

1 **Forchheimer Flow to a Well Considering Time-Dependent Critical**

2 **Radius**

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1 **Abstract:**

2 Previous studies on the non-Darcian flow into a pumping well assumed that critical radius (R_{CD})
3 was a constant or infinity, where R_{CD} represents the location of the interface between the
4 non-Darcian flow region and Darcian flow region. In this study, a two-region model considering
5 time-dependent R_{CD} was established, where the non-Darcian flow was described by the
6 Forchheimer equation. A new iteration method was proposed to estimate R_{CD} based on the
7 finite-difference method. The results showed that R_{CD} increased with time until reaching the
8 quasi-steady state flow, and the asymptotic value of R_{CD} only depended on the critical specific
9 discharge beyond which flow became non-Darcian. A larger inertial force would reduce the
10 change rate of R_{CD} with time, and resulted in a smaller R_{CD} at a specific time during the transient
11 flow. The difference between the new solution and previous solutions were obvious in the early
12 pumping stage. The new solution agreed very well with the solution of previous two-region
13 model with a constant R_{CD} under quasi-steady flow. It agreed with the solution of the fully
14 Darcian flow model in the Darcian flow region.

15 **Key words:** Confined aquifer; Finite-difference method; Iteration method; Two-region model

1 **Nomenclature**

2	B	aquifer thickness (L)
3	D_p	characteristic grain diameter (L)
4	K	hydraulic conductivity of the aquifer (LT^{-1})
5	K_β	apparent hydraulic conductivity, an empirical constant in the Forchheimer law
6		(LT^{-1})
7	q	specific discharge in the aquifer (LT^{-1})
8	q_C	critical specific discharge (LT^{-1})
9	q_Y, q_N	specific discharges for Darcian flow and non-Darcian flow (LT^{-1}), respectively
10	Q	well discharge (L^3T^{-1})
11	s	drawdown (L) for aquifer
12	s_Y, s_N	drawdowns (L) for Darcian flow and non-Darcian flow, respectively
13	s_w	drawdown (L) inside well
14	S	storage coefficient of the aquifer (dimensionless)
15	r	distance from the center of the well (L)
16	r_w	radius of the well screen (L)
17	R_C	critical radius for non-Darcian flow (L)
18	Re	Reynolds number (dimensionless)
19	Re _C	critical Reynolds number (dimensionless)
20	t	pumping time (T)
21	β	an empirical constant in the Forchheimer law (TL^{-1}), named as inertial force

1		coefficient in this study
2	ν	kinematic viscosity of the fluid (L^2T^{-1})
3	q_{ND}, q_{YD}	dimensionless specific discharges defined in Table 1 in the non-Darcian flow and
4		Darcian flow regions, respectively
5	q_{CD}	dimensionless critical specific discharge defined in Table 1
6	r_D	dimensionless distance defined in Table 1
7	r_{wD}	dimensionless radius of the well screen defined in Table 1
8	R_{CD}	dimensionless critical radius defined in Table 1
9	s_{ND}, s_{YD}	dimensionless drawdown s defined in Table 1 in the non-Darcian flow and Darcian
10		flow regions, respectively
11	s_{wD}	dimensionless drawdown inside the well defined in Table 1
12	t_D	dimensionless time defined in Table 1
13	β_D	dimensionless inertial force coefficient defined in Table 1
14	λ	ratio of the hydraulic conductivity and apparent hydraulic conductivity defined in
15		Table 1

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17 The subscript D refers to terms in dimensionless form.

18 The subscripts N and Y refer to terms related to non-Darcian flow and Darcian flow regions,

19 respectively.

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2 **1. Introduction**

3 Darcy's law indicates a linear relationship between the fluid velocity and the hydraulic
4 gradient (Bear, 1972), which is a basic assumption used to handle a great deal of problems
5 related to flow in porous and fractured media. However, many evidences from the laboratory and
6 field experiments show that this linear law may be invalid in some situations, especially when
7 the groundwater flow velocity is sufficiently high or sufficiently low, where non-Darcian flow
8 prevails (Basak, 1977;Bordier and Zimmer, 2000;Engelund, 1953;Forchheimer, 1901;Izbash,
9 1931;Liu et al., 2012;Soni et al., 1978).

10 Darcy's law considers kinematic forces but excludes inertial forces of flow. However, the
11 inertia forces become significant with respect to the kinematic forces when the velocity is great,
12 leading to non-Darcian flow (Engelund, 1953;Forchheimer, 1901;Irmay, 1959;Izbash, 1931).
13 Forchheimer (1901) proposed a heuristic Forchheimer law describing the non-Darcian flow,
14 which was an extension of Darcy's law by adding a second-order velocity term, representing the
15 inertial effect. To verify the applicability of the Forchheimer law, many approaches were
16 introduced, such as the dimensional analysis (Ward, 1964), the capillary model (Dullien and
17 Azzam, 1973), the hybrid mixture theory (Hassanizadeh and Gray, 1987), and the volume
18 averaging method (Whitaker, 1996). Recently, Giorgi (1997) and Chen et al. (2001) analytically
19 derived the Forchheimer law from the Navier-Stokes equation. Another widely used model
20 describing the non-Darcian flow was the Izbash equation (Izbash, 1931). This equation was a

1 fully empirical power-law function obtained through analyzing experimental data. The Izbash
2 equation was preferred for modeling purpose, since the power index in the Izbash equation can
3 be parameterized depending on flow conditions (Basak, 1977). George and Hansen (1992)
4 demonstrated that the Forchheimer and Izbash equations were identical for some cases.

5 Due to the high velocities, non-Darcian flow is likely to occur near pumping/injecting wells
6 (Yeh and Chang, 2013; Wen et al., 2008b). Several studies showed that the non-Darcian effect
7 had significant influence on hydraulic parameter estimations. For instance, Theis solution cannot
8 be used to explain the pumping test data in the Chaj-Doab area near Gujrat water distributory in
9 Pakistan (Ahmad, 1998), while Birpinar and Sen (2004) and Wen et al. (2011) found that the
10 Forchheimer law worked very well. Quinn et al. (2013) demonstrated that non-Darcian flow
11 effect increased as the initial applied head differential increased in a series of slug tests.
12 Specifically, Quinn et al. (2013) showed that the hydraulic conductivity was underestimated by
13 Darcy's law when the initial applied head differentials were greater than 0.2 m. They pointed out
14 that Darcian flow conditions can be maintained in the sandstone when the initial applied head
15 differentials were less than 0.2 m (Quinn et al., 2013). Mathias and Todman (2010) showed that
16 the Jacob method, based on Darcy's law, cannot fit the step-drawdown tests of van Tonder et al.
17 (2001) when the pumping rate was greater than $10 \text{ m}^3\text{hour}^{-1}$. However, the Forchheimer law
18 fitted the step-drawdown tests data very well (Mathias and Todman, 2010). In this study, we will
19 focus on the non-Darcian flow into a pumping well by the Forchheimer law.

20 Although many efforts have been devoted to study the non-Darcian flow around the well,

1 the exact solutions have not been obtained due to the non-linearity of the problem (Mathias et al.,
2 2008; Yeh and Chang, 2013). For example, Sen (1990, 2000) employed the Boltzmann transform
3 method to analytically solve the problems related to the non-Darcian flow. This method was
4 showed to be problematic, since both initial and boundary conditions cannot be simultaneously
5 transformed into a form only containing the Boltzmann variable (Camacho and Vasquez,
6 1992; Wen et al., 2008a). Wen et al. (2008a; 2008b) derived the semi-analytical solutions of the
7 non-Darcian flow model by combining the Linearization procedure and the Laplace transform
8 method (LL method), assuming that the flow in the non-Darcian flow region was in quasi-steady
9 state flow. Wen et al. (2008a; 2008b) pointed out that solutions by the Boltzmann transform and
10 the LL method coincided at late time. To test the accuracy of the semi-analytical solutions (Wen
11 et al., 2008a; Sen, 2000), Mathias et al. (2008) and Wen et al. (2009) employed the
12 finite-difference method to study the non-Darcian flow problems, and their results showed that
13 the semi-analytical solution only agreed very well with the numerical solution at late pumping
14 stage.

15 All above-mentioned investigations assume that the non-Darcian flow occurs over the entire
16 domain, which is called a fully non-Darcian flow (F-ND) model hereinafter. In fact, the regime
17 of the flow to the pumping well can be divided into two regions: non-Darcian flow occurs within
18 a narrow region around well, due to the relatively high velocity of flow there, and Darcian flow
19 prevails over the rest domain. One may think that such two-region flow could be described by
20 the Forchheimer law, which would automatically reduce to the Darcy's law at the location far

1 from the well (because the second-order velocity term in the Forchheimer law will be negligible
2 if velocity approaches zero). However, Forchheimer law (or F-ND model) may not work very
3 well for moderate velocity under which that Darcian flow prevails. Mackie (1983) demonstrated
4 that the two-region model could fit the experimental data in the laboratory better than the F-ND
5 model. Huyakorn and Dudgeon (1976) employed a two-region model to study flow near a
6 pumping well. Basak (1978) presented analytical solutions of the two-region model for
7 steady-state flow to a fully penetrating well. Sen (1988) and Wen et al. (2008b) derived the
8 analytical solutions of the two-region model for transient flow to a pumping well, and both
9 solutions were valid for the groundwater flow in the quasi-steady state.

10 All researches mentioned above implied that the critical radius is a constant, where the
11 critical radius represents the location separating the non-Darcian and Darcian flows (Sen,
12 1988;Wen et al., 2008b). For example, the critical radius is infinity for the F-ND model and is
13 zero for the fully Darcian flow model, while it is a finite constant for the two-region model in
14 which the critical radius is determined under the quasi-steady state flow condition (Sen,
15 1988;Wen et al., 2008b). Actually, the critical radius changes continuously with time for the
16 transient flow, and cannot be determined straightforwardly. For example, the initial critical
17 radius is zero for an initially hydrostatic aquifer, and it gradually increases with time until the
18 system becomes quasi-steady state near a constant-rate pumping well. The movement of critical
19 radius may be more complex for the variable-rate pumping tests (Bear, 1972;Mishra et al., 2012),
20 the slug tests (Quinn et al., 2013) or the step-drawdown tests (Louwyck et al., 2010;Mathias and

1 Todman, 2010). Therefore, the two-region model with time-dependent critical radius is more
2 reasonable for transient flow near a pumping well, and it is particularly true when the pumping
3 rate changes greatly.

4 In this study, we will investigate non-Darcian flow into a fully penetrating pumping well
5 considering a time-dependent critical radius using the finite-difference method. A new iteration
6 procedure will be proposed to estimate the moving critical radius. This new model reduces to the
7 F-ND model when the critical radius is infinite and it becomes the fully Darcian flow model
8 when the critical radius is 0.

9 **2. Problem statement and mathematic model**

10 *2.1. Location of the critical radius of the two-region model*

11 Previous researches showed that the porous media flow may be divided into four regimes,
12 such as A) non-Darcy pre-linear laminar flow, B) Darcy flow, C) non-Darcy post-linear laminar
13 flow, and D) non-Darcy post-linear turbulent flow (Basak, 1977; Bear, 1972). For radial flow to a
14 pumping well, the velocity in the aquifer decreases with the distance from the well. Therefore,
15 the radial flow might experience all four-flow regimes. To simplify the problem, we use a
16 two-region model that considers a non-Darcian flow region near the well and a Darcian flow
17 region away from the well. A unique feature of the two-region model used in this study is that
18 the critical radius is allowed to vary with time whereas it was assumed to be constant in previous
19 studies (Dudgeon et al., 1972b, a; Huyakorn and Dudgeon, 1976; Mackie, 1983; Sen, 1988; Wen et
20 al., 2008b).

1 Generally, the start of the non-Darcian flow can be determined by the critical Reynolds
2 number (Re_C), where the Reynolds number is defined as

$$3 \quad Re(r, t) = D_p q(r, t) / \nu, \quad (1)$$

4 where ν is the kinematic viscosity of the fluid (L^2T^{-1}); D_p is the characteristic grain diameter
5 (L); $q(r, t)$ is specific discharge (LT^{-1}) at distance r (L) and time t (T); Re is Reynolds number
6 which depends on time and space (dimensionless). The critical Reynolds number (Re_C) refers to
7 Re at the start of non-Darcian flow. Up to present, there is still considerable debate on Re_C for
8 the initiation of non-Darcian flow in porous media. Scheidegger (1974) gave Re_C to be 0.1 to 75;
9 Zeng and Grigg (2006) suggested the range of Re_C from 1 to 100. Re_C will be set to 100 to make
10 sure non-Darcian flow happen in this study. According to Eq. (1), one can see that the specific
11 discharge has a linear relationship to Re . Therefore, the critical specific discharge (q_C) can also
12 be used to determine the start of the non-Darcian flow, since one can calculate q_C for a given
13 Re_C . When the specific discharge is less than or equal to q_C (or $Re \leq Re_C$), the flow is
14 considered as Darcian. When the specific discharge is greater than q_C (or $Re > Re_C$), the flow is
15 taken as non-Darcian. Denoting $R_C(t)$ as the critical radius at which $q = q_C$ (or $Re = Re_C$), then
16 it is non-Darcian flow when $r \leq R_C(t)$ and Darcian flow when $r > R_C(t)$, as shown in Fig. 1.

17 For the quasi-steady state flow around a fully penetrating well in a homogeneous and
18 isotropic formation, one has (Sen, 1988; Wen et al., 2008b)

$$19 \quad R_C = Q / (2\pi B q_C), \quad (2)$$

20 where B is the thickness of the aquifer (L); Q is the well discharge (L^3T^{-1}). In the case of a

1 constant pumping rate, R_C is also a constant for a specific Re_C . This constant R_C was used in
 2 previous two-region models of transient non-Darcian flow (Sen, 1988;Wen et al., 2008b).
 3 Actually, R_C is not a constant for transient flow, and it cannot be determined directly since the
 4 velocity distribution changes with time. In this study, a new iteration method will be proposed to
 5 determine R_C as described below.

6 **2.2. Mathematic model**

7 Fig. 1 shows the physical model investigated in this study, where a pumping well fully
 8 penetrates a confined aquifer. The origin of the cylindrical coordinate system is at the center of
 9 the well. The r -axis is horizontal and outward from the well, and the z -axis is upward vertical.
 10 Three assumptions are made in this study. First, the non-Darcian and Darcian flow may coexist
 11 and the critical radius is time-dependent, and the non-Darcian flow is governed by the
 12 Forchheimer law. Second, the system is hydrostatic before the pumping starts, so $R_C(t=0)=0$.
 13 Third, the aquifer is homogeneous, isotropic, infinitely extensive and with a constant thickness.
 14 These assumptions, although quite idealized, are standard in well hydraulic study (Papadopoulos
 15 and Cooper, 1967;Sen, 1988;Wen et al., 2008b). Based on these assumptions, the governing
 16 equations of the two-region flow model can be described as follows

$$17 \quad \frac{\partial q_N(r,t)}{\partial r} + \frac{q_N(r,t)}{r} = \frac{S}{B} \frac{\partial s_N(r,t)}{\partial t}, \text{ if } r \leq R_C(t), \quad (3)$$

$$18 \quad \frac{\partial q_Y(r,t)}{\partial r} + \frac{q_Y(r,t)}{r} = \frac{S}{B} \frac{\partial s_Y(r,t)}{\partial t}, \text{ if } r > R_C(t), \quad (4)$$

19 where $s_Y(r,t)$ and $s_N(r,t)$ are drawdowns (L) at distance r and time t in Darcian flow and
 20 non-Darcian flow regions, respectively; S is the aquifer storage coefficient (dimensionless).

1 Initial condition is

$$2 \quad s_Y(r,0) = s_N(r,0) = 0. \quad (5)$$

3 The outer boundary condition is

$$4 \quad s_Y(\infty, t) = 0. \quad (6)$$

5 Assuming that the pumping rate is large enough to induce non-Darcian flow near the well,
6 the boundary condition at the wellbore, considering the wellbore storage with a finite diameter
7 well, can be written as

$$8 \quad 2\pi r B q_N(r, t) \Big|_{r \rightarrow r_w} - \pi r_w^2 \frac{ds_w(t)}{dt} = -Q, \quad (7)$$

9 where Q is positive for the pumping rate; r_w is the radius of the well (L); s_w is the drawdown
10 inside the well (L). Notice that well loss is not considered so the drawdown is continuous across
11 the well screen

$$12 \quad s_w(t) = s_N(r_w, t). \quad (8)$$

13 The drawdown and the discharge from the Darcian flow region into the non-Darcian flow
14 region are continuous at the critical radius

$$15 \quad s_N[R_C(t), t] = s_Y[R_C(t), t], \quad (9)$$

$$16 \quad q_N[R_C(t), t] = q_Y[R_C(t), t]. \quad (10)$$

17 In the non-Darcian flow region, we use the Forchheimer law to describe the flow
18 (Forchheimer, 1901)

$$1 \quad q_N + \beta q_N |q_N| = K_\beta \frac{\partial s_N}{\partial r}, \quad (11)$$

2 in which β (TL^{-1}) and K_β (LT^{-1}) are empirical constants depending on the properties of the
3 medium (Sidiropoulou et al., 2007). K_β is called the apparent hydraulic conductivity and it
4 reduces to the hydraulic conductivity when $\beta = 0$ (Chen et al., 2001; Sidiropoulou et al.,
5 2007). β is called the inertial force coefficient. Many studies demonstrated that the value of β
6 was related to the porous media and the fluid properties (Scheidegger, 1958; Moutsopoulos et al.,
7 2009). For example, Ergun equation (Ergun, 1952) was widely used to estimate β

$$8 \quad \beta = \frac{1.75 D_p}{150 \nu (1 - \phi)}, \quad (12)$$

9 where ϕ is porosity. When the kinematic viscosity of water (ν) at 20 °C is $10^{-6} m^2 \cdot s^{-1}$,
10 $D_p = 0.001 m$, $\phi = 0.3$, one has $\beta = 2.0 \times 10^{-4} m^2/day$.

11 In the Darcian flow region, one has

$$12 \quad q_Y(r, t) = K \frac{\partial s_Y(r, t)}{\partial r}, \quad r > R_c. \quad (13)$$

13 The mathematical model of Eqs. (3) - (13) can be used to describe the groundwater flow in the
14 aquifer with a time-dependent critical radius $R_c(t)$. This new model is an extension of the
15 previous model by Sen (1988). When $R_c(t) \rightarrow \infty$, this model becomes the F-ND model. When
16 $R_c(t) = 0$, it reduces to the fully Darcian flow model.

17 **2.3. Dimensionless transformation**

18 Defining the dimensionless variables in Table 1, Eqs. (3) - (13) can be rewritten as

$$1 \quad \frac{\partial q_{ND}}{\partial r_D} + \frac{q_{ND}}{r_D} = -\frac{\partial s_{ND}}{\partial t_D}, \quad r_D \leq R_{CD}, \quad (14)$$

$$2 \quad \frac{\partial q_{YD}}{\partial r_D} + \frac{q_{YD}}{r_D} = -\frac{\partial s_{YD}}{\partial t_D}, \quad r_D > R_{CD}, \quad (15)$$

$$3 \quad s_{ND}(r_D, 0) = s_{YD}(r_D, 0) = 0, \quad (16)$$

$$4 \quad s_{YD}(\infty, t_D) = 0, \quad (17)$$

$$5 \quad s_{ND}[R_{CD}(t_D), t_D] = s_{YD}[R_{CD}(t_D), t_D], \quad (18)$$

$$6 \quad q_{ND}[R_{CD}(t_D), t_D] = q_{YD}[R_{CD}(t_D), t_D]. \quad (19)$$

7 Notice that a negative sign has been used for defining q_D in Table 1. The subscript “ D ” means
 8 the dimensionless variables. The boundary condition with the wellbore storage (Eq. (7)) in the
 9 dimensionless form is

$$10 \quad (r_D q_{ND}) \Big|_{r_D \rightarrow r_{wD}} + \frac{r_{wD}^2}{2S} \frac{ds_{wD}(t_D)}{dt_D} = 1. \quad (20)$$

11 The dimensionless Forchheimer law becomes

$$12 \quad q_{ND} + \beta_D q_{ND} |q_{ND}| = -\frac{\partial s_{ND}}{\partial r_D}, \quad r_D \leq R_{CD}, \quad (21)$$

13 where β_D is the dimensionless inertial force coefficient. When the pumping rate is $0.628 \text{ m}^3 \cdot \text{s}^{-1}$,
 14 aquifer thickness is 10 m, and $\beta = 2.0 \times 10^{-4} \text{ m}^2/\text{day}$, one has $\beta_D = 0.02$ according to the definition
 15 of β_D , as shown in Table 1.

16 When $r_D > R_{CD}$, groundwater flow follows the Darcy’s law in the dimensionless format as

$$q_{yD}(r,t) = -\lambda \frac{\partial s_{yD}}{\partial r_D}, \quad r_D > R_{CD}, \quad (22)$$

where λ is the ratio of the hydraulic conductivity and apparent hydraulic conductivity, and it is usually taken as unity (Sidiropoulou et al., 2007).

3. Numerical solution

Because of the non-linearity of the problem, it is not easy to obtain the analytical solution of drawdown even if $R_{CD}(t_D)$ is constant. In this study, we will employ the finite-difference method to investigate the problem considering a time-dependent $R_{CD}(t_D)$. Due to the axisymmetric nature of the problem, the numerical simulation will be conducted with a non-uniform grid system, where the spatial steps are smaller near the well and become progressively greater away from the well. Similar to previous studies (Mathias et al., 2008; Wen et al., 2009), we discretize the dimensionless space r_D logarithmically. The dimensionless space domain $[r_{wD}, r_{eD}]$ is discretized into N nodes excluding the two boundary nodes r_{wD} and r_{eD} , where r_{eD} is a relatively large dimensionless distance used to approximate the infinite boundary (Mathias et al., 2008; Wen et al., 2009). For any node of r_i , $r_{wD} < r_i < r_{eD}$, $i=1, 2, \dots, N$, one has

$$r_i = (r_{i-1/2} + r_{i+1/2})/2, \quad i=1, 2, \dots, N, \quad (23)$$

where $r_{i+1/2}$ is calculated as follows

$$\log_{10}(r_{i+1/2}) = \log_{10}(r_{wD}) + i \left[\frac{\log_{10}(r_{eD}) - \log_{10}(r_{wD})}{N} \right], \quad i=0, 1, \dots, N. \quad (24)$$

After spatial discretization, Eqs. (14) - (15) become

$$1 \quad \frac{ds_{YD,i}}{dt_D} \approx \frac{r_{i-1/2}q_{YD,i-1/2} - r_{i+1/2}q_{YD,i+1/2}}{r_i(r_{i+1/2} - r_{i-1/2})}, \quad i=2, 3 \dots N_s-1, \quad r_D \leq R_{CD}, \quad (25)$$

$$2 \quad \frac{ds_{ND,i}}{dt_D} \approx \frac{r_{i-1/2}q_{ND,i-1/2} - r_{i+1/2}q_{ND,i+1/2}}{r_i(r_{i+1/2} - r_{i-1/2})}, \quad i= N_s, 3 \dots N-1, \quad r_D > R_{CD}, \quad (26)$$

3 where $q_{YD,i}$ and $s_{YD,i}$ are the dimensionless specific discharge q_{YD} and dimensionless drawdown
4 s_{YD} at node i for the Darcian flow, respectively; $q_{ND,i}$ and $s_{ND,i}$ are the dimensionless specific
5 discharge q_{ND} and dimensionless drawdown s_{ND} at node i for the non-Darcian flow, respectively.
6 In terms of the Forchheimer equation of Eq. (21), one can obtain

$$7 \quad q_{ND,i-1/2} \approx \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{s_{ND,i-1} - s_{ND,i}}{r_i - r_{i-1}} \right) \right]^{\frac{1}{2}} \right\}, \quad i=2, 3 \dots N_s-1, \quad (27)$$

8 and

$$9 \quad q_{ND,i+1/2} \approx \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{s_{ND,i} - s_{ND,i+1}}{r_{i+1} - r_i} \right) \right]^{\frac{1}{2}} \right\}, \quad i=2, 3 \dots N_s-1, \quad (28)$$

10 where node N_s means the location of $R_{CD}(t_D)$. At the well-aquifer boundary, one has

$$11 \quad q_{ND,1-1/2} \approx \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{s_{wD} - s_{ND,1}}{r_1 - r_{wD}} \right) \right]^{\frac{1}{2}} \right\}, \quad (29)$$

12 where s_{wD} is the dimensionless drawdown inside the well. Considering Eq. (20), s_{wD} can be
13 approximated as follows

$$14 \quad \frac{ds_{wD}}{dt_D} \approx \frac{2S}{r_{wD}^2} (1 - r_{wD}q_{ND,1-1/2}). \quad (30)$$

1 When $r_D > R_{CD}$, the finite-difference scheme of the specific discharge can be obtained from

2 Eq. (22)

3
$$q_{YD,i-1/2} \approx \lambda \frac{S_{YD,i-1} - S_{YD,i}}{r_i - r_{i-1}}, i = N_s, N_s+1 \dots N-1, \quad (31)$$

4
$$q_{YD,i+1/2} \approx \lambda \frac{S_{YD,i} - S_{YD,i+1}}{r_{i+1} - r_i}, i = N_s, N_s+1 \dots N-1. \quad (32)$$

5 As for the boundary at the infinity, the finite-difference scheme is

6
$$q_{YD,N+1/2} \approx \lambda \frac{S_{YD,N}}{r_{eD} - r_N}. \quad (33)$$

7 Now one obtains a set of ordinary differential equations. It is notable that R_{CD} or N_s which is
8 related to the index i in Eqs. (27) - (28) and Eqs. (31) - (32) is time-dependent. In the following
9 section, a new iteration method will be proposed to determine the values of R_{CD} or N_s .

10 **4. Iteration method to determine R_{CD} or N_s**

11 Before introducing the new iteration method, the relationship between R_{CD} and the
12 velocity distribution will be investigated first, based on the two-region model with a constant
13 R_{CD} . The values of the constant R_{CD} are set to 0, 0.02, 0.04, 0.08 and 0.50, respectively. The
14 other parameters are $r_{wD} = 1 \times 10^{-4}$, $\beta_D = 20$, $\lambda = 1$. The mathematic model with a constant
15 R_{CD} will be solved by the finite-difference method.

16 Fig. 2a shows the specific discharge distributions with different R_{CD} of 0, 0.02, 0.04, 0.08
17 and 0.50. The curve of $R_{CD} = 0$ represents the fully Darcian flow model. One can find that the

1 specific discharge decreases with increasing R_{CD} at a given r_D , starting from its maximum at
 2 $R_{CD}=0$ (Darcian flow). This observation is understandable. The increasing R_{CD} implies a
 3 stronger contribution of the inertial effect, which also means a larger resistance to flow, thus it
 4 leads to a smaller specific discharge. After trying many different sets of aquifer parameters, such
 5 as $\beta_D=0.002, 0.02, 0.2$, and $R_{CD}=0.01, 0.03, 0.1$, numerical simulation indicates that this
 6 observation is universally valid. This observation will serve as the basis for the new iteration
 7 method to seek the location of $R_{CD}(t_D)$.

8 Similar to the use of Re_C to determine the start of the non-Darcian flow, one can use q_{CD}
 9 for the initiation of the non-Darcian flow, where q_{CD} is the dimensionless critical specific
 10 discharge defined in Table 1. We denote r_{jCD} as the newly computed critical radius at the j^{th}
 11 step of the new iteration method, where $j=1,2,3,\dots$. Since the aquifer system is initially
 12 hydrostatic, the initial critical radius r_{0CD} is set to 0. For a given dimensionless time t_{1D} , the
 13 detailed procedures of the iteration method for searching $R_{CD}(t_{1D})$ will be introduced as follows.
 14 Firstly, the specific discharge distribution in the aquifer can be calculated using Eqs. (25) - (33)
 15 with $R_{CD}(t_{1D})=r_{0CD}$, as shown in Fig. 2b. Based on the computed specific discharge distribution,
 16 one can find the new critical radius r_{1CD} according to a given constant q_{CD} . Secondly, the new
 17 specific discharge distribution can be similarly calculated using Eqs. (25) - (33) with
 18 $R_{CD}(t_{1D})=r_{1CD}$, and the new critical radius r_{2CD} can be obtained according to q_{CD} . It is notable
 19 that r_{1CD} and r_{2CD} serve as the upper and lower limits for searching $R_{CD}(t_{1D})$, as illustrated in
 20 Fig. 2b. Similarly, one can estimate the new critical radius r_{3CD} using r_{2CD} , where r_{3CD} is

1 located somewhere between r_{1CD} and r_{2CD} . Following the same procedures, a new critical
 2 radius r_{4CD} can be calculated based on r_{3CD} , and r_{4CD} is between r_{2CD} and r_{3CD} . One can
 3 repeat above computations until the new critical radius finally converges. For the actual
 4 problems, we define a convergence criterion $|R_{CD}^{old} - R_{CD}^{new}| \leq \xi$, where R_{CD}^{old} and R_{CD}^{new} are the
 5 critical radius for the previous step and present step, respectively; ξ is a small positive value
 6 such as 0.001. If this criterion is satisfied, the new critical radius r_{jCD} is thought as the
 7 estimation of $R_{CD}(t_{1D})$. We develop a MATLAB program named as Two-Region Model with
 8 Moving critical radius (MTRM) to facilitate the computation. By the way, this iteration method
 9 is convergent. Fig. 3 represents the flow chart of the MTRM algorithm, where t_k is the time at
 10 time step k ; k_{max} is the total number of the time steps; dt_D is the dimensionless time step; $s_{D,i}$ and
 11 $q_{D,i}$ are the dimensionless specific drawdown and dimensionless discharge at node i in the aquifer
 12 respectively.

13 **5. Results and discussions**

14 **5.1. Comparison with the previous solutions**

15 To test the new solution, the fully Darcian flow solution of Papadopoulos and Cooper
 16 (1967), the fully non-Darcian flow solution of Mathias et al. (2008) and the two-region model of
 17 Sen (1988) will be introduced. Figs. 4a and 4b show the distance-drawdown curves of the four
 18 mentioned-above models in the early and late pumping stages, respectively. In these two figures,
 19 “Papadopoulos and Cooper (1967)” represents the analytical solution of the fully Darcian flow
 20 model, “Sen (1988)” is the analytical solution of the two-region model by the Boltzmann

1 transform method, and “Mathias et al. (2008)” represents the numerical solution of the fully
2 non-Darcian flow model. The deflection point of the curve is the location of the critical radius.

3 In the early stage, the differences among three previous solutions and the new solution of
4 this study are obvious, as shown in Fig. 4a. Firstly, the solution of Papadopoulos and Cooper
5 (1967) is smaller than the others near the well. This is because the inertial forces of the
6 non-Darcian flow increase the resistance for flow, thus resulting in drawdown greater than those
7 for the Darcian flow near the well. The second is that the F-ND solution agrees with the new
8 solution near the well. The third is that the solution of Sen (1988) does not agree with the new
9 solution near the well at the early time. This is probably because of the Boltzmann transform
10 method used by Sen (1988) to deal with the non-Darcian flow at the early time, which has been
11 discussed in several previous studies (Camacho and Vasquez, 1992; Wen et al., 2008b). The
12 fourth is that there is a deflection point on the new solution, leading to discontinuity of the
13 drawdown slope. This observation may be reasonable, as also reported by Moutsopoulos et al.
14 (2009), who named it non-uniform hydraulic behavior.

15 In the late pumping stage, the transient flow approaches the quasi-steady state, and the
16 specific discharge distribution is invariant with time according to Eqs. (3) - (4) or Eqs. (14) - (15),
17 regardless of the Darcian flow or non-Darcian flow. Under the quasi-steady state flow condition,
18 the critical radius obtained by this new solution becomes a constant which is the same as the one
19 used by previous two-region models such as Sen (1988) and Wen et al. (2009). Therefore, the
20 new solution agrees with that of Sen (1988) very well at late time (see Fig. 4b). Another fact that

1 can be seen in Fig. 4b is that the new solution agrees with the solution of Papadopoulos and
2 Cooper (1967) in the Darcian flow region.

3 **5.2 Effect of the inertial force coefficient to the critical radius**

4 The inertial force coefficient (β) is of primary concern for the non-Darcian flow described
5 by the Forchheimer equation, and the values of β_D are chosen as 0.001, 0.01, and 0.1. Fig. 5
6 shows the critical radius (R_{CD}) changes with time for different dimensionless inertial force
7 coefficients. Several observations can be seen. Firstly, R_{CD} increases with time until the flow
8 approaching the quasi-steady state condition. In the early pumping stage, the specific discharge
9 is very large near the well and decreases quickly with the distance from the well, so R_{CD} is very
10 small. With time going, the cone of depression will expand along the radial direction and the
11 slope of the cone of depression becomes flatter, so R_{CD} becomes greater. Secondly, a larger β_D
12 would reduce the rate of change R_{CD} versus time, thus result in longer time to approach its
13 asymptotic value, and consequently leads to a smaller R_{CD} at a specific time in the transient
14 state (see Fig. 5). This is because a larger β_D implies a stronger inertial force, which increases
15 the resistance of flow. The third interesting observation is that the asymptotic value of R_{CD} is
16 the same for different β_D . This can be explained using Eq. (2). Based on the definition of the
17 dimensionless parameters defined in Table 1, Eq. (2) becomes

$$18 \quad q_{CD} = 1/R_{CD}. \quad (34)$$

19 Therefore, the value of R_{CD} does not depend on β_D under the quasi-state state flow condition,
20 while it only reciprocally depends on the critical specific discharge.

1 **5.3 Effect of the critical specific discharge to the critical radius**

2 The criterion to judge the initiation of the non-Darcian flow is an important factor of
3 concern. Up to now, there is still considerable debate on what value of Re_C to use for the start
4 of non-Darcian flow. The recommended values of Re_C range from 0.1 to 100 for porous media
5 flow (Bear, 1972;Scheidegger, 1974;Zeng and Grigg, 2006). To check the influence of Re_C on
6 R_{CD} during the transient flow, the values of q_{CD} are chosen as 100, 50 and 10 considering the
7 direct relationship of q_{CD} and R_{CD} in Eq. (2). The other parameters are $\beta_D=0.01$,
8 and $r_{wD} = 1 \times 10^{-4}$.

9 Fig. 6 shows the effect of q_{CD} on R_{CD} . It is obvious that the asymptotic value of R_{CD} is
10 equal to $1/q_{CD}$, as reflected in Eq. (34). Another interesting observation is that R_{CD} decreases
11 with increasing q_{CD} , and it takes shorter time for R_{CD} to approach its asymptotic value.

12 **5.4. Type curves in the non-Darcian flow region and Darcian flow region**

13 Type curves are a series of curves that reveal the functional relationship between the well
14 functions (or drawdown) and the dimensionless time factors (Sen, 1988;Wen et al., 2011). Type
15 curve is one of the common approaches to identify the aquifer parameters or to predict the
16 drawdown(Sen, 1988;Wen et al., 2011). Sen (1988) presented different type curves in the
17 Darcian flow region and non-Darcian flow region based on a two-region model. In that model
18 (Sen, 1988), R_{CD} was a fixed value which only depends on the rate of pumping but independent
19 of time. In this study, R_{CD} changes with time, and the type curves might be different from the
20 ones generated by Sen (1988). To investigate the behaviors of the type curves of the new

1 solution, the two observation locations will be chosen, $r_D=0.005$ and 0.02 . According to Eq. (34),
2 the maximum of R_{CD} is 0.001 at the quasi-steady state, so the flow at $r_D=0.005$ will experience
3 both Darcian flow (at the early time) and non-Darcian flow (at late time), while the flow at
4 $r_D=0.02$ is always Darcian.

5 Fig. 7 shows the time-drawdown at $r_D=0.005$ for different dimensionless inertial force
6 coefficients in the log-log scale. Two interesting observations can be seen from this figure. The
7 first observation is that there is a deflection point in the curve of $\beta_D=0.1$ or 1 , that becomes
8 larger in time with increasing β_D . This is because a larger β_D implies a stronger inertial effect,
9 which leads to a larger drawdown and longer time to approach the quasi-steady state condition.
10 This observation is not found in the F-ND model (Wen et al., 2011) and in the two-region model
11 (Sen, 1988). The second observation is that the drawdown in the quasi-steady state increases
12 with increasing β_D , and the reason for this has been explained in previous studies (Wen et al.,
13 2011).

14 Fig. 8 represents the time-drawdown at $r_D=0.02$ in the log-log scale. One notable point is
15 that flow at $r_D=0.02$ is always Darcian, so there is no deflection point in the type curves. The
16 differences among the curves with different β_D are obvious at the beginning, and then they
17 approach the same value at the quasi-steady state.

18 **6. Summary and conclusions**

19 In this study, a new two-region flow model considering the time-dependent critical radius
20 (R_{CD}) is established to investigate the groundwater flow into a pumping well, and a new iteration

1 method is proposed to estimate R_{CD} , based on the finite-difference method. Results show that
2 this iteration method is convergence although it has not been analytical verified using rigorous
3 mathematic model. In the non-Darcian flow region, the flow is governed by the Forchheimer
4 equation, and the start of the non-Darcian flow is determined by the critical specific discharge,
5 which is calculated by the critical Reynolds number. The new solution is compared with
6 previous solutions, such as the fully Darcian flow model, the two-region model with a constant
7 critical radius, and the fully non-Darcian flow model. The impacts of the dimensionless inertial
8 force coefficient (β_D) and dimensionless critical specific discharge (q_{CD}) on the critical radius
9 and flow field have been analyzed. Several findings can be drawn from this study:

10 (1) In the early stage, the new solution agrees with the fully non-Darcian flow solution near
11 the well, differs with the fully Darcian flow model of Papadopoulos and Cooper (1967) and the
12 two-region model of Sen (1988).

13 (2) In the quasi-steady flow stage, the new solution agrees with the solution of Sen (1988)
14 very well. It agrees very well with the solution of the fully Darcian flow model (Papadopoulos and
15 Cooper, 1967) in the Darcian flow region.

16 (3) R_{CD} increases with time until reaching the quasi-steady state flow, and the asymptotic
17 value of R_{CD} only depends on q_{CD} . A larger β_D would reduce the rate of change of R_{CD}
18 with time, and result in a smaller R_{CD} at a specific time during the transient flow state.

19 (4) There is a deflection point in the type curve when the observation well location is within
20 the non-Darcian flow region in the quasi-steady state when $\beta_D \geq 0.1$, and the time associated

1 with this deflection point becomes larger with a larger β_D .

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11 **References**

- 12 Ahmad, N.: Evaluation of groundwater resources in the upper middle part of Chaj-Doak area,
13 Pakistan., PhD, Istanbul Technical Univ., Turkey, 1998.
- 14 Basak, P.: Non-penetrating well in a semi-infinite medium with nonlinear flow, Journal of
15 Hydrology, 33, 375-382, 10.1016/0022-1694(77)90047-6, 1977.
- 16 Basak, P.: Analytical solutions for two-regime well flow problems, Journal of Hydrology, 38,
17 147-159, 1978.
- 18 Bear, J.: Dynamics of fluids in porous media., Elsevier, New York, 1972.
- 19 Birpinar, M., and Sen, Z.: Forchheimer groundwater flow law type curves for leaky aquifers,
20 Journal of Hydrologic Engineering, 9, 51-59, doi:10.1061/(ASCE)1084-0699(2004)9:1(51),
21 2004.

1 Bordier, C., and Zimmer, D.: Drainage equations and non-Darcian modelling in coarse porous
2 media or geosynthetic materials, *Journal of Hydrology*, 228, 174-187,
3 10.1016/s0022-1694(00)00151-7, 2000.

4 Camacho, R. G., and Vasquez, M.: Analytical solution incorporating nonlinear radial flow in
5 confined aquifers - comment, *Water Resources Research*, 28, 3337-3338,
6 10.1029/92wr01646, 1992.

7 Chen, Z. X., Lyons, S. L., and Qin, G.: Derivation of the Forchheimer law via homogenization,
8 *Transport in Porous Media*, 44, 325-335, 10.1023/a:1010749114251, 2001.

9 Dudgeon, C. R., Huyakorn, P. S., and Swan, W. H. C.: Hydraulics of flow near wells in
10 unconsolidated sediments, *Field studies*, The university of new south wales, Australia, 1972a.

11 Dudgeon, C. R., Huyakorn, P. S., and Swan, W. H. C.: Hydraulics of flow near wells in
12 unconsolidated sediments, *Theoretical and experimental studies*, The university of new south
13 wales, Australia, 1972b.

14 Dullien, F. A. L., and Azzam, M. I. S.: Flow rate pressure gradient measurements in periodically
15 nonuniform capillary tubes, *Aiche Journal*, 19, 222-229, 10.1002/aic.690190204, 1973.

16 Engelund, F.: On the laminar and turbulent flows of ground water through homogeneous sand,
17 *Tech. Univ. Denmark, Copenhagen, Denmark*, 1953.

18 Ergun, S.: Fluid flow through packed columns, *Chemical engineering progress*, 48, 89-94, 1952.

19 Forchheimer, P. H.: *Wasserbewegung durch boden*, *Zeitsch-rift des Vereines Deutscher*
20 *Ingenieure*, 49, 1736-1749, 1901.

21 George, G. H., and Hansen, D.: Conversion between quadratic and power law for non-Darcy
22 Flow, *Journal of Hydraulic Engineering-Asce*, 118, 792-797,
23 10.1061/(asce)0733-9429(1992)118:5(792), 1992.

1 Giorgi, T.: Derivation of the Forchheimer law via matched asymptotic expansions, *Transport in*
2 *Porous Media*, 29, 191-206, 10.1023/a:1006533931383, 1997.

3 Hassanizadeh, S. M., and Gray, W. G.: High-velocity flow in porous-media, *Transport in Porous*
4 *Media*, 2, 521-531, 1987.

5 Huyakorn, P., and Dudgeon, C. R.: Investigation of 2-regime well flow, *Journal of the hydraulics*
6 *division-asce*, 102, 1149-1165, 1976.

7 Irmay, S.: On the theoretical derivations of Darcy and Forchheimer formulas - discussion - reply,
8 *Journal of Geophysical Research*, 64, 486-487, 10.1029/JZ064i004p00486, 1959.

9 Izbash, S. V.: *O filtracii V Kropnozernstom Materiale*, Leningrad, USSR (in Russian), 1931.

10 Liu, H. H., Li, L. C., and Birkholzer, J.: Unsaturated properties for non-Darcian water flow in
11 clay, *Journal of Hydrology*, 430, 173-178, 10.1016/j.jhydrol.2012.02.017, 2012.

12 Louwyck, A., Vandenbohede, A., and Lebbe, L.: Numerical analysis of step-drawdown tests:
13 Parameter identification and uncertainty, *Journal of Hydrology*, 380, 165-179,
14 10.1016/j.jhydrol.2009.10.034, 2010.

15 Mackie, C. D.: Determination of Nonlinear formation losses in pumping wells, *International*
16 *Conference on Groundwater and Man*, 5-9 December 1983.

17 Mathias, S. A., Butler, A. P., and Zhan, H. B.: Approximate solutions for Forchheimer flow to a
18 well, *Journal of Hydraulic Engineering-Asce*, 134, 1318-1325,
19 10.1061/(asce)0733-9429(2008)134:9(1318), 2008.

20 Mathias, S. A., and Todman, L. C.: Step-drawdown tests and the Forchheimer equation, *Water*
21 *Resources Research*, 46, W0751410.1029/2009wr008635, 2010.

22 Mishra, P. K., Vessilinov, V., and Gupta, H.: On simulation and analysis of variable-rate pumping
23 tests, *Ground Water*, 51, 469-473, 10.1111/j.1745-6584.2012.00961.x, 2012.

1 Moutsopoulos, K. N., Papaspyros, I. N. E., and Tsihrintzis, V. A.: Experimental investigation of
2 inertial flow processes in porous media, *Journal of Hydrology*, 374, 242-254,
3 <http://dx.doi.org/10.1016/j.jhydrol.2009.06.015>, 2009.

4 Papadopoulos, I. S., and Cooper, H. H.: Drawdown in a well of large diameter, *Water Resources*
5 *Research*, 3, 241-244, 10.1029/WR003i001p00241, 1967.

6 Quinn, P. M., Parker, B. L., and Cherry, J. A.: Validation of non-Darcian flow effects in slug tests
7 conducted in fractured rock boreholes, *Journal of Hydrology*, 486, 505-518,
8 10.1016/j.jhydrol.2013.02.024, 2013.

9 Scheidegger, A. E.: The physics of flow through porous media, *Soil Science*, 86, 355, 1958.

10 Scheidegger, A. E.: The physics of flow through porous media, University of Toronto Press,
11 152-170 pp., 1974.

12 Sen, Z.: Type curves for two-region well flow, *Journal of Hydraulic Engineering*, 114, 1461-1484,
13 10.1029/WR024i004p00601, 1988.

14 Sen, Z.: Nonlinear radial flow in confined aquifers toward large-diameter wells, *Water Resources*
15 *Research*, 26, 1103-1109, 10.1029/WR026i005p01103, 1990.

16 Sen, Z.: Non-Darcian groundwater flow in leaky aquifers, *Hydrological Sciences Journal-Journal*
17 *Des Sciences Hydrologiques*, 45, 595-606, 10.1080/02626660009492360, 2000.

18 Sidiropoulou, M. G., Moutsopoulos, K. N., and Tsihrintzis, V. A.: Determination of Forchheimer
19 equation coefficients a and b , *Hydrological Processes*, 21, 534-554, 10.1002/hyp.6264, 2007.

20 Soni, J. P., Islam, N., and Basak, P.: A experimental evaluation of non-Darcian flow in porous
21 media, *Journal of Hydrology*, 38, 231-241, 10.1016/0022-1694(78)90070-7, 1978.

22 van Tonder, G. J., Botha, J. F., and van Bosch, J.: A generalised solution for step-drawdown tests
23 including flow dimension and elasticity, *Water S.A.*, 27, 345-354, 2001.

- 1 Ward, J. C.: Turbulent flow in porous media, Journal of Hydraulics Division, American Society
2 of Civil Engineers, 90, 1-12, 1964.
- 3 Wen, Z., Huang, G. H., and Zhan, H. B.: An analytical solution for non-Darcian flow in a
4 confined aquifer using the power law function, Advances in Water Resources, 31, 44-55,
5 10.1016/j.advwatres.2007.06.002, 2008a.
- 6 Wen, Z., Huang, G. H., Zhan, H. B., and Li, J.: Two-region non-Darcian flow toward a well in a
7 confined aquifer, Advances in Water Resources, 31, 818-827,
8 10.1016/j.advwatres.2008.01.014, 2008b.
- 9 Wen, Z., Huang, G. H., and Zhan, H. B.: A numerical solution for non-Darcian flow to a well in a
10 confined aquifer using the power law function, Journal of Hydrology, 364, 99-106,
11 10.1016/j.jhydrol.2008.10.009, 2009.
- 12 Wen, Z., Huang, G. H., and Zhan, H. B.: Non-Darcian flow to a well in a leaky aquifer using the
13 Forchheimer equation, Hydrogeology Journal, 19, 563-572, 10.1007/s10040-011-0709-2,
14 2011.
- 15 Whitaker, S.: The Forchheimer equation: A theoretical development, Transport in Porous Media,
16 25, 27-61, 10.1007/bf00141261, 1996.
- 17 Yeh, H.-D., and Chang, Y.-C.: Recent advances in modeling of well hydraulics, Advances in
18 Water Resources, 51, 27-51, 10.1016/j.advwatres.2012.03.006, 2013.
- 19 Zeng, Z. W., and Grigg, R.: A criterion for non-Darcy flow in porous media, Transport in Porous
20 Media, 63, 57-69, 10.1007/s11242-005-2720-3, 2006.

21

22

1 **Table 1 Dimensionless variables used in this study**

$r_D = \frac{r}{B}$	$r_{wD} = \frac{r_w}{B}$
$R_{CD} = \frac{R_C}{B}$	$t_D = \frac{K_\beta t}{SB}$
$\beta_D = \frac{Q\beta}{2\pi B^2}$	$\lambda = \frac{K}{K_\beta}$
$s_{wD} = \frac{2\pi K_\beta B}{Q} s_w$	$s_{yD} = \frac{2\pi K_\beta B}{Q} s_Y(r, t)$
$s_{ND} = \frac{2\pi K_\beta B}{Q} s_N(r, t)$	$q_{ND} = -\frac{2\pi B^2}{Q} q_N(r, t)$
$q_{yD} = -\frac{2\pi B^2}{Q} q_Y(r, t)$	$q_{CD} = -\frac{2\pi B^2}{Q} q_C$

2

1 **Figure captions**

2 Fig. 1. The schematic diagram of the non-Darcian flow into a fully penetrating pumping well
3 considering the time-dependent critical radius.

4 Fig. 2a. Specific discharge distributions with different critical radius R_{CD} .

5 Fig. 2b. The schematic diagram showing the iterative process of seeking R_{CD} .

6 Fig. 3. Flow chat of the MTRM algorithm.

7 Fig. 4a. Comparison of the distance-drawdowns by the fully Darcian flow model (Papadopoulos
8 and Cooper, 1967), the fully non-Darcian flow model (Mathias et al., 2008), the two-region
9 flow model (Sen, 1988), and the new model in early pumping stage.

10 Fig. 4b. Comparison of the distance-drawdowns by the fully Darcian flow model (Papadopoulos
11 and Cooper, 1967), the fully non-Darcian flow model (Mathias et al., 2008), the two-region
12 flow model (Sen, 1988), and the new model in late pumping stage.

13 Fig. 5. Time-dependent critical radius (R_{CD}) for different inertial force coefficients.

14 Fig. 6. Time-dependent critical radius (R_{CD}) for different critical specific discharges.

15 Fig. 7. Time-drawdown at $r_D=0.005$ for different inertial force coefficients in a log-log scale.

16 Fig. 8. Time-drawdown at $r_D=0.02$ for different inertial force coefficients in a log-log scale.