

## ***Interactive comment on “Bringing it all together” by J. C. I. Dooge***

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### **HYSTERESIS, THE MISSING NON-LINEARITY**

Dooge (2004) in “Bringing it all together” concludes by posing “interesting problems deserving of attention by the present generation of young [and not so young] hydrologists”. He states that “the most useful strategy to follow is that based on the rigorous analysis of simplified equations of motion” and quotes with approval Pedloski (1979) on geophysical fluid dynamics: “The approximations are necessary to make any progress possible, while precision in analysis is demanded to make the progress meaningful”. The paper exemplifies this strategy by removing the “concentrated non-linearities” at the soil surface and by applying to what is left, the techniques of linearising mathematics and scaling. This has been, and continues to be, a very fruitful approach. However, there remains a significant hydrological non-linearity not yet incorporated in the strategy: the rate-independent hysteresis of unsaturated soil water. The purpose of this

discussion is to highlight its importance and its difficulty; and to draw attention to the progress recently made by the discussor and his co-workers in the application of novel mathematical concepts and tools for treating this kind of hysteresis.

Rate-dependent hysteresis has been known in hydrology for a long time, for example, in the looped-rating curve, and in similar plots of state and rate variables from systems of linear and non-linear differential equations that model hydrological processes. These rate-dependent hysteretic loops do not show *affine similarity with respect to time*, i.e. when the time argument of an input function is stretched, or speeded up, the corresponding output function is *not* stretched in the same way. For example, the attenuation of a sinusoidal input to a linear system depends on its frequency. Stretching the frequency changes the attenuation. Hence, any looped plot is rate dependent. In contrast to this, rate-independent hysteretic loops exhibit *affine similarity with respect to time*, i.e. when the time argument of an input function is stretched, the corresponding output function is stretched in the same way. No smooth differential or integro-differential operator has this property. Consequently, this kind of hysteresis cannot be represented by a Volterra series expansion. New mathematical concepts and analytical and numerical tools for treating hysteresis have recently been developed. This discussion outlines their application to hydrology. Two results are presented: a conceptual model for the hysteretic soil-moisture characteristic and a hysteretic linear reservoir. Both are based on the Preisach model.

## Hysteresis and the Preisach Model

Hysteresis is a collective name for strongly nonlinear, rate-independent, phenomena. The modern mathematical concept of hysteresis was suggested by M. Krasnosel'skii and his co-workers (Krasnosel'skii & Pokrovskii, 1989). See also Visintin (1994), Brokate & Sprekels (1996), and Krejci (1996).

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## The Hysteron

The simplest hysteretic system is the hysteron shown in Figure 2. It is denoted by the operator  $R_{\alpha,\beta}$ . The input to  $R$ ,  $x(t)$ , is compared to two threshold values  $\alpha < \beta$ . The output from  $R$ ,  $y(t)$ , can take one of two values, 0 or 1, depending on the history of  $x(t)$  as it crosses the thresholds; at any moment in time the hysteron is either switched "on" or "off". In Figure 2, the bold lines represent the set of possible input-output pairs. They overlap in the interval  $\alpha < x < \beta$ . If the input is increasing and crosses the lower threshold, the output remains at 0. If the input increases further and crosses the upper threshold, the output jumps from 0 to 1. If the input then decreases, the jump back to 0 occurs when the input crosses the lower threshold, completing an anti-clockwise hysteretic loop that is rate-independent.

## The Preisach Model

The Preisach Model was originally conceived to describe ferromagnetism (Preisach, 1935). It was subsequently found to be an excellent model for other phenomena (often due to the fundamental Mayergoz Identification Theorem, Mayergoz, 1991). The basis of the model is the following:

Suppose we have a finite set of hysterons  $R^j = R_{\alpha_j, \beta_j}$ ,  $1 \leq j \leq N$ . We can consider a parallel connection of these hysterons  $R^j$  with the weights  $\mu_j = \mu(j) > 0$ . The output is

$$y(t) = y[t_0, \eta_0] = \sum_{j=1}^N \mu_j R^j[t_0, \eta_0(j)] x(t), \quad t > t_0.$$

where  $\eta_0$  is the initial state of each hysteron. This idea can be represented graphically as shown in Figure 3.

Hystérons can also be connected in series. The final outputs from  $n$  hystérons in series or in parallel are fundamentally different, unlike linear sub-systems. Given  $n$  linear sub-systems in series (or in parallel), the Equivalence Theorem (Dooge and O’Kane, 2003, pages 11-13) shows how to find the equivalent sub-systems in parallel (or in series).

The standard way to represent the Preisach model is with the Preisach  $(\alpha, \beta)$  half-plane, where  $\beta \geq \alpha$ . Each point on the plane represents a hysteron. The coloured regions in Figure 4 represent hystérons that are “switched on”. In the continuous case the parallel summation of the hystérons can be taken as the integral over the region where the hystérons are “switched on” with some probability density. The figures below illustrate the Preisach Model. Here the input  $x(t)$  moves along the x-axis, and controls the point on the diagonal above itself. The output  $y(t)$  is the “area” of the coloured domain with respect to some density.

The region that is integrated depends on the past history of the system. This history is recorded by a series of horizontal and vertical lines forming a staircase shape that can be seen in Figure 4. The evolution of the “staircase” is as follows: When an input  $x(t)$  moves along the horizontal axis, it controls a point on the diagonal  $\beta = \alpha$  above itself. When moving the point towards the upper right corner in Figure 4a, this point on the diagonal drags the *horizontal line* upwards, and colours in the domain between this horizontal line and the diagonal (this yellow triangle will be added to the region that is to be integrated). When moving the point towards the bottom left corner in Figure 4b, the diagonal point drags the *vertical line* to the left and decolours everything to the right of this line and above the diagonal (in moving from  $u$  to  $v$  the yellow region will be cleared and hence removed from the region that is to be integrated). The output  $y(t)$  is the integral of the region defined by the blue domain with respect to some density. For more details see the following webpage: <http://phys.ucc.ie/~oll/hysteresis/node17.htm>.

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## The Soil-Moisture characteristic

It has been known for seventy years in soil physics that the soil-water characteristic usually exhibits hysteresis (Haines, 1930; Poulouvassilis and Kargas, 1950; Miller and Miller, 1956; Poulouvassilis, 1962; Philip, 1964; Mualem, 1974; Parlange, 1976; Bear, 1988; Dullien, 1992; Lehmann et al., 1998; Stauffer and Kinzelbach, 2001; Haverkamp et al., 2002; Hillel, 2004).

Haverkamp et al. (2002) used similarity-scaling of soil-moisture characteristic curves, arranged in scanning order, together with an algorithm that stores and updates the hysteretic state of soil water. They successfully applied it to the Grenoble GRIZZLY database assembled for this purpose<sup>1</sup>. This is the largest database of hysteretic soil-water characteristic curves in the world.

Recently, O'Kane, Pokrovskii et al. (2003) successfully modelled the soil-water characteristic using a Preisach operator in the case of all the soils in the GRIZZLY database. The identification problem for the Preisach operator on its own has been solved successfully using a Preisach density (nick-named the Wedge density, see Figures 6 and 7), with a very small number of parameters. A single parameter in that model surpassed all previous models containing as many parameters as all the characteristic scanning curves determined in the laboratory experiments. This is a significant result and justifies the use of the Preisach Operator (see Figure 7 for the results for one of the fittings). For more details see Flynn, (2004a, b), Flynn et al. (2003, 2004), Flynn & O'Kane (2004), Flynn, D., McNamara, H., O'Kane, J.P., Pokrovskii, A. (2003-5).

The extension of this work to the more difficult problem of identifying the Preisach operator in the presence of a differential equation describing the conservation of water mass is the next step.

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<sup>1</sup>Prof. Randel Haverkamp kindly supplied us with data from his GRIZZLY database.

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## The hysteretic linear reservoir

The linear reservoir is defined by a conservation law

$$\frac{ds}{dt} = x(t) - y(t),$$

$$s(0) = s_0$$

where  $s(t)$  is the instantaneous mass in storage at time  $t$ . It responds to the difference in the rates of inflow  $x(t)$  and outflow  $y(t)$  starting from a given initial condition. A second principle is required to close the dynamics of this process. The simplest case is a linear dynamic equation relating the outflow rate to the instantaneous mass in storage

$$y(t) = \frac{s(t)}{k}, \quad k > 0.$$

This model can be read as a single negative feedback-loop where  $s(t)$  is continuously adjusted to bring the output closer to the input. The speed and completeness of the adjustment is controlled by  $k$ . We can introduce hysteresis by making  $k$  a hysteron (O’Kane, 2005). This can be represented in a feedback diagram, see Figure 8.

In the interval  $(\alpha, \beta)$  on the  $s$  axis, the output from the hysteron is either of two possible values of  $k$  depending on the history of  $s(t)$ . As  $s(t)$  crosses the  $\alpha$  or  $\beta$  thresholds,  $k$  may jump from one value to the other, generating a hysteretic loop on a cone through the origin in the  $(y, s)$  plane. The slopes of the upper and lower lines of the cone are equal to the reciprocal values of the  $k$ s. A single hysteretic loop centred on the point  $(1, 1)$  in the  $(y, s)$  plane is presented in Figure 9. The corresponding periodic variation in  $x(t)$ ,  $s(t)$  and  $y(t)$  is shown in Figures 10 and 11 for the cases of clockwise and anticlockwise looping. A positive sinusoid of amplitude 2 and average value 1 drives the linear reservoir. The hysteretic jumps in  $y(t)$  are produced by a 20% deviation about the reference unit values of all parameters. Numerical integration using the simple

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Euler scheme was used. The next step is the introduction of a parallel collection of hysterons that can be represented as points on the Preisach two-dimensional positive orthant

$$\Pi = (\alpha, \beta) : \beta > \alpha > 0$$

analogous to the classical Preisach model described already. If these hysterons are weighted with an integrable positive density  $p(\alpha, \beta)$  their combined contribution to  $k$  is

$$R_p(t) = \int_{0 \leq \alpha \leq \beta} p(\alpha, \beta) R_{\alpha, \beta}[t_0, \eta(\alpha, \beta)] s(t) \alpha d\beta$$

The integration makes the output  $y(t)$  continuous. Conceptual models with a small number of parameters and consisting of hysteretic reservoirs in series and parallel can now be explored.

For the figures please see <http://www.copernicus.org/EGU/hess/hessd/1/41/figures.zip>.

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