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Technical Note: Improved partial wavelet coherency for understanding scale-specific and localized bivariate relationships in geosciences

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Introduction

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S1 Derivation of the complex PWC Eq.(1)

Complex partial spectrum from frequency (scale)domain (Makhtar et al., 2014) can be used to define that of time-frequency (location-scale) domain, $\underset{W}{\leftrightarrow}^{y,x\cdot Z}(s,\tau)$, which is expressed as

$$\underset{W}{\leftrightarrow}^{y,x\cdot Z}(s,\tau) = \underset{W}{\leftrightarrow}^{y,x}(s,\tau) - \frac{\underset{W}{\leftrightarrow}^{y,Z}(s,\tau)\overline{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)} \tag{S1}$$

where $\cdot$ is the notation for excluding variables, $\underset{W}{\leftrightarrow}$ is the smoothed cross spectrum, $\overline{()}$ is the complex conjugate operator, y, x, and Z ($Z = \{Z_1, Z_2, \cdots, Z_q\}$) refer to the response variable, predictor variable, and excluding variables, respectively. s and τ refer to scale (frequency) and location (time), respectively.

Given the definition of coherence between two variables y and x, their complex coherence $\gamma_{y,x}(s,\tau)$ (Eq.(5)) can be re-written as

$$\gamma_{y,x}(s,\tau) = \frac{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)}{\sqrt{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\,\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}} \tag{S2}$$

Then we can define complex partial coherence as

$$\gamma_{y,x\cdot Z}(s,\tau) = \frac{\underset{W}{\leftrightarrow}^{y,x\cdot Z}(s,\tau)}{\sqrt{\underset{W}{\leftrightarrow}^{y,y\cdot Z}(s,\tau)\,\underset{W}{\leftrightarrow}^{x,x\cdot Z}(s,\tau)}} \tag{S3}$$

Based on Eq. (S1) and Eqs 2, 3, and 4 ($R^2_{y,x,Z}(s,\tau) = \frac{\underset{W}{\leftrightarrow}^{y,Z}(s,\tau)\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)^{-1}\overline{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)}$,

$R^2_{y,Z}(s,\tau) = \frac{\underset{W}{\leftrightarrow}^{y,Z}(s,\tau)\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)^{-1}\overline{\underset{W}{\leftrightarrow}^{y,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)}$, and $R^2_{x,Z}(s,\tau) = \frac{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)^{-1}\overline{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}$)

we obtain

$$\underset{W}{\leftrightarrow}^{y,x\cdot Z}(s,\tau) = \underset{W}{\leftrightarrow}^{y,x}(s,\tau)\left(1 - \frac{\underset{W}{\leftrightarrow}^{y,z}(s,\tau)\,\overline{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)\,\underset{W}{\leftrightarrow}^{y,x}(s,\tau)}\right) = \underset{W}{\leftrightarrow}^{y,x}(s,\tau)\,\left(1 - R^2_{y,x,Z}(s,\tau)\right) \tag{S4}$$

$$\underset{W}{\leftrightarrow}^{y,y\cdot Z}(s,\tau) = \underset{W}{\leftrightarrow}^{y,y}(s,\tau)\left(1 - \frac{\underset{W}{\leftrightarrow}^{y,z}(s,\tau)\,\overline{\underset{W}{\leftrightarrow}^{y,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)\,\underset{W}{\leftrightarrow}^{y,y}(s,\tau)}\right) = \underset{W}{\leftrightarrow}^{y,y}(s,\tau)\,\left(1 - R^2_{y,Z}(s,\tau)\right) \tag{S5}$$

$$\underset{W}{\leftrightarrow}^{x,x\cdot Z}(s,\tau) = \underset{W}{\leftrightarrow}^{x,x}(s,\tau)\left(1 - \frac{\underset{W}{\leftrightarrow}^{x,z}(s,\tau)\,\overline{\underset{W}{\leftrightarrow}^{x,Z}(s,\tau)}}{\underset{W}{\leftrightarrow}^{Z,Z}(s,\tau)\,\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}\right) = \underset{W}{\leftrightarrow}^{x,x}(s,\tau)\,\left(1 - R^2_{x,Z}(s,\tau)\right) \tag{S6}$$

Inserting Eqs S4, S5, and S6 into Eq. (S3), we have

$$\gamma_{y,x\cdot \mathrm{Z}}(s,\tau)=\frac{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)\left(1-R^2_{y,x,Z}(s,\tau)\right)}{\sqrt{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\left(1-R^2_{y,Z}(s,\tau)\right)\underset{W}{\leftrightarrow}^{x,x}(s,\tau)\left(1-R^2_{x,Z}(s,\tau)\right)}}=$$

$$\frac{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)\left(1-R^2_{y,x,Z}(s,\tau)\right)}{\sqrt{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}\sqrt{\left(1-R^2_{y,Z}(s,\tau)\right)\left(1-R^2_{x,Z}(s,\tau)\right)}}=\frac{\left(1-R^2_{y,x,Z}(s,\tau)\right)\gamma_{y,x}(s,\tau)}{\sqrt{\left(1-R^2_{y,Z}(s,\tau)\right)\left(1-R^2_{x,Z}(s,\tau)\right)}} \quad \text{(S7)}$$

Obviously, Eq. (S7) and Eq. (1) are identical.

S2 Derivation of the PWC in case of one excluding variable (Eq.14) from Eq. (9)

When only one variable (e.g., Z_1) is excluded, Eq.(9) ($\rho^2_{y,x\cdot Z}=\frac{\left|1-R^2_{y,x,Z}(s,\tau)\right|^2 R^2_{y,x}(s,\tau)}{\left(1-R^2_{y,Z}(s,\tau)\right)\left(1-R^2_{x,Z}(s,\tau)\right)}$)

can be written as

$$\rho^2_{y,x\cdot Z_1}=\frac{\left|1-R^2_{y,x,Z_1}(s,\tau)\right|^2 R^2_{y,x}(s,\tau)}{\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)} \quad \text{(S8)}$$

Based on Eq. (2),

$$\rho^2_{y,x\cdot Z_1}=\frac{\left|1-\frac{\underset{W}{\leftrightarrow}^{y,Z_1}(s,\tau)\underset{W}{\leftrightarrow}^{Z_1,Z_1}(s,\tau)^{-1}\overline{\underset{W}{\leftrightarrow}^{x,Z_1}(s,\tau)}}{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)}\right|^2\frac{\left|\underset{W}{\leftrightarrow}^{y,x}(s,\tau)\right|^2}{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}}{\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)}$$

$$=\frac{\left|\underset{W}{\leftrightarrow}^{y,x}(s,\tau)-\frac{\underset{W}{\leftrightarrow}^{y,Z_1}(s,\tau)\overline{\underset{W}{\leftrightarrow}^{x,Z_1}(s,\tau)}}{\underset{W}{\leftrightarrow}^{Z_1,Z_1}(s,\tau)}\right|^2}{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\underset{W}{\leftrightarrow}^{x,x}(s,\tau)\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)}$$

$$=\frac{\frac{1}{\sqrt{\left(\underset{W}{\leftrightarrow}^{y,y}(s,\tau)\underset{W}{\leftrightarrow}^{x,x}(s,\tau)\right)}^2}\left|\underset{W}{\leftrightarrow}^{y,x}(s,\tau)-\frac{\underset{W}{\leftrightarrow}^{y,Z_1}(s,\tau)\overline{\underset{W}{\leftrightarrow}^{x,Z_1}(s,\tau)}}{\left(\sqrt{\underset{W}{\leftrightarrow}^{Z_1,Z_1}(s,\tau)}\right)^2}\right|^2}{\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)}$$

$$=\frac{\left|\frac{\underset{W}{\leftrightarrow}^{y,x}(s,\tau)}{\sqrt{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)}\sqrt{\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}}-\frac{\underset{W}{\leftrightarrow}^{y,Z_1}(s,\tau)\overline{\underset{W}{\leftrightarrow}^{x,Z_1}(s,\tau)}}{\sqrt{\underset{W}{\leftrightarrow}^{y,y}(s,\tau)}\sqrt{\underset{W}{\leftrightarrow}^{x,x}(s,\tau)}\sqrt{\underset{W}{\leftrightarrow}^{Z_1,Z_1}(s,\tau)}\sqrt{\underset{W}{\leftrightarrow}^{Z_1,Z_1}(s,\tau)}}\right|^2}{\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)}$$

$$= \frac{\left| \frac{\overset{\leftrightarrow}{W}^{y,x}(s,\tau)}{\sqrt{\overset{\leftrightarrow}{W}^{y,y}(s,\tau)}\sqrt{\overset{\leftrightarrow}{W}^{x,x}(s,\tau)}} - \frac{\overset{\leftrightarrow}{W}^{y,Z_1}(s,\tau)}{\sqrt{\overset{\leftrightarrow}{W}^{y,y}(s,\tau)}\sqrt{\overset{\leftrightarrow}{W}^{Z_1,Z_1}(s,\tau)}} \frac{\overline{\overset{\leftrightarrow}{W}^{x,Z_1}(s,\tau)}}{\sqrt{\overset{\leftrightarrow}{W}^{x,x}(s,\tau)}\sqrt{\overset{\leftrightarrow}{W}^{Z_1,Z_1}(s,\tau)}} \right|^2}{\left(1 - R^2_{y,Z_1}(s,\tau)\right)\left(1 - R^2_{x,Z_1}(s,\tau)\right)}$$

$$= \frac{\left|\gamma_{y,x}(s,\tau) - \gamma_{y,Z_1}(s,\tau)\overline{\gamma_{x,Z_1}(s,\tau)}\right|^2}{\left(1-R^2_{y,Z_1}(s,\tau)\right)\left(1-R^2_{x,Z_1}(s,\tau)\right)} \quad \text{(S9)}$$

S3 Supplementary data and analyses

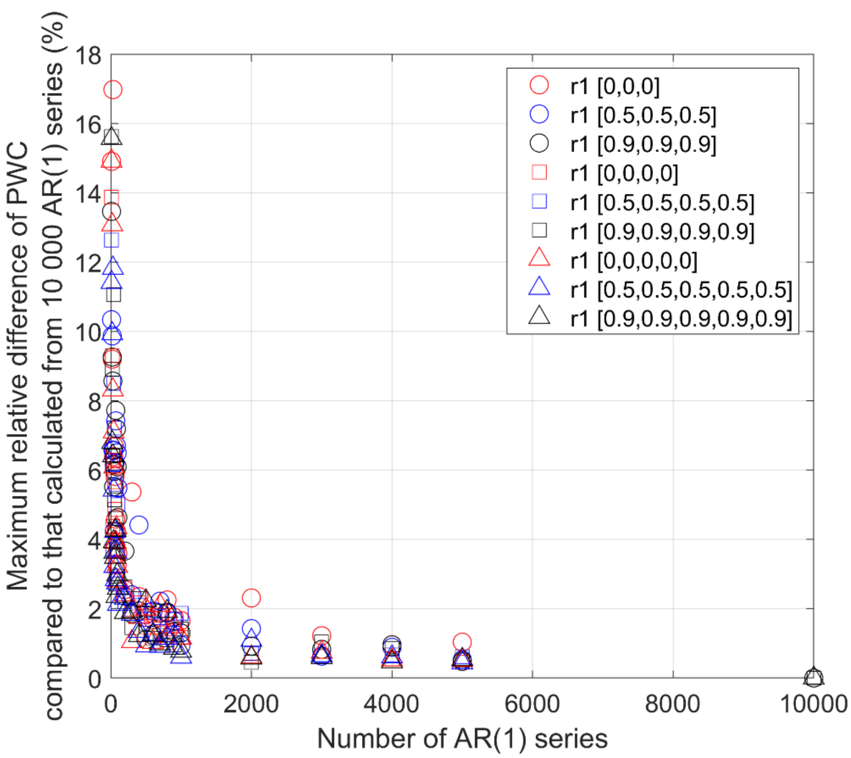

Figure S1. Relationship between maximum relative difference (%) of PWC compared to that calculated from 10 000 AR(1) series (surrogate dataset) versus the number of AR(1) series during the significance test using the Monte Carlo test. Number of scales per octave is 12. The first-order autocorrelation coefficients (r1) in brackets refer to those for the response variable (first), predictor variable (second), and excluding variables (third and onwards).

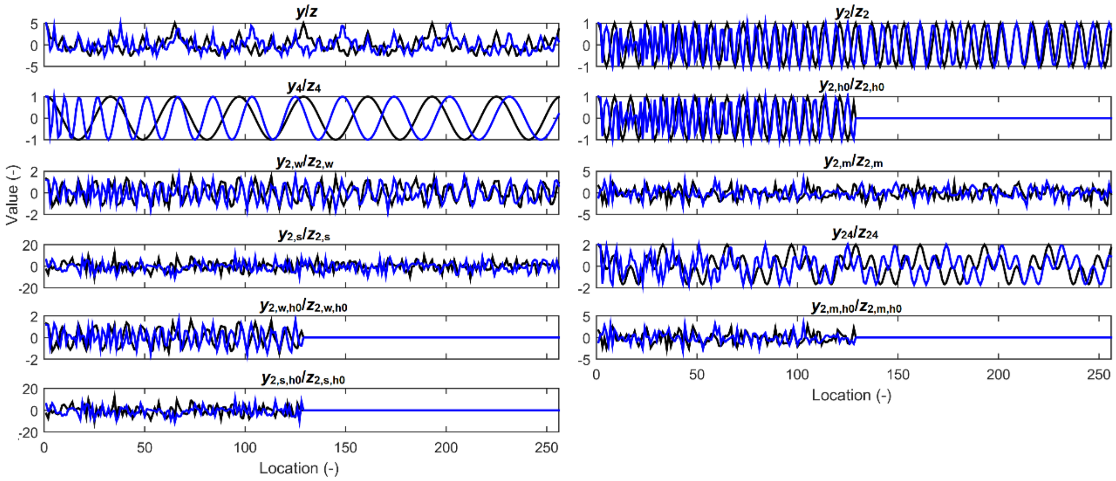

Figure S2. Variables used to test partial wavelet coherency. Black and blue lines represent variables for stationary (i.e., y, y_2, y_4, $y_{2,h0}$, $y_{2,w}$, $y_{2,m}$, $y_{2,s}$, y_{24}, $y_{2,w,h0}$, $y_{2,m,h0}$, and $y_{2,s,h0}$) and non-stationary (e.g., z, z_2, z_4, $z_{2,h0}$, $z_{2,w}$, $z_{2,m}$, $z_{2,s}$, z_{24}, $z_{2,w,h0}$, $z_{2,m,h0}$, and $z_{2,s,h0}$) cases, respectively. All variables are explained in Sect. 3.1 of the main body of the paper.

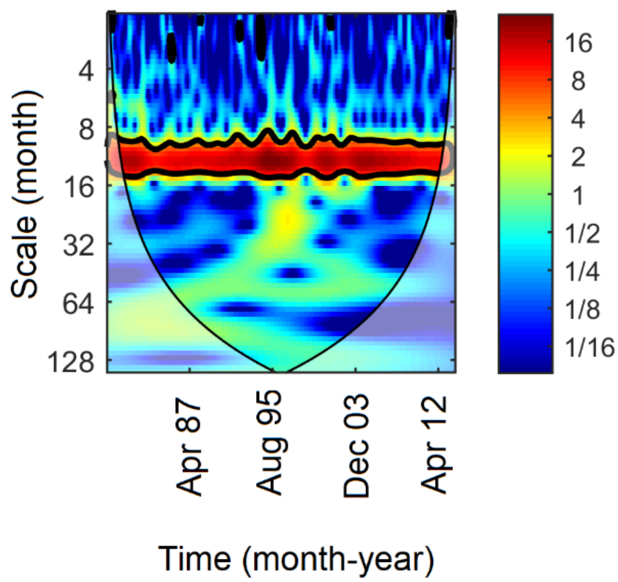

Figure S3. Wavelet power spectrum of free water evaporation (E) at the Changwu site in Shaanxi, China. Thin solid lines demarcate the cones of influence, and thick solid lines show the 95% confidence levels.

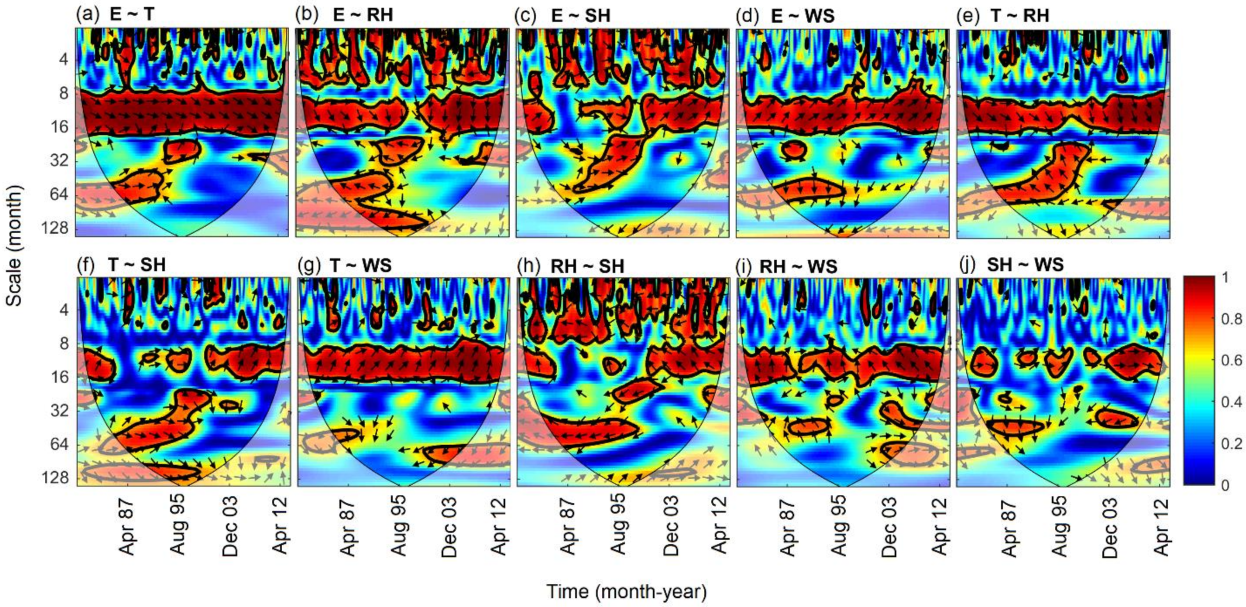

Figure S4. Bivariate wavelet coherency between every two meteorological factors (E, evaporation; T, mean temperature; RH, relative humidity; SH, sun hours; WS, wind speed) at the Changwu site in Shaanxi, China. Arrows show the phase angles of the wavelet spectra. Thin solid lines demarcate the cones of influence, and thick solid lines show the 95% confidence levels.

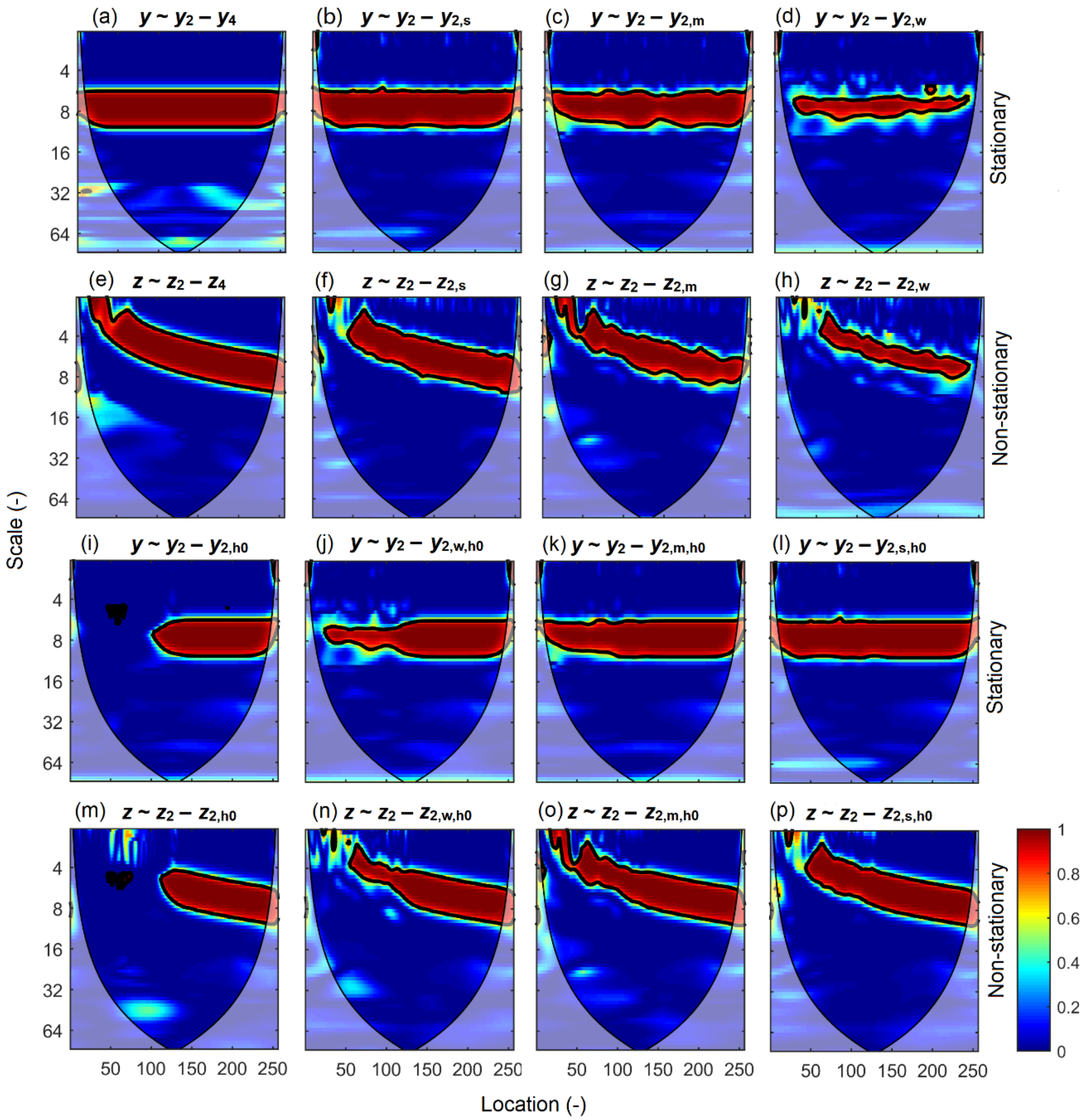

Figure S5. Partial wavelet coherency (PWC) between response variable y (or z) and predictor variable y_2 (or z_2) after excluding the effect of variables y_4 (or z_4), $y_{2,s}$ (or $z_{2,s}$), $y_{2,m}$ (or $z_{2,m}$), $y_{2,w}$ (or $z_{2,w}$), $y_{2,h0}$ (or $z_{2,h0}$), $y_{2,w,h0}$ (or $z_{2,w,h0}$), $y_{2,m,h0}$ (or $z_{2,m,h0}$), and $y_{2,s,h0}$ (or $z_{2,s,h0}$) for the stationary (or non-stationary) case using the classical implementation (Eq. 15). Thin and thick solid lines show the cones of influence and the 95% confidence levels, respectively. All variables were generated by following Yan and Gao (2007) and Hu and Si (2016) and are explained in Sect. 3.1 and are shown in Fig. S2 of Sect. S3 in the Supplement.

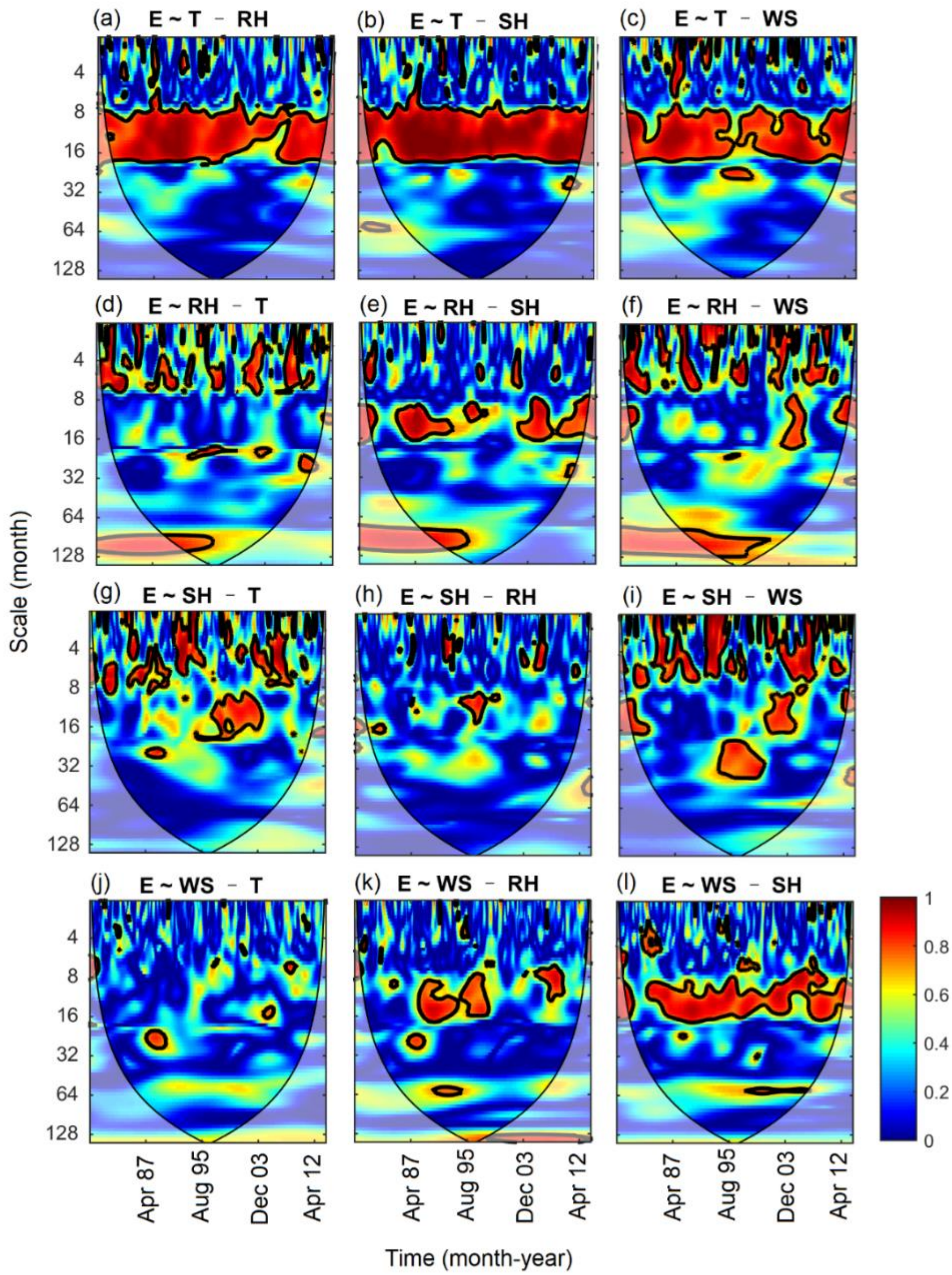

Figure S6. Partial wavelet coherency (PWC) between evaporation (E) and each meteorological factor (T, mean temperature; RH, relative humidity; SH, sun hours; WS, wind speed) after excluding the effect of each of other three meteorological factors using the classical calculation (Eq. 15).

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