(4)} \cdot \tilde{\varphi}^{(3)} - \gamma^{(4)} \\
&= T \cdot (\beta^{(4)} \cdot \sum_{j=1}^{2} SUF^{(j)} + \alpha^{(4)} \cdot SUF^{(3)} - \beta^{(4)}) + (\beta^{(4)} \cdot \sum_{j=1}^{2} \tilde{\varphi}^{(j)} + \alpha^{(4)} \cdot \tilde{\varphi}^{(3)} - \gamma^{(4)}) \\
&= T \cdot SUF^{(4)} + \tilde{\varphi}^{(4)}
\end{aligned}
$$

in which we use the recursive definitions

$$SUF^{(i)} = \beta^{(i)} \cdot \sum_{j=1}^{i-2} SUF^{(j)} + \alpha^{(i)} \cdot SUF^{(i-1)} - \beta^{(i)} \quad [-] \tag{A.2-28}$$

$$\tilde{\varphi}^{(i)} = \beta^{(i)} \cdot \sum_{j=1}^{i-2} \tilde{\varphi}^{(j)} + \alpha^{(i)} \cdot \tilde{\varphi}^{(i-1)} - \gamma^{(i)} \quad [m^3/s] \tag{A.2-29}$$

Please note that the SUF are dimensionless and depend only on the time invariant plant hydraulic parameters, whereas the $\tilde{\varphi}$ have units of a flux rate and depend on the transient soil water statusses. For a full accordance with the notation from Couvreur et al. (2012), we rearrange the equations using the "compensatory root water uptake" φ as follows

$$\varphi^{(i)} := \frac{\tilde{\varphi}^{(i)}}{SUF^{(i)}} \quad [m^3/s] \tag{A.2-30}$$

$$Q_{Rad}^{(i)} = T \cdot SUF^{(i)} + \tilde{\varphi}^{(i)} = SUF^{(i)} \cdot (T + \varphi^{(i)}) \quad [m^3/s]$$