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The Budyko functions under non-steady-state conditions

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S.1 Scaling ΔS by P instead of E_p

Figures S1, S2 and S3 are obtained with the equations developed in Appendix A using the parameter $H_P = -\Delta S/P$. They correspond respectively to Figs. 2, 3 and 4 obtained with the parameter $H_E = -\Delta S/E_p$. Figure S1 shows the upper and lower limits of the feasible domain of evaporation in the Turc and the Budyko spaces, first under steady state conditions and then under non-steady state conditions, with a storage term ΔS either positive or negative. Figure S2 shows the ML formulation applied to the Fu-Zhang equation (Eqs. A11a, b) in the Budyko space with $\omega = 1.5$ and different values of H_P. Figure S3 shows the ML formulation with the Fu-Zhang equation in the space $[E_p/(P-\Delta S), E/(P-\Delta S)]$ (Eqs. A18a, b) with $\omega = 1.5$ and different values of H_P.

S.2 Applications of the ML formulation to different Budyko curves

Tables S1 and S3 give the ML formulation applied to the different Budyko curves of Table 1, respectively with the parameter H_E (Eqs. 9 and 12) and H_P (Eqs. A8 and A10). Tables S2 and S4 give the ML formulation applied to the different Budyko curves of Table 1 in the space $[E_p/(P-\Delta S), E/(P-\Delta S)]$, respectively with the parameter H_E (Eqs. 20a and 20b) and H_P (Eqs. A17a and A17b).

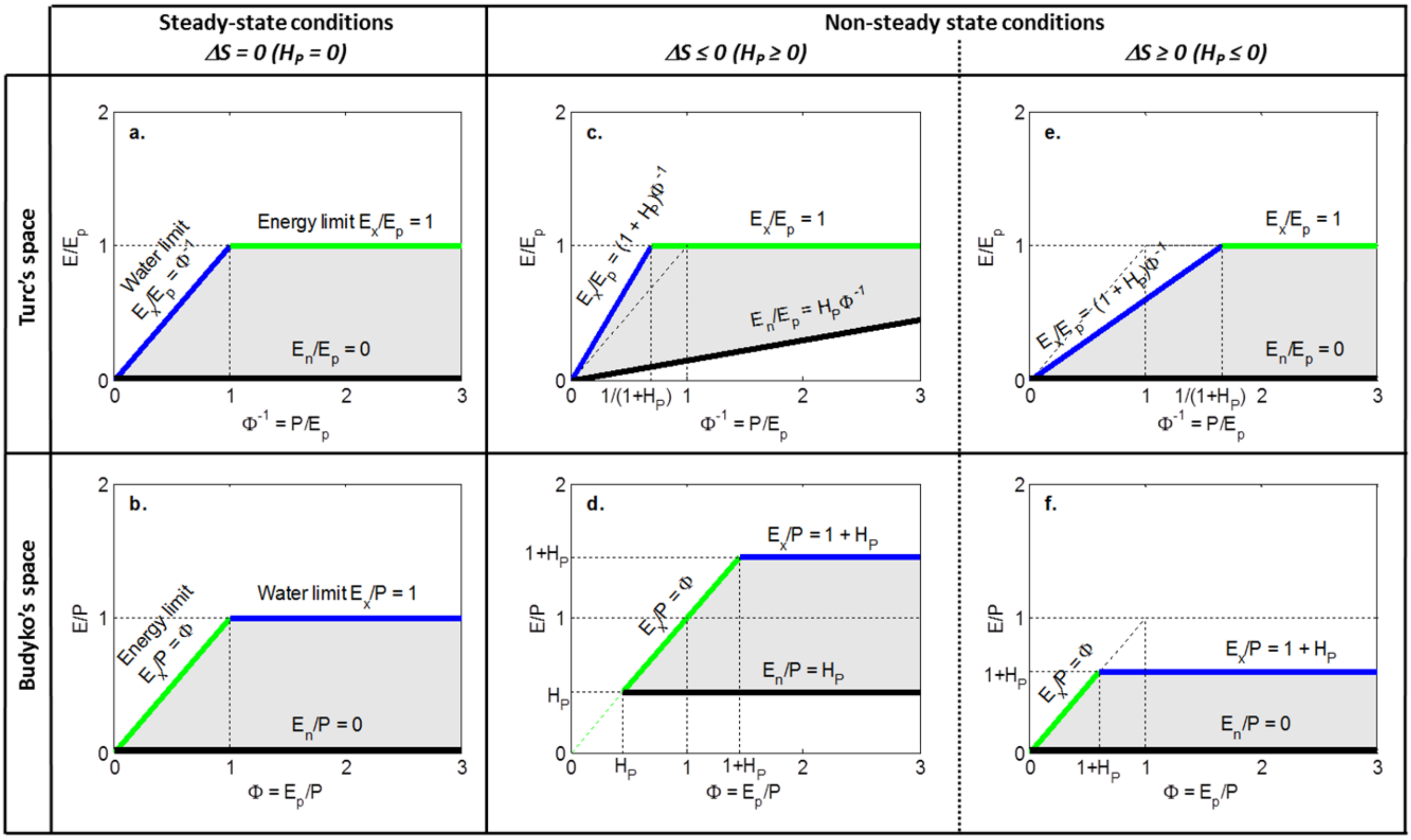

Figure S1: Upper and lower limits of the feasible domain of evaporation (in grey) in the Turc space (*P/E_p, E/E_p*) and in the Budyko space *(E_p/P, E/P)* (water limit in blue, energy limit in green and lower limit in black) when using the non-dimensional parameter *H_P*: (a and b) for steady state conditions; (c, d, e and f) for non-steady state conditions with a storage term *ΔS* either negative (c and d) or positive (e and f).

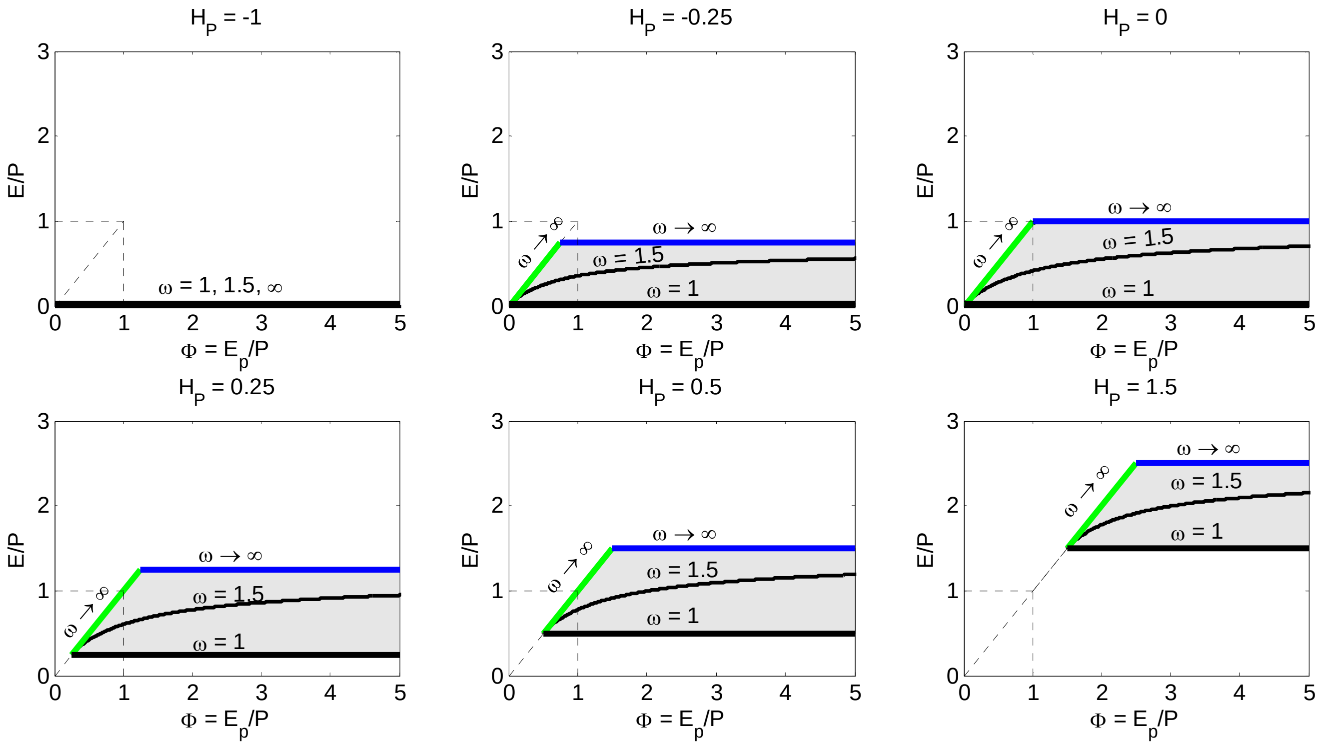

Figure S2: The ML formulation in the Budyko space with the Fu-Zhang relationship (Eqs. A11a, b) for $\omega = 1.5$ and different values of H_P. The bold lines indicate the upper and lower limits of the feasible domain of evaporation (in grey).

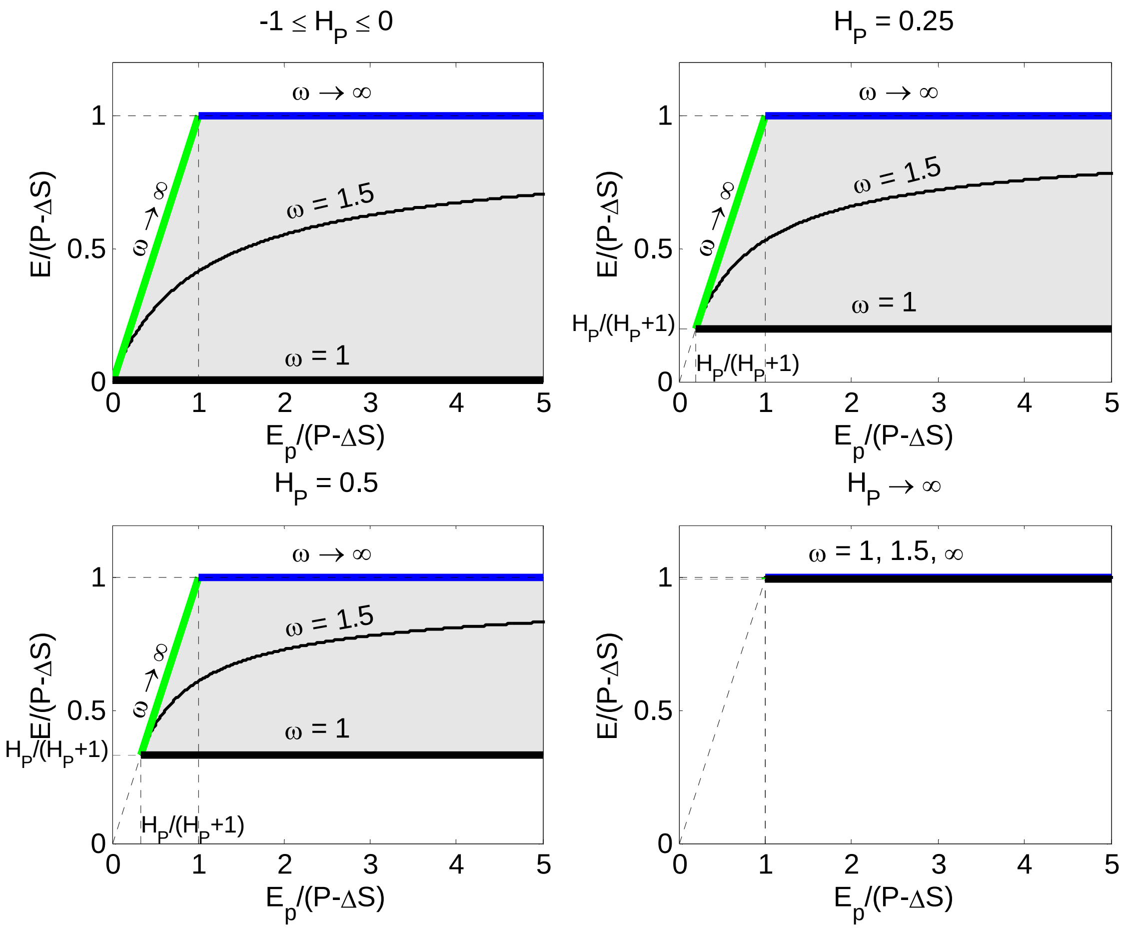

Figure S3: The ML formulation Eqs. (A18a, b) with the Fu-Zhang equation in the space $[E_p/(P\text{-}\Delta S),\ E/(P\text{-}\Delta S)]$ **for** $\omega = 1.5$ **and four values of** H_P**. The bold lines indicate the upper and lower limits of the feasible domain shown in grey. For** $-1 \leq H_P \leq 0$ **the curve is similar to the one under steady state conditions.**

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Table S1: The ML formulation under non-steady state conditions using the parameter H_E applied to the different Budyko curves of Table 1 in the standard Budyko space *(E_p/P, E/P)*.

Reference Budyko curve under steady state conditions $\Delta S = 0$	The ML formulation under non-steady state conditions using Eq. (9) for $\Delta S \leq 0$ and Eq. (12) for $\Delta S \geq 0$
Budyko (1974)	*if* $\Delta S \leq 0$, $\frac{E}{P} = \left\{(1-H_E)\Phi \tanh\left[\frac{1}{(1-H_E)\Phi}\right]\left[1-\exp(-\Phi + H_E\Phi)\right]\right\}^{1/2} + H_E\Phi$
	if $\Delta S \geq 0$, $\frac{E}{P} = \left\{\Phi(1+H_E\Phi)\tanh\left(\frac{1+H_E\Phi}{\Phi}\right)\left[1-\exp\left(\frac{-\Phi}{1+H_E\Phi}\right)\right]\right\}^{1/2}$
Turc (1954), Mezentsev (1955), Yang et al. (2008)	*if* $\Delta S \leq 0$, $\frac{E}{P} = \left[1 + (1-H_E)^{-\lambda}\Phi^{-\lambda}\right]^{-\frac{1}{\lambda}} + H_E\Phi$
	if $\Delta S \geq 0$, $\frac{E}{P} = \left[\Phi^{-\lambda} + (1+H_E\Phi)^{-\lambda}\right]^{-\frac{1}{\lambda}}$
Fu (1981), Zhang et al. (2004)	*if* $\Delta S \leq 0$, $\frac{E}{P} = 1 + \Phi - \left[1 + (1-H_E)^{\omega}\Phi^{\omega}\right]^{\frac{1}{\omega}}$
	if $\Delta S \geq 0$, $\frac{E}{P} = 1 + (1+H_E)\Phi - \left[(1+H_E\Phi)^{\omega} + \Phi^{\omega}\right]^{\frac{1}{\omega}}$.
Zhang et al. (2001)	*if* $\Delta S \leq 0$, $\frac{E}{P} = \frac{(1-H_E)\Phi + w(1-H_E)^2\Phi^2}{(1-H_E)\Phi + w(1-H_E)^2\Phi^2 + 1} + H_E\Phi$
	if $\Delta S \geq 0$, $\frac{E}{P} = \frac{\Phi(1+H_E\Phi)[1+(w+H_E)\Phi]}{1+(1+2H_E)\Phi+\left(w+H_E+H_E^2\right)\Phi^2}$
Zhou et al. (2015)	*if* $\Delta S \leq 0$, $\frac{E}{P} = (1-H_E)\Phi\left[\frac{k}{1+k(1-H_E)^n\Phi^n}\right]^{1/n} + H_E\Phi$
	if $\Delta S \geq 0$, $\frac{E}{P} = \Phi(1+H_E\Phi)\left[\frac{k}{(1+H_E\Phi)^n + k\Phi^n}\right]^{1/n}$

Table S2: The ML formulation under non-steady state conditions using the parameter H_E applied to the different Budyko curves of Table 1 in the space *[E_p/(P-ΔS), E/(P-ΔS)]*.

Reference Budyko curve under steady state conditions *ΔS = 0*	The ML formulation under non-steady state conditions using Eq. (20a) for *ΔS ≤ 0* and Eq. (20b) for *ΔS ≥ 0*
Budyko (1974)	*if ΔS ≤ 0,* $\frac{E}{P-\Delta S}=\left\{(1-H_E\Phi')(1-H_E)\Phi'\tanh\left[\frac{1-H_E\Phi'}{(1-H_E)\Phi'}\right]\left[1-\exp\left(\frac{-\Phi'+H_E\Phi'}{1-H_E\Phi'}\right)\right]\right\}^{1/2}+H_E\Phi'$ *if ΔS ≥ 0,* $\frac{E}{P-\Delta S}=\left\{\Phi'\tanh\left(\frac{1}{\Phi'}\right)[1-\exp(-\Phi')]\right\}^{1/2}$
Turc (1954), Mezentsev (1955), Yang et al. (2008)	*if ΔS ≤ 0,* $\frac{E}{P-\Delta S}=\left\{[(1-H_E)\Phi']^{-\lambda}+(1-H_E\Phi')^{-\lambda}\right\}^{-\frac{1}{\lambda}}+H_E\Phi'$ *if ΔS ≥ 0,* $\frac{E}{P-\Delta S}=\Phi'\left(1+\Phi'^{\lambda}\right)^{-\frac{1}{\lambda}}$
Fu (1981), Zhang et al. (2004)	*if ΔS ≤ 0,* $\frac{E}{P-\Delta S}=1+(1-H_E)\Phi'-[(1-H_E\Phi')^{\omega}+(1-H_E)^{\omega}(\Phi')^{\omega}]^{1/\omega}$ *if ΔS ≥ 0,* $\frac{E}{P-\Delta S}=1+\Phi'-(1+\Phi'^{\omega})^{\frac{1}{\omega}}$.
Zhang et al. (2001)	*if ΔS ≤ 0,* $\frac{E}{P-\Delta S}=\frac{1+(w-wH_E-H_E)\Phi'}{1+w\frac{(1-H_E)\Phi'}{1-H_E\Phi'}+\frac{1-H_E\Phi'}{(1-H_E)\Phi'}}+H_E\Phi'$ *if ΔS ≥ 0,* $\frac{E}{P-\Delta S}=\frac{1+w\Phi'}{1+w\Phi'+\Phi'^{-1}}$
Zhou et al. (2015)	*if ΔS ≤ 0,* $\frac{E}{P-\Delta S}=(1-H_E\Phi')(1-H_E)\Phi'\left[\frac{k}{(1-H_E\Phi')^{n}+k(1-H_E)^{n}\Phi'^{n}}\right]^{\frac{1}{n}}+H_E\Phi'$ *if ΔS ≥ 0,* $\frac{E}{P-\Delta S}=\Phi'\left(\frac{k}{1+k\Phi'^{n}}\right)^{1/n}$

Table S3: The ML formulation under non-steady state conditions using the parameter H_P applied to the different Budyko curves of Table 1 in the standard Budyko space *(E_p/P, E/P)*.

Reference Budyko curve under steady state conditions $\Delta S = 0$	The ML formulation under non-steady state conditions using Eq. (A8) for $\Delta S \leq 0$ and Eq. (A10) for $\Delta S \geq 0$
Budyko (1974)	$if\ \Delta S \leq 0,\ \frac{E}{P} = \left\{(\Phi - H_P)\tanh\left(\frac{1}{\Phi - H_P}\right)[1 - \exp(-\Phi + H_P)]\right\}^{1/2} + H_P$ $if\ \Delta S \geq 0,\ \frac{E}{P} = \left\{\Phi(1 + H_P)\tanh\left(\frac{1+H_P}{\Phi}\right)\left[1 - \exp\left(\frac{-\Phi}{1+H_P}\right)\right]\right\}^{1/2}$
Turc (1954), Mezentsev (1955), Yang et al. (2008)	$if\ \Delta S \leq 0,\ \frac{E}{P} = \left[1 + (\Phi - H_P)^{-\lambda}\right]^{-\frac{1}{\lambda}} + H_P$ $if\ \Delta S \geq 0,\ \frac{E}{P} = \left[\Phi^{-\lambda} + (1 + H_P)^{-\lambda}\right]^{-\frac{1}{\lambda}}$
Fu (1981), Zhang et al. (2004)	$if\ \Delta S \leq 0,\ \frac{E}{P} = 1 + \Phi - [1 + (\Phi - H_P)^{\omega}]^{\frac{1}{\omega}}$ $if\ \Delta S \geq 0,\ \frac{E}{P} = 1 + (\Phi + H_P) - [(1 + H_P)^{\omega} + \Phi^{\omega}]^{\frac{1}{\omega}}$.
Zhang et al. (2001)	$if\ \Delta S \leq 0,\ \frac{E}{P} = \frac{(\Phi - H_P) + w(\Phi - H_P)^2}{(\Phi - H_P) + w(\Phi - H_P)^2 + 1} + H_P$ $if\ \Delta S \geq 0,\ \frac{E}{P} = \frac{\Phi(1 + H_P)[1 + (w\Phi + H_P)]}{1 + (\Phi + 2H_P) + (w\Phi^2 + H_P\Phi + H_P^2)}$
Zhou et al. (2015)	$if\ \Delta S \leq 0,\ \frac{E}{P} = (\Phi - H_P)\left[\frac{k}{1 + k(\Phi - H_P)^n}\right]^{1/n} + H_P$ $if\ \Delta S \geq 0,\ \frac{E}{P} = \Phi(1 + H_P)\left[\frac{k}{(1 + H_P)^n + k\Phi^n}\right]^{1/n}$

Table S4: The ML formulation under non-steady state conditions using the parameter H_P applied to the different Budyko curves of Table 1 in the space *[E_p/(P-ΔS), E/(P-ΔS)]*.

Reference Budyko curve under steady state conditions *ΔS = 0*	The ML formulation under non-steady state conditions using Eq. (A17a) for $\Delta S \leq 0$ and Eq. (A17b) for $\Delta S \geq 0$
Budyko (1974)	*if* $\Delta S \leq 0$, $\frac{E}{P-\Delta S} = \left\{\left[\frac{\Phi'}{1+H_P} - \frac{H_P}{(1+H_P)^2}\right] \tanh\left[\frac{1}{(1+H_P)\Phi' - H_P}\right]\left[1 - \exp(-(1+H_P)\Phi' + H_P)\right]\right\}^{1/2} + \frac{H_P}{1+H_P}$ *if* $\Delta S \geq 0$, $\frac{E}{P-\Delta S} = \left\{\Phi' \tanh\left(\frac{1}{\Phi'}\right)\left[1 - \exp(-\Phi')\right]\right\}^{1/2}$
Turc (1954), Mezentsev (1955), Yang et al. (2008)	*if* $\Delta S \leq 0$, $\frac{E}{P-\Delta S} = \left\{\left[\Phi' - \frac{H_P}{1+H_P}\right]^{-\lambda} + \left(\frac{1}{1+H_P}\right)^{-\lambda}\right\}^{-\frac{1}{\lambda}} + \frac{H_P}{1+H_P}$ *if* $\Delta S \geq 0$, $\frac{E}{P-\Delta S} = \Phi'\left(1 + \Phi'^{\lambda}\right)^{-\frac{1}{\lambda}}$
Fu (1981), Zhang et al. (2004)	*if* $\Delta S \leq 0$, $\frac{E}{P-\Delta S} = 1 + \Phi' - \frac{H_P}{1+H_P} - \left[\left(\frac{1}{1+H_P}\right)^{\omega} + \left(\Phi' - \frac{H_P}{1+H_P}\right)^{\omega}\right]^{1/\omega}$ *if* $\Delta S \geq 0$, $\frac{E}{P-\Delta S} = 1 + \Phi' - \left(1 + \Phi'^{\omega}\right)^{\frac{1}{\omega}}$
Zhang et al. (2001)	*if* $\Delta S \leq 0$, $\frac{E}{P-\Delta S} = \frac{1 + w\Phi' - (w+1)\frac{H_P}{1+H_P}}{1 + w[(1+H_P)\Phi' - H_P] + \frac{1}{(1+H_P)\Phi' - H_P}} + \frac{H_P}{1+H_P}$ *if* $\Delta S \geq 0$, $\frac{E}{P-\Delta S} = \frac{1 + w\Phi'}{1 + w\Phi' + \Phi'^{-1}}$
Zhou et al. (2015)	*if* $\Delta S \leq 0$, $\frac{E}{P-\Delta S} = \frac{1}{1+H_P}\left(\Phi' - \frac{H_P}{1+H_P}\right)\left[\frac{k}{\left(\frac{1}{1+H_P}\right)^{n} + k\left(\Phi' - \frac{H_P}{1+H_P}\right)^{n}}\right]^{\frac{1}{n}} + \frac{H_P}{1+H_P}$ *if* $\Delta S \geq 0$, $\frac{E}{P-\Delta S} = \Phi'\left(\frac{k}{1+k\Phi'^{n}}\right)^{1/n}$