Reply to Pierluigi Furcolo's comments

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We thank very much Pierluigi Furcolo for his kind and helpful review, which can be summarized in two main comments that we address below.

1. *The paper correctly shows the evaluation of uncertainty of sample moments at different orders and at different scales. It is evident how uncertainty and bias can significantly alter the numerical results. Nevertheless, in the multifractal analysis, what actually is investigated is the scaling of the moments, which is related to the scaling of the partition coefficients or, equivalently, to ratios of moments at different scales. My feeling is that correlation among sample moments at different scales can play a "positive" role, in this case, allowing for an estimation of the scaling exponent which is less uncertain than the values of the moments themselves. More generally, an assessment of how uncertainty of the moments affects uncertainty of the scaling exponents may be included in the paper (even just by numerical investigation on the generated samples)*;

Our work is mainly aimed to explore the information content in estimates of raw moments which are used in multifractal analyses of hydrological processes. Therefore, studying the scaling of raw moments is relevant but not really crucial to the focus of our manuscript. However, since this study is the ultimate aim of a multifractal analysis, we agree with the Reviewer's suggestion to carry out numerical investigations on the generated samples by simply taking an average slope of linear regressions of sample moments at different scales Δ in log-log diagrams (actually, this is commonly the case when dealing with real world data). This analysis will be included in the revised version of the manuscript (Lombardo et al., 2013) with the objective of exploring the variability of the moment scaling function $K(q)$ when increasing q.

As mentioned by the Reviewer, multifractal analyses are often carried out using non-dimensional quantities (e.g. de Lima and Grasman, 1999; Serinaldi, 2010). We define the scale ratio λ so that λ=1 for the largest scale of interest $\Delta_{\max}$, i.e. $\lambda=\Delta_{\max}/\Delta$. In our case, we assume that $\Delta_{\max}=\lfloor n/8 \rfloor=128$ where the sample size n=1024, so that sample moments can be estimated from at least 8 data values, while the generic aggregated scale Δ is bounded in [1, 128]. Similarly, we form the non-dimensional process $\underline{\varepsilon}(\lambda)$ dividing the local average of the continuous-time process $\underline{x}(t)$ by its mean at the largest scale $\Delta_{\max}$ (or equivalently λ=1); then:

$$\underline{\varepsilon}(\lambda)=\frac{\underline{x}_j^{\left(\frac{\Delta_{\max}}{\lambda}\right)}}{\mathrm{E}\left[\underline{x}_j^{(\Delta_{\max})}\right]}\approx\frac{\underline{x}_j^{\left(\frac{\Delta_{\max}}{\lambda}\right)}}{m};\quad \lambda=\frac{\Delta_{\max}}{\Delta} \tag{1}$$

where m is the temporal mean of the data series.

The scaling behaviour of the process is characterized as follows:

$$\mathrm{E}\left[(\underline{\varepsilon}(\lambda))^q\right]\approx\lambda^{K(q)} \tag{2}$$

The scaling of the moments of the process is described by the moment scaling function $K(q)$. If $K(q)$ linearly increases with q, then the process is said to be "simple scaling", otherwise it exhibits a "multiple scaling" behaviour.

In Fig. 1, we graphically show how uncertainty in sample moments is reflected in the uncertainty in the scaling exponents estimates (determined as explained in Sect. 3 by Lombardo et al. (2013)).

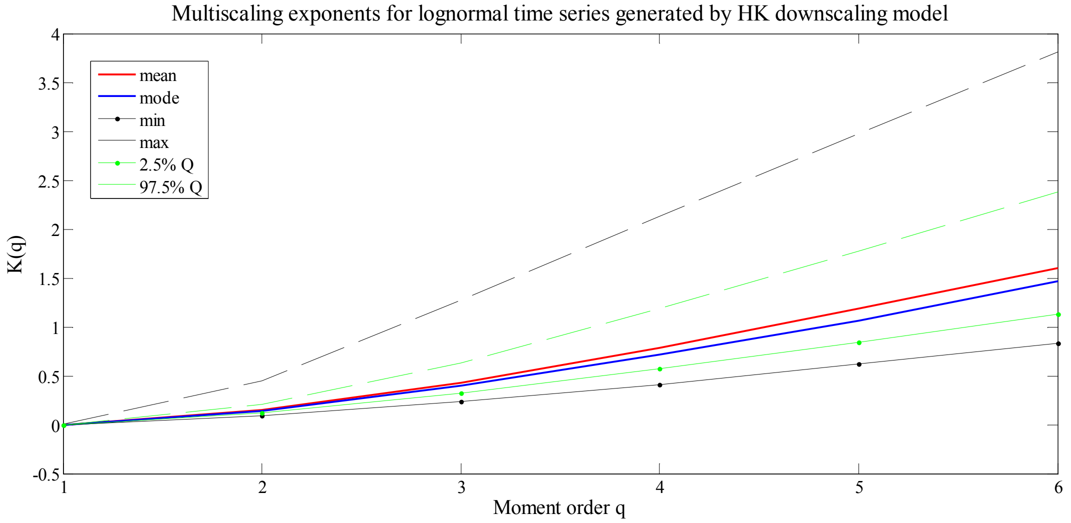

Figure 1. Prediction intervals of the moment scaling function $K(q)$ versus the order q for the lognormal series generated by our downscaling model (Lombardo et al., 2012).

It can be noticed that the function $K(q)$ shows a nonlinear behaviour for the lognormal series generated by our model, thus suggesting a multifractal behaviour. Furthermore, the prediction intervals spread out widely while increasing the moment order q, which is consistent with an enhancement of uncertainty. Analogous considerations apply to the series generated by the other Monte Carlo experiments shown in our manuscript (not reported here).

Furthermore, note that we used the ratios of moment estimates in all calculations to compute $\underline{\varepsilon}(\lambda)$. Nonetheless, recalling that we assumed unit ensemble mean μ=E[$\underline{x}(t)$]=1 in all our Monte Carlo experiments, we found (not shown here) the same results if using raw moments without taking any ratios. This is to stress that ratios of moments do not seem to play any significant role in the estimation of multiscaling exponents in our case.

To conclude the reply to Reviewer's comment, we would like to put the emphasis on some important theoretical considerations. The theory of multifractals depends on the fact that raw moments obey power laws as the scale $\Delta \to 0$ (or equivalently $\lambda \to \infty$) (Falconer, 1990; Gneiting and Schlather, 2004), and so it depends on taking limits which cannot be achieved in reality. For most experimental purposes, the multifractal behaviour of a process $\underline{x}(t)$ is usually found by estimating the gradient of a graph of $\log(\mathrm{E}[(\underline{\varepsilon}(\lambda))^q])$ against $\log\lambda$ over an "appropriate" range of scales, where empirical points are closely matched by a straight line of slope $K(q)$. Being the latter an asymptotic slope, it is difficult to find the "appropriate" range of scales to estimate $K(q)$, because we could be misled by some artificial slopes which do not indicate the multifractal behaviour of the underlying process (see e.g. Koutsoyiannis, 2013). In addition, we should emphasize that the empirical moment scaling function $K(q)$ varies across scales for ergodic processes. The simple proof for this is given below in the special case of q=2.

According to eqs. (1) and (2) we could write:

$$\mathrm{E}\left[\left(\underline{x}_j^{\left(\frac{\Delta_{\max}}{\lambda}\right)}\right)^2\right] \approx \lambda^{K(2)}\left(\mathrm{E}\left[\underline{x}_j^{(\Delta_{\max})}\right]\right)^2 = \lambda^{K(2)}\mu^2 \quad (3)$$

where μ is the mean of the process.

On the other hand, we know that:

$$\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^2\right] = \gamma(\Delta) + \mu^2 \quad (4)$$

where $\gamma(\Delta)$ is the variance of the local average process at the scale Δ. If we assume that the process is ergodic, then we must have $\gamma(\Delta) \to 0$ as $\Delta \to \infty$ (Papoulis, 1991, p. 430).

Recalling that $\Delta = \Delta_{\max}/\lambda$, from eqs. (3) and (4) we have:

$$\lambda^{K(2)}\mu^2 = \gamma\left(\frac{\Delta_{\max}}{\lambda}\right) + \mu^2 \quad (5)$$

dividing both sides by μ^2 and taking the logarithms, we obtain:

$$K(2) = \frac{\log\left(\gamma\left(\frac{\Delta_{\max}}{\lambda}\right)\Big/\mu^2 + 1\right)}{\log\lambda} \quad (6)$$

Clearly then, as $\lambda \to 0$ (i.e., as the scale grows to infinity $\Delta \to \infty$), the numerator $\to$ 0 and the denominator $\to \infty$. So, $K(2)=0$ asymptotically. Note that we have not made any assumption about the dependence structure or the marginal probability of the process, the only assumption is that the process is ergodic. In summary, for scales tending to infinity the $K(q)$ should tend to zero, while for scales tending to zero the $K(q)$ will take nonzero values.

All the above considerations apply to our analysis shown in Fig. 1.

In the model by Lombardo et al. (2012) the theoretical moment scaling function is given by:

$$K_{\mathrm{Th}}(q) = q(q-1)(1-H) \tag{7}$$

where H is the Hurst coefficient.

Before discussing the results in Fig. 1, let us prove the eq. (7).

It can be shown that if the local average $\underline{x}_j^{(\Delta)}$ is lognormally distributed, its q-order raw moment is given by (Kottegoda and Rosso, 2008, p. 216):

$$\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^q\right] = \exp\left(q\mu_{\ln\left(\underline{x}_j^{(\Delta)}\right)} + \frac{1}{2}q^2\gamma_{\ln\left(\underline{x}_j^{(\Delta)}\right)}\right) \tag{8}$$

where the two parameters can be determined in terms of the mean μ=E[$\underline{x}_j^{(\Delta)}$] and the variance $\gamma(\Delta)$=Var[($\underline{x}_j^{(\Delta)}$)] of the local average process as follows:

$$\mu_{\ln\left(\underline{x}_j^{(\Delta)}\right)} = \log\mu - \frac{1}{2}\log\left(\frac{\gamma(\Delta)}{\mu^2} + 1\right) \tag{9}$$

$$\gamma_{\ln\left(\underline{x}_j^{(\Delta)}\right)} = \log\left(\frac{\gamma(\Delta)}{\mu^2} + 1\right) \tag{10}$$

In the downscaling model by Lombardo et al. (2012), the function $\gamma(\Delta)$ obeys the following power law:

$$\gamma(\Delta) = \gamma\Delta^{2H-2} \tag{11}$$

where $\gamma \equiv \gamma(\Delta=1)$ is the variance of the reference local average process $\underline{x}_j^{(\Delta=1)}$.

In order to derive the theoretical moment scaling function $K_{\mathrm{Th}}(q)$, we should investigate the following limiting behaviour (Falconer, 1990, p. 257):

$$K_{\mathrm{Th}}(q) = \lim_{\Delta\to 0}\frac{\log\left(\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^q\right]\right)}{-\log\Delta} \tag{12}$$

where, according to eq. (8), the numerator of the right-hand side can be written as:

$$\log\left(\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^q\right]\right) = q\mu_{\ln\left(\underline{x}_j^{(\Delta)}\right)} + \frac{1}{2}q^2\gamma_{\ln\left(\underline{x}_j^{(\Delta)}\right)} \tag{13}$$

Substituting eqs. (9) and (10) in the right-hand side of eq. (13), we obtain:

$$\log\left(\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^q\right]\right) = q\log\mu + \frac{q}{2}(q-1)\log\left(\frac{\gamma(\Delta)}{\mu^2} + 1\right) \tag{14}$$

From eq. (11) and using the properties of the logarithm, the eq. (14) becomes

$$\log\left(\mathrm{E}\left[\left(\underline{x}_j^{(\Delta)}\right)^q\right]\right) = \log\left(\mu^q\left(\frac{\gamma}{\mu^2}\Delta^{2H-2} + 1\right)^{\frac{q}{2}(q-1)}\right) \tag{15}$$

Recall that the Hurst coefficient is a parameter satisfying $0<H<1$ (Mandelbrot and Van Ness, 1968), then the exponent $2H-2<0$.

Substituting eq.(15) in eq. (12), we easily obtain eq. (7). Based on the above theoretical findings, the empirical results in Fig. 1 do not seem to agree well with their theoretical counterparts. For example, in our case H=0.85, for q=4 the theoretical value should be $K_{\mathrm{Th}}(q)$=1.8, while the estimated mean value is about $K(q)$=0.5 in the scale range of our Monte Carlo experiments. Hence, not finding the "appropriate" range of scales, in addition to estimation problems reported in Lombardo et al. (2013), may lead to remarkable underestimation of the moment scaling function.

Therefore, we can conclude that even if the generated process is multifractal, the sample estimates of the q-moments from a unique sample can provide misleading results.

2. *It is not clear from the text if the generated data correspond to a "bare" realization of the multifractal process or to a "dressed" one. The algebraic behaviour of the tail of the marginal distribution of the process at a given scale (and the divergence of the moments) generally depends on the dressing and can stem also from thin tailed generators. Furthermore, when analyzing synthetic data at different scales, starting from a bare realization at the smallest scale produces some bias in the scaling of the moments because of the different amount of dressing that arises at the different scales. This point is not really crucial to the focus of the paper, yet it is quite relevant in the multifractal modelling. This reviewer therefore suggests to add a better explanation of how the data are generated and (eventually) dressed and the possible implications on the numerical (as well as theoretical, if available) results*;

The synthetic series generated by our downscaling model (Lombardo et al., 2012) are "bare" quantities, in the sense that the cascade is developed from a large scale to our scale of interest. We agree with the Reviewer that the "dressing" (averaging) procedure could have in general an impact on the sample moment estimation. However, the issue of investigating the different impact of bare and dressed quantities on the scaling of the moments is out of the scope of the paper under review (Lombardo et al., 2013), but it could be really worth investigating further in future works.

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