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Flexible forecast value metric suitable for a wide range of decisions: application using probabilistic subseasonal streamflow forecasts

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Supplement

- Derivation of relative economic value (REV)
- Derivation of equivalence of relative economic value (REV) and relative utility value (RUV)
- Additional economic metrics used to quantify forecast value

1 Derivation of relative economic value (REV)

This section derives the standard relative economic value (REV) metric using information in the following contingency table (Richardson, 2000).

Table S1: Contingency table for the cost-loss decision problem with expenses from each possible combination of action and occurrence. Here C is the cost of the mitigating action, L_u is the unavoidable portion of loss L from the event occurring, and L_a is avoidable portion of loss from the action.

	Event occurred	Event did not occur
Action taken	Hit rate (h) $C+L_u$	False alarm rate (f) C
Action not taken	Miss rate (m) $L=L_a+L_u$	Quiets/correct rejection rate (q) 0

The expected long run expense for decisions based on forecast information depends on the rate at which each combination of action and occurrence occurred over a historical period. The net expense of that combination:

$$\text{Average net expense} = \sum_{\text{outcomes}} \text{rate of combination} \times \text{net expense of combination} \tag{S1}$$

This can be quantified by substituting the terms from Table S1 into Eq. (S1) and noting that the rate and net expense of some elements will be zero. For forecast information this leads to

$$E_{\text{forecast}} = h\cdot\left(C+L_u\right)+f\cdot C+m\cdot\left(L_a+L_u\right) \tag{S2}$$

A user with access solely to climatological historical average information will take protective action either always or never. It is assumed that the user will choose the action that leads to the smallest net expense according to the climatological frequency $\overline{o}=h+m$ of the event.

$$\begin{aligned} E_{\text{climate}} &= \min\left(\overline{o}\left(L_a+L_u\right), C+\overline{o}L_u\right) \\ &= \overline{o}L_u+\min\left(\overline{o}L_a, C\right) \end{aligned} \tag{S3}$$

A user with access solely to climatological information will act when $C/L_a<\overline{o}$. That is, a user will either always act or never act depending on whether their particular C/L_a is smaller or larger than the event frequency.

If perfect information is available, then the combination will always be either a hit or correct rejection:

$$E_{\text{perfect}} = \overline{o}\left(C+L_u\right) \tag{S4}$$

The Relative Economic Value (REV) metric can be constructed comparing the relative difference between the forecast and perfect information relative to the climatological baseline.

$$\text{REV} = \frac{E_{\text{climate}} - E_{\text{forecast}}}{E_{\text{climate}} - E_{\text{perfect}}} \tag{S5}$$

Noting the climatological frequency of the event, we can rewrite Eq. (S2).

$$E_{\text{forecast}} = \bar{o}L_u + \left(h + f\right)C + mL_a \tag{S6}$$

Substituting Eq. (S3), (S4) and (S6) into Eq. (S5).

$$\text{REV} = \frac{\min\left(\bar{o}L_a, C\right) - \left(h + f\right)C - mL_a}{\min\left(\bar{o}L_a, C\right) - \bar{o}C} \tag{S7}$$

Noting the following parameter which is known as the cost-loss ratio.

$$\alpha = \frac{C}{L_a} \tag{S8}$$

Substituting the cost-loss ratio into Eq. (S7) and dividing by L_a leads to the following standard equation for REV (Zhu et al., 2002).

$$\text{REV} = \frac{\min\left(\bar{o}, \alpha\right) - \left(h + f\right)\alpha - m}{\min\left(\bar{o}, \alpha\right) - \bar{o}\alpha} \tag{S9}$$

2 Derivation of equivalence of relative economic value (REV) and relative utility value (RUV)

This section demonstrates the equivalence of the REV metric as detailed in Eq. (S9) and the RUV metric when 5 assumptions are applied to the decision context. We first define the general decision context using expected utility theory, the RUV metric, and then introduce the specific assumptions required to constrain RUV to REV and prove equivalence. Thereby demonstrating that the REV metric can be considered a special case of the more general RUV metric.

2.1 Expected utility framework with cost-loss economic model

The von Neumann-Morgenstern expected utility for a single timestep over M states.

$$U_t\left(E_t\right)=\sum_{m=1}^{M} p_{t,m}\mu\left(E_t\left(m\right)\right) \quad \text{(S10)}$$

where $p_{t,m}$ is the probability of state m occurring in timestep t and $E_t\left(m\right)$ is the outcome associated with that state. The outcome is typically but not necessarily in monetary units.

For a state of the world m at a specific timestep t, with damages $d_t\left(m\right)$, cost to mitigate the damages C_t, and amount of damages avoided $b_t\left(m\right)$, the outcome is given by

$$E_t\left(m\right)=b_t\left(m\right)-d_t\left(m\right)-C_t \quad \text{(S11)}$$

The benefit function $b_t\left(m\right)$ specifies the damages avoided from taking action to mitigate them,

$$b_t\left(m\right)=\min\left(\beta\cdot C_t, d_t\left(m\right)\right) \quad \text{(S12)}$$

The optimal amount $\overline{C}_t$ to spend at timestep t can be found by maximising the expected utility

$$\overline{C}_t=\underset{C_t}{\operatorname{argmax}}\ U_t\left(E_t\right) \quad \text{(S13)}$$

This leads to the following expression for the ex post utility after substitutions Eq. (S11), (S12), and (S13) into Eq. (S10)

$$\Upsilon\left(E_t\right)=\mu\left(\min\left(\beta\cdot\overline{C}_t, d_t\left(\overline{m}_t\right)\right)-d_t\left(\overline{m}_t\right)-\overline{C}_t\right) \quad \text{(S14)}$$

where $\Upsilon\left(E_t\right)$ is the ex post utility, $\overline{C}_t$ is the spend amount that was found ex ante, $\overline{m}_t$ is the state of the world associated with the observed flow at timestep t.

Note that a specific damage function $d_t\left(m\right)$ can be used for each individual timestep, however using a common damage function $d\left(m\right)$ for all timesteps is both more practically feasible and likely representative of damages for most case studies. A single damage function was used for the sensitivity analysis, and incidentally was a required assumption for equivalence between REV and RUV.

2.2 Relative utility value (RUV)

RUV is defined as follows

$$\mathrm{RUV} = \frac{\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^r\right)\right] - \underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^f\right)\right]}{\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^r\right)\right] - \underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^p\right)\right]} \tag{S15}$$

where $\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^{\bullet}\right)\right]$ is the expected value of the ex post expected utility from Eq. (S14) over a set of observations and either forecast (f), reference climatology (r), or perfect information (p).

2.3 Equivalence of REV and RUV with specific assumptions

In a cost-loss decision problem the two relevant states are "flow above" and "flow below" a decision threshold Q_d .

$$\begin{aligned} m &= \text{above} \qquad && \text{if } Q_t \geq Q_d \\ m &= \text{below} \qquad && \text{if } \mathrm{Q}_t < Q_d \end{aligned} \tag{S16}$$

Assumption 1: A step damage function with binary values of 0 and L is used to specify the losses above and below the decision threshold for all timesteps,

$$d(m;L) = \begin{cases} L & \text{when } m = \text{above} \\ 0 & \text{when } m = \text{below} \end{cases} \tag{S17}$$

To calculate the net outcome when action is taken to mitigate the loss, we substitute Eq. (S12) and (S17) into Eq. (S11) and make a change of notation, which leads to the following net outcomes for the states above and below.

$$\begin{aligned} E_{t,above} &= \min\left(\beta \cdot C_t, L\right) - L - C_t \\ E_{t,below} &= -C_t \qquad \text{since } \beta \cdot C_t > 0 \end{aligned} \tag{S18}$$

Assumption 2: Linear utility function is assumed which implies no aversion to risk,

$$\mu\left(E\right) = E \tag{S19}$$

Substituting Eq. (S18) into Eq. (S10), applying the linear utility function assumption in Eq. (S19), and simplifying for only two states of the world using p_t , the forecast probability of flow above the flow threshold for timestep t , leads to.

$$\begin{aligned} U(E_t) &= p_t \cdot E_{t,above} + (1-p_t)\cdot E_{t,below} \\ &= p_t \cdot \left[\min\left(\beta \cdot C_t, L\right) - L - C_t\right] + (1-p_t)\cdot\left[-C_t\right] \end{aligned} \tag{S20}$$

Assumption 3: Probability of flow above the threshold will always be either 1 or 0,

$$p_t \in \left\{0,1\right\} \tag{S21}$$

Assumption 3 is required because the core REV formulation is only valid for categorical forecasts. When REV is used with probabilistic forecasts they are quantified by converting them to categorical forecasts using the threshold-approach with a threshold which maximises REV for each value of α. We can now determine the single timestep ex ante utility for the four possible outcomes; forecast probability is 1 or 0, and an action has been taken or not. The derivation for the case where $p_t = 1$ and $C_t \neq 0$ is as follows. Consider the avoided losses to be.

$$L_t^a = \min\left(\beta \cdot C_t, L\right) \tag{S22}$$

Substituting this into Eq. (S20) and noting that the total loses at each timestep are fixed and consist of avoided and un-avoided components $L = L_t = L_t^a + L_t^u$

$$\begin{aligned} U\left(E_t\right) &= L_t^a - L - C_t \\ &= -\left(L - L_t^a\right) - C_t \\ &= -\left(C_t + L_t^u\right) \end{aligned} \tag{S23}$$

Simplifying Eq. (S20) for the other 3 outcomes and presenting all in a tabular form leads to the following table of ex ante utility value.

Table S2: Ex ante utility values for a time-step of Expected Utility Theory with REV assumptions

	Event is forecast to occur	Event is not forecast to occur
Action taken $C_t \neq 0$	$-\left(C_t + L_t^u\right)$	$-C_t$
Action not taken $C_t = 0$	$-L$	0

Note that the elements of Table S2 are identical but negative to Table S1 used in the derivation of the REV. Applying Eq. (S13) to Eq. (S20) will lead to an optimal amount $\overline{C}_t$ to spend on the mitigating action for each timestep. By considering that the probability is always either 1 or 0 due to assumption 3 and that all costs and losses are positive values we can formulate $\overline{C}_t$ as follows.

When $p_t = 0$:

$$\overline{C}_t = \underset{C_t}{\operatorname{argmax}}\left(-C_t\right) = 0 \tag{S24}$$

When $p_t = 1$:

$$\bar{C}_t = \underset{C_t}{\operatorname{argmax}}\left[\min\left(\beta \cdot C_t, L\right) - L - C_t\right] = \frac{L}{\beta} \tag{S25}$$

Therefore, for any timestep the cost will be either $\bar{C}_t = 0$ when $p_t = 0$ or $\bar{C}_t = \frac{L}{\beta}$ when $p_t = 1$.

The ex post utility for each timestep, shown in Table S3, can be found by substituting these optimal costs back into the elements of Table S2, and letting the probability be conditioned on the state of observed flow $\bar{Q}_t$ above the threshold, rather than the forecast flow used for the ex ante utility, according to the following.

$$p_t = \begin{cases} 1 & \text{when } \bar{Q}_t \geq Q_d \\ 0 & \text{when } \bar{Q}_t < Q_d \end{cases} \tag{S26}$$

Table S3: Ex post utility values for a time-step of Expected Utility Theory with REV assumptions

	Event occurred	Event did not occur
Action taken $C_t = \frac{L}{\beta}$	$-\left(\frac{L}{\beta} + L_t^u\right)$	$-\frac{L}{\beta}$
Action not taken $C_t = 0$	$-L$	0

Like the derivation of REV, a contingency table is used since every timestep can be mapped to one of the 4 outcomes in Table S2.

Determining the expected ex post utility for the reference climatology (event frequency) information requires an additional assumption.

Assumption 4: The frequency of the binary decision event $\bar{o}$ is used for the reference baseline.

Since the reference climatology is a single value, the decision-maker will either always take action or never take action.

$$\mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^r\right)\right] = \max\left\{\mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^c\right)\right]_{\text{never}}, \mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^c\right)\right]_{\text{always}}\right\} \tag{S27}$$

$$\begin{aligned} \mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^r\right)\right]_{never} &= \bar{o}\left(-L\right) + \left(1-\bar{o}\right)\left(0\right) \\ &= -\bar{o}L \end{aligned} \tag{S28}$$

$$\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^r\right)\right]_{always} = -\bar{o}\left(\frac{L}{\beta}+L_t^u\right)+\left(1-\bar{o}\right)\left(-L\right) \\ = -\bar{o}L_t^u - \frac{L}{\beta} \tag{S29}$$

$$\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^r\right)\right] = \max\left\{-\bar{o}L_t^u - \frac{L}{\beta}, -\bar{o}L\right\} \\ = -\min\left\{\bar{o}L_t^u + \frac{L}{\beta}, \bar{o}L_t^u + \bar{o}L_t^a\right\} \\ = -\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} - \bar{o}L_t^u \tag{S30}$$

Expected ex post utility for perfect information is

$$\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^p\right)\right] = -\bar{o}\left(\frac{L}{\beta}+L_t^u\right) \tag{S31}$$

Expected ex post utility for forecast information is

$$\underset{t\in T}{\mathbb{E}}\left[\Upsilon\left(E_t^f\right)\right] = h\left[-\left(\frac{L}{\beta}+L_t^u\right)\right]+m\left[-L\right]+f\left[-\frac{L}{\beta}\right]+q\left[0\right] \\ = -h\frac{L}{\beta} - hL_t^u - mL - f\frac{L}{\beta} \\ = -\left(h+f\right)\frac{L}{\beta} - hL_t^u - m\left(L_t^a + L_t^u\right) \\ = -\left(h+f\right)\frac{L}{\beta} - \left(h+m\right)L_t^u - mL_t^a \\ = -\left(h+f\right)\frac{L}{\beta} - \bar{o}L_t^u - mL_t^a \tag{S32}$$

where h is the hit rate, m is the miss rate, f is the false alarm rate, and q is the correct rejection (quiets) rate from the contingency table.

Substitute Eq. (S31), (S32) and (S30) into Eq. (S15) and simplify.

$$\text{RUV} = \frac{\left[-\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} - \bar{o}L_t^u\right] - \left[-(h+f)\frac{L}{\beta} - \bar{o}L_t^u - mL_t^a\right]}{\left[-\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} - \bar{o}L_t^u\right] - \left[-\bar{o}\left(\frac{L}{\beta} + L_t^u\right)\right]}$$

$$= \frac{\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} + \bar{o}L_t^a - (h+f)\frac{L}{\beta} - \bar{o}L_t^u - mL_t^a}{\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} + \bar{o}L_t^u - \bar{o}\frac{L}{\beta} - \bar{o}L_t^u} \tag{S33}$$

$$= \frac{\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} - (h+f)\frac{L}{\beta} - mL_t^a}{\min\left\{\frac{L}{\beta}, \bar{o}L_t^a\right\} - \bar{o}\frac{L}{\beta}}$$

Assumption 5: At each timestep the avoided losses are equal to the total possible losses.

$$L_t^a = L \qquad \text{for } t \in T \tag{S34}$$

Applying this assumption by substitution into Eq. (S33) and factoring out L leads to.

$$\text{RUV} = \frac{\min\left(\frac{1}{\beta}, \bar{o}\right) - (h+f)\frac{1}{\beta} - m}{\min\left(\frac{1}{\beta}, \bar{o}\right) - \bar{o}\frac{1}{\beta}} \tag{S35}$$

Noting the relationship $\beta = \frac{1}{\alpha}$ Eq. (S35) becomes

$$\text{RUV} = \frac{\min(\alpha, \bar{o}) - (h+f)\alpha - m}{\min(\alpha, \bar{o}) - \bar{o}\alpha} \tag{S36}$$

which is identical to the definition of the REV metric in Eq. (S9).

3 Additional economic metrics used to quantify forecast value

Three metrics were used by Matte et al. (2017) to quantify forecast value.

Utility-difference: the average ex post expected utility using perfect information is subtracted from the average ex post expected utility with forecast information to give a relative measure of utility. A negative value indicates that decisions made using forecast information led to worse utility on average than using perfect information, and a positive value the converse.

$$\text{utility\_difference} = \mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^f\right)\right] - \mathop{\mathbb{E}}_{t\in T}\left[\Upsilon\left(E_t^p\right)\right] \quad \text{(S37)}$$

Benefits-hitrate: the average avoided damages (benefits) over the observed streamflow record using spending decisions based on forecast information relative to average avoided damages with perfect spending decisions. A higher benefits-hitrate is better, a value of 1 is equivalent to using observations to make spending decisions which lead to a maximum amount of damages avoided. This should not be confused with the hitrate used in constructing a contingency table. $b_t^f\left(\overline{m}_t\right)$ is the avoided losses for the observed state $\overline{m}_t$ at timestep t using the optimal cost $\overline{C}_t$ found ex ante using forecast information.

$$\text{hitrate} = \frac{\mathop{\mathbb{E}}_{t\in T}\left[b_t^f\left(\overline{m}_t\right)\right]}{\mathop{\mathbb{E}}_{t\in T}\left[b_t^p\left(\overline{m}_t\right)\right]} \quad \text{(S38)}$$

Overspending: the percentage difference of the average forecast spend amount to the perfect spend amount. It is a standard measure of economic performance**.** An overspend closer to 0 is better which is equivalent to using observations to make perfect spending decisions and a value of 1 is equivalent to overspending by 100% (paying twice what was necessary).

$$\text{overspending} = \frac{\mathop{\mathbb{E}}_{t\in T}\left[\overline{C}_t\right] - \mathop{\mathbb{E}}_{t\in T}\left[C_t^p\right]}{\mathop{\mathbb{E}}_{t\in T}\left[C_t^p\right]} \quad \text{(S39)}$$

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