Supplement of

Novel Keeling-plot-based methods to estimate the isotopic composition of ambient water vapor

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Proposition. In the traditional linear Keeling plot system, denote \( \delta_a = f(t), \delta_v = g(t), \delta_{ET} = h(t) \) and \( C_a = l(t) > 0 \) as continuous functions of time. And for two definite moments \( t_1 \) and \( t_2 \) \((t_1 < t_2)\), \( \delta_{a_1} \neq \delta_{a_2} \neq \delta_{v_1} \neq \delta_{v_2} \neq \delta_{ET_1} \neq \delta_{ET_2} \). The slopes of corresponding keeling plot curve are \( k_1 = C_{a_1}(\delta_{a_1} - \delta_{ET_1}) \) and \( k_2 = C_{a_2}(\delta_{a_2} - \delta_{ET_2}) \), respectively. Then we have that when \( k_1 k_2 < 0 \), there exists \([t_1', t_2'] \subset [t_1, t_2] \), such that \( [\min(f(t_1'), f(t_2')), \max(f(t_1'), f(t_2'))] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \).

Remark: To make a proof of the proposition, classical Intermediate Value Theorem (IVT) was used. It states that if \( f \) is a continuous function from the interval \( I = [a, b] \) to real number (R). Then Version I, if \( u \) is a number between \( f(a) \) and \( f(b) \), there is \( c \in (a, b) \) such that \( f(c) = u \).

Version II. the image set \( f(I) \) is also an interval, and it contains \([\min(f(a), f(b)), \max(f(a), f(b))] \). While in this study, IVT was able to be explained as follows: if \( f \) is a continuous function from the interval \( I = [t_1, t_2] \) to \( R \) with \( \min(f(t_1), f(t_2)) < \delta_v \) and \( \max(f(t_1), f(t_2)) > \delta_v \), then Version I implies that there is \( t' \in (t_1, t_2) \) such that \( f(t') = \delta_v \). And Version II implies that the image set \( f(I) \) is also an interval, and it contains \([\min(f(t_1), f(t_2)), \max(f(t_1), f(t_2))] \).

Proof. Since \( k_1 k_2 < 0 \), we have \( \delta_{a_1} < \delta_{v_1} \) and \( \delta_{a_2} > \delta_{v_2} \), or \( \delta_{a_1} > \delta_{v_1} \) and \( \delta_{a_2} < \delta_{v_2} \). As a result, the cases \( \delta_{a_1} < \delta_{v_1} < \delta_{a_2} < \delta_{v_2}, \delta_{v_1} < \delta_{a_1} < \delta_{v_2} < \delta_{a_2}, \delta_{v} < \delta_{a_2} < \delta_{v_1} < \delta_{a_1}, \delta_{a_1} < \delta_{v_1} < \delta_{v_2} < \delta_{a_2} \) and \( [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \cap [\min(\delta_{a_1}, \delta_{a_2}), \max(\delta_{a_1}, \delta_{a_2})] = \emptyset \) do not meet the precondition \( k_1 k_2 < 0 \). There are only four cases below. We will prove the proposition in each of the four cases.

Case 1: \([\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \subset [\min(\delta_{a_1}, \delta_{a_2}), \max(\delta_{a_1}, \delta_{a_2})] \) \( \text{(Fig. 1 a)} \).

According to IVT Version I, there exists \( t_1' \in [t_1, t_2] \), such that \( f(t_1') = \delta_{v_1} \); similarly, there exists \( t_2' \in [t_1, t_2] \), such that \( f(t_2') = \delta_{v_2} \). Based on IVT Version II, there exists \([t_1', t_2'] \subset [t_1, t_2]\) such that \( [\min(f(t_1'), f(t_2')), \max(f(t_1'), f(t_2'))] = [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \).

Case 2: \([\min(\delta_{a_1}, \delta_{a_2}), \max(\delta_{a_1}, \delta_{a_2})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \) \( \text{(Fig. 1 b)} \).

According to IVT Version I, there exists \( t_1' \in [t_1, t_2] \), such that \( f(t_1') = \delta_{a_1} \); similarly, there exists \( t_2' \in [t_1, t_2] \), such that \( f(t_2') = \delta_{a_2} \). Based on IVT Version II, there exists \([t_1', t_2'] \subset [t_1, t_2]\) such that \( [\min(f(t_1'), f(t_2')), \max(f(t_1'), f(t_2'))] = [\min(\delta_{a_1}, \delta_{a_2}), \max(\delta_{a_1}, \delta_{a_2})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \).

Case 3: \( \delta_{v_2} < \delta_{a_1} < \delta_{v_1} < \delta_{a_2}, \) or \( \delta_{a_2} < \delta_{v_1} < \delta_{a_1} < \delta_{v_2} \) \( \text{(Fig. 1 c and Fig. 1 d)} \).
According to IVT Version I, there exists \( t' \in [t_1, t_2] \), such that \( f(t'_2) = \delta_{v_i} \). Given case (2), when \([\min(\delta_{a_1}, \delta_{v_1}), \max(\delta_{a_1}, \delta_{v_1})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \), there exists \([t_1, t_2] \subset [t_1, t_2] \), such that \([\min( f(t'_1), f(t'_2) ), \max( f(t'_1), f(t'_2) )]\] \([\min(\delta_{a_1}, \delta_{v_1}), \max(\delta_{a_1}, \delta_{v_1})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \).

Case 4: \( \delta_{v_1} < \delta_{a_2} < \delta_{v_2} < \delta_{a_1} \), or \( \delta_{a_1} < \delta_{v_2} < \delta_{a_2} < \delta_{v_1} \) (Fig. 1 e and Fig. 1 f).

According to IVT Version I, there exists \( t'_1 \in [t_1, t_2] \), such that \( f(t'_1) = \delta_{v_2} \). Based on case (2), when \([\min(\delta_{a_2}, \delta_{v_2}), \max(\delta_{a_2}, \delta_{v_2})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \), there exists \([t'_1, t'_2] \subset [t_1, t_2] \), such that \([\min( f(t'_1), f(t'_2) ), \max( f(t'_1), f(t'_2) )]\] \([\min(\delta_{a_2}, \delta_{v_2}), \max(\delta_{a_2}, \delta_{v_2})] \subset [\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})] \).

Thus the proposition is true for all four possible scenarios, which make the estimation of \( \delta_a \) theoretically feasibly when \( k_1 k_2 < 0 \) and \( \delta_{v_1} \) and \( \delta_{v_2} \) adequately close. Actual \( \delta_a \) between \( t_1 \) and \( t_2 \) can be ensured in the interval \([\min(\delta_{v_1}, \delta_{v_2}), \max(\delta_{v_1}, \delta_{v_2})]\). To simplify the result, actual \( \delta_a \) between \( t_1 \) and \( t_2 \) can be approximately regarded as what Eq. (7) reveals.