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Technical note: Statistical generation of climate-perturbed flow duration curves

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Abstract. This supplementary information demonstrates that for triplets (M,V,L) of streamflow statistics representing average behavior, variability, and low flows, there is unique parameterisation of the flow duration curve (FDC) according to the Kosugi model. We consider the "mean" case where $(M,V,L) = (\mu,\sigma,q_{low})$ where μ is the mean, σ is the standard deviation and q_{low} is the 1^{st} or 5^{th} percentile of flow, and the "median case" $(M,V,L) = (m,CV,q_{low})$ where m is the median and $CV = \mu/\sigma$ is the coefficient of variation. It also provides conditions on (M,V,L) for the existence of a parameterisation.

1 Kosugi function reminders

We model the flow duration curve (FDC) with the Kosugi equation, as proposed by Sadegh et al. (2016). The equation models streamflow q as a function of the flow quantile $u \in [0,1]$:

$$q(u) = c + (a-c)\,z(u)^b \tag{1}$$

where (a,b,c) are parameters, with a and c in the same units as q, and b unitless. We need $a-c>0$ and $b>0$ for $z(u)$ is defined as follows, and represented in Figure S1:

$$z(u) = \exp\left[\sqrt{2}\,\mathrm{erfc}^{-1}(2u)\right] \tag{2}$$

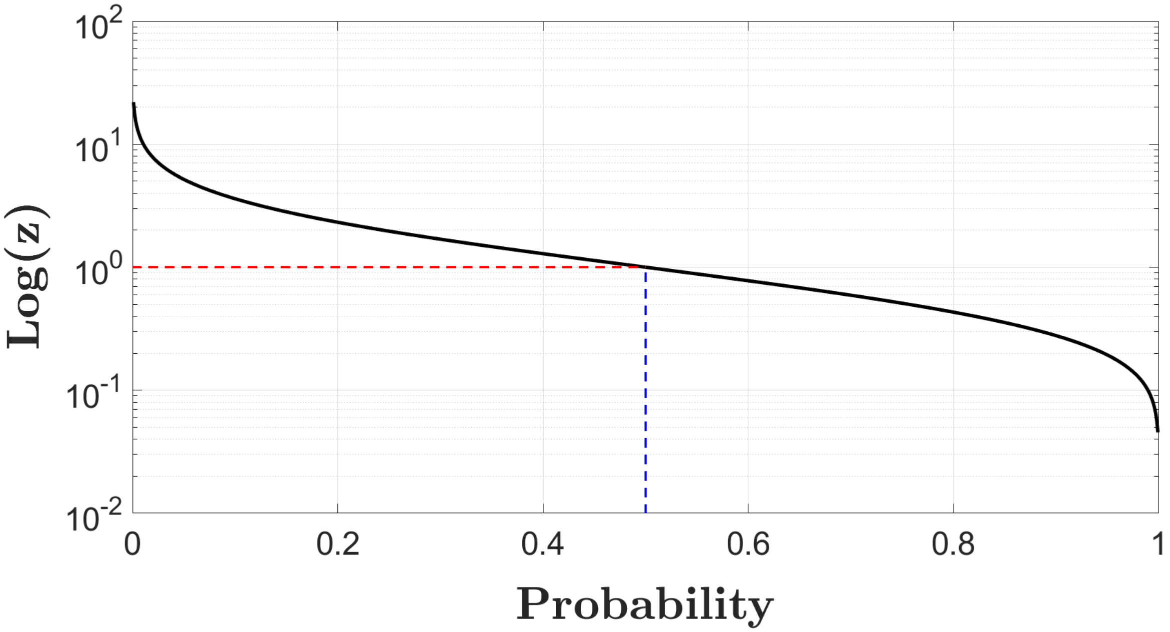

Figure S1. The $z(e)$ function.

This supplementary information will relate parameters (a,b,c) to triplets of streamflow statistics (M,V,L) representing average behavior, variability, and low flows. It will do so first in the "mean" case of $(M,V,L) = (\mu,\sigma,q_{low})$ in Section 2, where μ is the mean, σ is the standard deviation and q_{low} is the 1^{st} or 5^{th} percentile of flow. Then in Section 3 we will examine the "median case" of $(M,V,L) = (m,CV,q_{low})$, where m is the median and $CV = \mu/\sigma$ is the coefficient of variation.

2 "Mean" case $(M,V,L) = (\mu,\sigma,q_{low})$

In this section we assume we know the streamflow mean μ, standard deviation σ and low flow percentile q_{low}. We assume we have $\mu > q_{low}$. We will prove the (a,b,c) triplet of Kosugi parameter is unique, and give a sufficient condition on (μ,σ,q_{low}) for its existence.

2.1 Relating parameter triplets

Writing the definition of mean, standard deviation and low flow quantile for the Kosugi FDC yields three equations. For this, let us introduce the auxiliary function f:

$$f(b) = \int_0^1 z(u)^b \, du \tag{3}$$

We can then write the mean μ according to its definition as the integral of $q(u)$ for $u \in [0,1]$. Using the linearity properties of the integral yields:

$$\mu = c + (a-c) f(b) \tag{4}$$

Similarly, by definition of the variance, and using the definition of the mean above, we have:

$$\sigma^2 = \int_0^1 \left[c + (a-c) z(u)^b\right]^2 du - \left[c + (a-c) f(b)\right]^2 \tag{5}$$

Developing the squares and exploiting again the linearity of the integral enables us to simplify this into this definition of V:

$$\sigma = (a-c)\sqrt{f(2b) - f(b)^2} \tag{6}$$

Lastly, introducing $\varepsilon = z(q_{low})$ where $q_{low} = 0.99$ (respectively 0.95) if we are interested in the first (resp. fifth) flow percentile, we have the following relationship for q_{low}:

$$q_{low} = c + (a-c)\,\varepsilon^b \tag{7}$$

2.2 Solution strategy

Clearly, for b fixed, (a,c) is the solution of a system of two linear equations. For instance, from equations (4) and (7), we get:

$$\begin{cases} \mu = c + (a-c) f(b) \\ q_{low} = c + (a-c)\,\varepsilon^b \end{cases} \tag{8}$$

which is equivalent to:

$$\begin{cases} a - c = \dfrac{\mu - q_{low}}{f(b) - \varepsilon^b} \\[2ex] a = \dfrac{q_{low}\left(f(b) - 1\right) + \mu(1-\varepsilon^b)}{f(b) - \varepsilon^b} \\[2ex] c = \dfrac{q_{low}\, f(b) - \mu\,\varepsilon^b}{f(b) - \varepsilon^b} \end{cases} \tag{9}$$

We can then relate streamflow parameters (μ, σ, q_{low}) to Kosugi parameter function of b alone, by replacing $(a-c)$ into equation (6):

$$\frac{\sigma}{\mu - q_{low}} = \frac{\sqrt{f(2b) - f(b)^2}}{f(b) - \varepsilon^b} = \mathcal{F}(b) \tag{10}$$

Thus, whether we can find a unique triplet (a, b, c) for (μ, σ, q_{low})hinges on whether $\mathcal{F}(b)$ is monotonous for $b > 0$. Then existence will depend on (1) proving that $f(b) > \varepsilon^b$ for $b > 0$ (so $\mathcal{F}(b)$ is defined and positive), and (2) establishing the lower bond for $\mathcal{F}(b)$. For all of this, it would be easier to work with a simpler expression for $f(b)$. This is the topic of the next paragraph.

2.3 Simplifying $f(b)$

Let us operate a variable change $x = \mathrm{erfc}^{-1}(2u)$ in the integral that defines $f(b)$. Then for $u = 0$, we have $x = +\infty$, for $u = 1$ we have $x = -\infty$. We also have $u = \mathrm{erfc}(x)/2 = (1 - \mathrm{erf}(x))/2$. Using the derivation of the error function to relate du and dx we can therefore write:

$$f(b) = \int_0^1 \exp\left[\sqrt{2}\,\mathrm{erfc}^{-1}(2u)\right]^b du = \int_{+\infty}^{-\infty} e^{\sqrt{2}bx} \frac{-e^{-x^2}}{\sqrt{\pi}} dx \tag{11}$$

Which directly leads to:

$$f(b) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-(x - \frac{b}{\sqrt{2}})^2\right) e^{b^2/2} dx \tag{12}$$

Then a further change of variable $y = x - b/\sqrt{2}$ leads to:

$$f(b) = e^{b^2/2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy \right) \tag{13}$$

and since the quantity between the parentheses is equal to 1 this simplifies into:

$$f(b) = e^{b^2/2} \tag{14}$$

This remarkable equation simplifies the calculations going forward. It also demonstrates that $f(b) - \varepsilon^b = \exp(b^2) - \exp(b \ln(\varepsilon))$ is positive for $b > 0$, because $\ln(\varepsilon) < 0$.

2.4 Unicity of the parameterisation

Using the result from equation (14) into equation (10), we can write:

$$\mathcal{F}(b) = \frac{g(b)}{h(b)} \tag{15}$$

with:

$$\begin{cases} g(b) = \sqrt{e^{b^2} - 1} \\ h(b) = 1 - e^{-b^2/2} \varepsilon^b \end{cases} \tag{16}$$

Recall that to demonstrate unicity of the Kosugi parameterisation, it is enough to show that for $b > 0$, $\mathcal{F}(b)$ grows monotonically with b. Derivation with respect to b yields:

$$\begin{cases} g'(b) = \dfrac{b\, e^{b^2}}{g(b)} \\ h'(b) = (b - \ln(\varepsilon))e^{-b^2/2}\varepsilon^{b} \end{cases} \tag{17}$$

where $\ln$ is the base e logarithm, ($\ln(e) = 1$). Since $g(b) > 0$, we can write for $b > 0$:

$$\mathcal{F}'(b) = \frac{1}{g(b)h^2(b)}\left(g(b)g'(b)h(b) - g^2(b)h'(b)\right) \tag{18}$$

Thanks to the above equation, $\mathcal{F}'(b)$ has the same sign as $\mathcal{U} = gg'h - g^2h'$. $\mathcal{U}(b)$ is given by:

$$\mathcal{U}(b) = be^{b^2} + \varepsilon^{b}\left[b(e^{-b^2/2} - 2e^{b^2/2}) + \ln(\varepsilon)\left(e^{b^2/2} - e^{-b^2/2}\right)\right] \tag{19}$$

Figure S2 graphically shows that $\ln(\mathcal{U}(b)) > 0$ for $b > 0$, in both cases where $\varepsilon = z(0.99)$ (if q_{low} is the first percentile) or $\varepsilon = z(0.95)$ (if q_{low} is the first percentile). We also represented e^{b^2} on Figure S2, since it becomes the dominant term in $\mathcal{U}(b)$ as b grows farther from 0. It is therefore clear that $\mathcal{F}(b)$ grows with b when $b > 0$, and that therefore, there is at most a unique b solution of equation (10). Equation (9) provides unique a and c for a value of b. This enables us to conclude on the uniqueness of the Kosugi parameterisation.

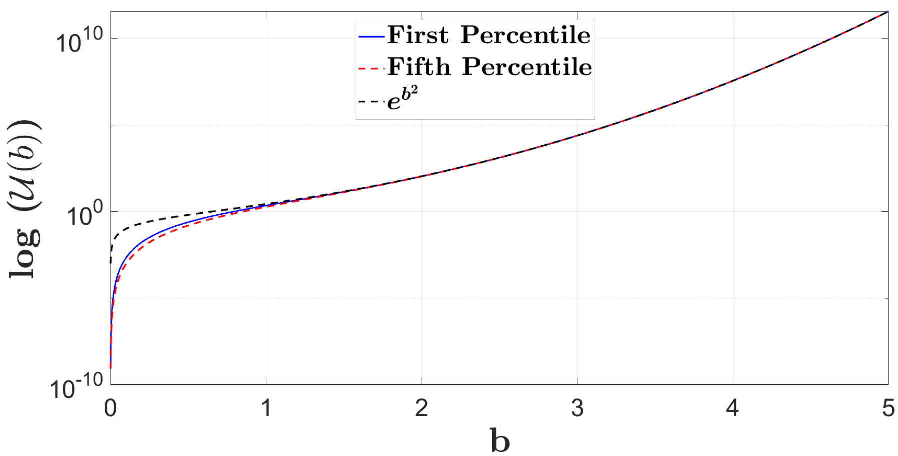

Figure S2. Representation of $\log(\mathcal{U}(b))$ to establish that $\mathcal{F}'(b) > 0$. Blue coloured line and dashed red line represent the derivation based on first percentile and fifth percentile of flow respectively. The dashed black line signifies e^{b^2}.

2.5 Condition for existence

$\mathcal{F}(b) = g(b)/h(b)$ grows monotonically with b when $b > 0$, and goes to $+\infty$ as $b \to +\infty$. Therefore, a solution exists if:

$$\frac{\sigma}{\mu - q_{low}} \lim_{b\to 0^+} h(b) > \lim_{b\to 0} g(b) \tag{20}$$

and these limits are defined because both g and h are continuously differentiable over $]0, +\infty)$. The respective first-order Taylor expansions of g and h at 0^+ yield:

$$\begin{cases} g(b) = b + o(b) \\ h(b) = -\ln(\varepsilon)\, b + o(b) \end{cases} \tag{21}$$

Since Taylor expansions are unique, the results from equation (21) into equation (20) yields the existence condition:

$$\frac{\sigma}{\mu - q_{low}} > \frac{-1}{\ln(\varepsilon)} \tag{22}$$

where $\varepsilon < 1$ so $\ln(\varepsilon) < 0$ and $-1/\ln(\varepsilon) \approx 0.43$ if q_{low} is the first percentile; 0.61 if q_{low} is the fifth percentile. Note this condition is sufficient: if it is met, one can find the unique b with equation (10), then a and c with equation (9).

3 Median, coefficient of variation and low flow quantile

In this section we assume we know the streamflow median m, coefficient of variation $CV = \sigma/\mu$, and low flow percentile q_{low}. We assume flow is not constant for large-stretches of the FDC (true for natural flows in perennial rivers) so we have $m > q_{low}$ and $CV > 0$. We will prove the (a,b,c) triplet of Kosugi parameter is unique, and exists given a condition on (μ,σ,q_{low}) that is often met in practice.

3.1 Relating parameter triplets

The median m corresponds to $q(0.5)$ in equation (1). Since for $u = 0.5$ we have $z(u) = 0$, we have:

$$M = a \tag{23}$$

CV is the ratio of standard deviation and mean. These two quantities are given by equations (6) and (4), respectively, so:

$$CV = \frac{(a-c)\sqrt{f(2b) - f(b)^2}}{c + (a-c)f(b)} \tag{24}$$

Finally, q_{low} still verifies equation (7).

3.2 Solution strategy

Finding a is immediate thanks to equation (23), and combined with equation (7), this directly leads to the following expression for c:

$$c = \frac{q_{low} - m\varepsilon^b}{1 - \varepsilon^b} \tag{25}$$

which means that c can be easily and uniquely computed once b is known. To find b, we use equation (24) and replace $f(b)$ with $e^{b^2/2}$ thanks to equation (14). This leads to:

$$CV = \frac{\sqrt{e^{b^2} - 1}}{e^{-b^2/2}\frac{c}{a-c} + 1} \tag{26}$$

Let us introduce R as the ratio of L by M:

$$R = \frac{L}{M} \tag{27}$$

Clearly, we have $0 < R < 1$. Equations (23) and (25) then become:

$$\frac{c}{a-c} = \frac{R - \varepsilon^b}{1 - R} \tag{28}$$

And finally:

$$CV = (1-R)\frac{\sqrt{e^{b^2}-1}}{1-R+(R-\varepsilon^b)e^{-b^2/2}} = \mathcal{G}(b) \tag{29}$$

where $\mathcal{G}$ only depends on b because R is a known parameter. As was the case in Section 2, we need to establish that there is (at most) a single $b > 0$ for a given value of V, and find the condition for existence. Then we can then deduce c. Yet, before establishing unicity and condition for existence, it is important to clarify on which range for $b > 0$ we can say that $\mathcal{G}(b)$ is defined.

3.3 Range of b for which the equation for CV is defined

Similar to equation (15), we can write:

$$\mathcal{G}(b) = (1-R)\frac{g(b)}{k(b)} \tag{30}$$

with $g(b)$ (and $g'(b)$) defined as in equations (16) and (17), and $k(b)$ defined as:

$$k(b) = 1-R+(R-\varepsilon^b)e^{-b^2/2} \tag{31}$$

$\mathcal{G}(b)$ is defined in the range for $b > 0$ in which $k(b) \neq 0$. Since $g(b) > 0$, it corresponds to a coefficient of variation in the range in which $k(b) > 0$. We have $k(0) = 0$ and $\lim_{b\to\infty} k(b) = 1-R > 0$, and need to understand variations to know what happens in between. $k(b)$ is derivated as follows:

$$k'(b) = \left[-bR+(b-\ln(\varepsilon))\varepsilon^b\right]e^{-b^2/2} \tag{32}$$

so that $k'(b)$ has the sign of $v(b) = \left[-bR+(b-\ln(\varepsilon))\varepsilon^b\right]$. In turn we have:

$$v'(b) = -R+\left[1+(b-\ln(\varepsilon))\ln(\varepsilon)\right]\varepsilon^b \tag{33}$$

Since $\ln(\varepsilon) < -1$, for any positive value of b, $[1+(b-\ln(\varepsilon))\ln(\varepsilon)] < 0$. This means that v is monotonously decreasing for $b \geq 0$. We have $v(0) = -\ln(\varepsilon) > 1$, and $\lim_{b\to\infty} k(b) = -\infty$ because $-bR$ is the dominant term. Therefore, there is a b_{lim} such that $v(b_{lim}) = 0$. For $b < b_{lim}$, $k(b)$ grows strictly and monotonously to a global maximum $k(b_{lim})$, then it degrows for $b > b_{lim}$ towards its limit value $1-R > 0$. This means that $k(b_{lim}) > 0$, and $k(b) > 0$ for $b > 0$.

3.4 Unicity of the parameterisation

For $b > b_{lim}$, We know that the numerator $g(b)$ always grows with $b > 0$, and for $b > b_{lim}$, the numerator decreases strictly, so $\mathcal{G}(b)$ is strictly and monotonously growing. To demonstrate this is the case for any $b > 0$ we need to prove that $\mathcal{G}'(b)$ never reaches 0. This will complete the proof that the Kosugi parameterisation is unique if it exists. The following equivalence is true:

$$\mathcal{G}'(b) = 0 \iff gg'k - g^2k' = 0 \tag{34}$$

This is equivalent to this linear equation in R:

$$R = \varepsilon^b + \frac{\ln(\varepsilon)\,\varepsilon^b\,(e^{b^2}-1)+b\,e^{3b^2/2}\,(1-\varepsilon^b)}{b\,(1-2\,e^{b^2}+e^{3b^2/2})} \tag{35}$$

The last expression is plotted in Figure S3 for both $L =$ first percentile (blue line) and fifth percentile (dashed red line). Clearly, stationarity requires $R < 0$ or $R > \lim_{b\to\infty} R = 1^+$. Both of these are impossible, because $0 < R < 1$ by definition of R as the ratio of q_{low} by m. Therefore $\mathcal{G}(b)$ is monotonic and strictly growing with b. This means that there is at most one b for a given CV.

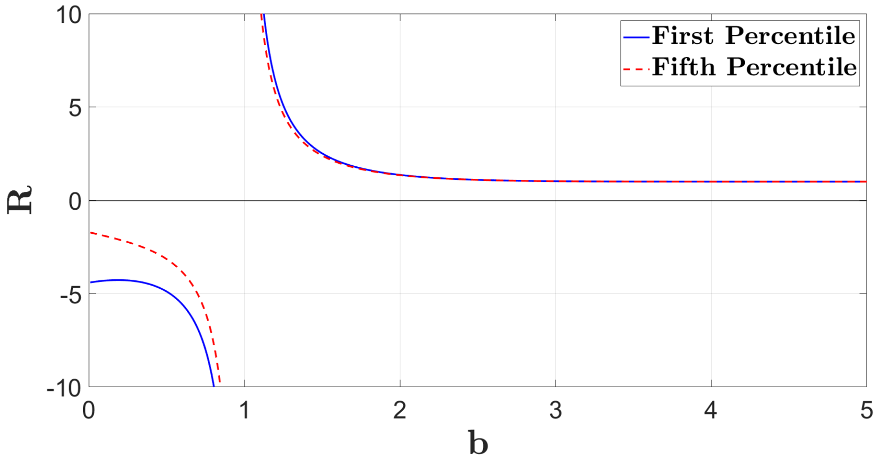

Figure S3. Plot of the stationary point locus in $b-R$ space on which $\mathcal{G}'(b) = 0$, as given by Equation 35. Blue coloured line and dashed red line represent the derivation based on first percentile and fifth percentile of flow respectively

3.5 Condition for existence

$\mathcal{G}(b) = (1-R)\, g(b)/k(b)$ is monotonous and grows with b when $b > 0$, and goes to $+\infty$ as $b \to +\infty$. Therefore, a solution exists if:

$$\frac{CV}{1-R} \lim_{b \to 0^+} k(b) > \lim_{b \to 0} g(b) \tag{36}$$

and these limits are defined because both g and k are continuously differentiable over $]0, +\infty)$. The respective first-order Taylor expansions of g at 0^+ is given in equation (21) and for k we have:

$$k(b) = -\ln(\varepsilon)\, b + o(b) \tag{37}$$

Since Taylor expansions are unique, the results from equation (37) into equation (36) yields the existence condition:

$$\frac{CV}{1-R} > \frac{-1}{\ln(\varepsilon)} \tag{38}$$

where $\varepsilon < 1$ so $\ln(\varepsilon) < 0$ and $-1/\ln(\varepsilon) \approx 0.43$ if q_{low} is the first percentile; 0.61 if q_{low} is the fifth percentile.

Note this condition is sufficient: if it is fulfilled, one can find the unique b with equation (29) then, 23 and 25 equations above directly lead to obtaining unique values of a and c.

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Code and data availability. All data needed and the Python scripts to reproduce the analysis in this manuscript are are available in Yildiz et al. (2022) at https://doi.org/10.5281/zenodo.7423056

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