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Matching the Budyko functions with the complementary evaporation relationship: consequences for the drying power of the air and the Priestley–Taylor coefficient

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S.1 Properties of the function $E/P = B_1'(\Phi_0)$

Putting $y = E/P$ with $0 \leqslant y \leqslant 1$, $x = \Phi_0$ and $a = \alpha_0/[(1+b)\alpha_w]$, function B_1' given by Eq. (20) can be rewritten as:

$$x = a\left[\left(y^{-\lambda} - 1\right)^{-1/\lambda} + by\right]. \tag{S1.1}$$

For $x = 0$, y is obviously equal to 0. When x tends to infinite the result is less evident. Eq. (S1.1) can be rewritten as:

$$\frac{x}{a} - by = \frac{1}{\left(\frac{1}{y^{\lambda}} - 1\right)^{\frac{1}{\lambda}}}. \tag{S1.2}$$

When x tends to infinite, given that y is limited by 1, the right-hand term of the equation should tend to infinite. This means that y should tend to 1 so that the denominator tends to zero.

The derivative of the function (Eq. S1.1) is given by:

$$\frac{dx}{dy} = a\left[b + y^{-(\lambda+1)}\left(y^{-\lambda} - 1\right)^{-\frac{1}{\lambda}-1}\right], \tag{S1.3}$$

which can be rewritten as:

$$\frac{dy}{dx} = \frac{1}{a}\left[b + \left(1 - y^{\lambda}\right)^{-(1+\lambda)/\lambda}\right]^{-1}. \tag{S1.4}$$

Close to $x = 0$, y is close to zero and the derivative can be approximated by:

$$\frac{dy}{dx} \approx \frac{1}{a(1+b)}\left[1 - \left(\frac{1+\lambda}{\lambda(1+b)}\right)y^{\lambda}\right] \approx \frac{\alpha_w}{\alpha_0}. \tag{S1.5}$$

If Eq. (22) is taken into account:

$$\frac{\alpha_w}{\alpha_0} = \frac{\left[1 + b\left(1 + x^{\lambda}\right)^{-\frac{1}{\lambda}}\right]}{(1+b)}, \tag{S1.6}$$

which means that α_w/α_0 and dy/dx tend to 1 when x tends to zero.

S.2 Properties of the function $E/E_0 = B_2'(\Phi_0^{-1})$

With $X = \Phi_0^{-1}$, $Y = E/E_0$ and the parameter a defined as above in S1, function B_2' (Eq. 21) can be written as

$$X^{-\lambda} = Y^{-\lambda} - \left(\frac{1}{a} - bY\right)^{-\lambda} . \qquad \text{(S2.1)}$$

When X tends to zero, Y (limited by 1) necessarily tends to zero, and when X tends to infinite Y tends to $1/[(1+b)a] = \alpha_w/\alpha_0$, which is equal to 1 according to Eq. (S1.6) ($x = 1/X = 0$).

The derivative of B_2' can be written as:

$$\frac{dY}{dX} = \frac{X^{-\lambda-1}}{Y^{-\lambda-1} + b\left(\frac{1}{a} - bY\right)^{-\lambda-1}} = \frac{1}{\left(\frac{X}{Y}\right)^{\lambda+1}\left[1 + \frac{b}{\left(\frac{1}{aY} - b\right)^{\lambda+1}}\right]} . \qquad \text{(S2.2)}$$

When X tends to zero, Y also tends to zero and the term into square brackets tends to 1 which means that:

$$\frac{dY}{dX} \rightarrow \left(\frac{Y}{X}\right)^{\lambda+1} . \qquad \text{(S2.3)}$$

Taking into account Eq. (S2.1), we have:

$$\left(\frac{Y}{X}\right)^{\lambda+1} = \left[1 - \left(\frac{1}{aY} - b\right)^{-\lambda}\right]^{\frac{\lambda+1}{\lambda}} , \qquad \text{(S2.4)}$$

which tends to 1.

S.3 Transcendental forms of the basic equations $E/P = B_1(\Phi_p)$ and $E/E_p = B_2(\Phi_p^{-1})$

Eqs. (4) and (5) have the same following form:

$$y = \left(1 + x^{-\lambda}\right)^{-1/\lambda} , \qquad \text{(S3.1)}$$

with $x = \Phi_p$ and $y = E/P$. Eq. (S3.1) can be also written as:

$$x = \left(y^{-\lambda} - 1\right)^{-1/\lambda} . \qquad \text{(S3.2)}$$

With similar notations, Eq. (23) can be written as:

$$y + \left(y^{-\lambda} - 1\right)^{-1/\lambda} = x + \left(1 + x^{-\lambda}\right)^{-1/\lambda} . \qquad \text{(S3.3)}$$

Eq. (S3.3) is equivalent to $y + x = x + y$, which means that S3.1 or S3.2 are solutions of Eq. (S3.3).

A similar reasoning can be conducted with Eq. (24), which can be written with $X = \Phi_p^{-1}$ and $Y = E/E_p$:

$$\left[1 - Y + \left(1 + X^{-\lambda}\right)^{-1/\lambda}\right]^{-\lambda} = Y^{-\lambda} - X^{-\lambda} . \qquad \text{(S3.4)}$$

Given that Eq. (S3.1) is verified by X and Y:

$$Y^{-\lambda} = 1 + X^{-\lambda} . \qquad \text{(S3.5)}$$

Eq. (S3.4) is equivalent to $1 = 1$, which means that Eq. (S3.1) or (S3.2) is solution of Eq. (S3.4).

S.4 Calculations made with the Fu-Zhang equation

The Fu-Zhang equation is written as:

$$\frac{E}{P} = 1 + \Phi_p - \left[1 + \left(\Phi_p\right)^{\omega}\right]^{\frac{1}{\omega}} \ . \qquad \text{(S4.1)}$$

First, we study the feasible domain of the drying power of the air E_a and the correspondence with the evaporation rate E. Inserting Eq. (S4.1) into Eq. (9) yields:

$$\frac{E_a}{E_P} = D(\Phi_P^{-1}) = \left(1 + \frac{\Delta}{\gamma}\right)\left(1 - \frac{1}{(1+b)\alpha_w}\left\{1 + b\left[1 + \Phi_P^{-1} - (1 + \Phi_P^{-\omega})^{\frac{1}{\omega}}\right]\right\}\right). \qquad \text{(S4.2)}$$

The limits given in Eqs. (11), (12) and (13) are independent from the Budyko function used. Consequently D^* remains unchanged:

$$D^* = \frac{b}{(1+b)\alpha_w}\left(1 + \frac{\Delta}{\gamma}\right) . \qquad \text{(S4.3)}$$

Using a similar reasoning as in Eqs (14), (15), (16) and (17), we obtain:

$$d^* = 2^{\frac{1}{\omega}} - 1 \ , \qquad \text{(S4.4)}$$

$$\omega = \frac{ln2}{\ln(d^* - 1)} \ , \qquad \text{(S4.5)}$$

$$\delta^* = \left(1 + \frac{\Delta}{\gamma}\right)\frac{b}{(1+b)\alpha_w}\left(1 - 2^{-\frac{1}{\lambda}}\right) = D^* d^* \ . \qquad \text{(S4.6)}$$

Second, we link the Priestley-Taylor coefficient α_0 to the Fu-Zhang shape parameter ω. Substituting E_p in Eq. (S4.1) by its value given by Eq. (18) and putting $\Phi_0 = E_0/P$ gives:

$$\frac{E}{P} = 1 + \frac{(1+b)\alpha_w}{\alpha_0}\Phi_0 - b\frac{E}{P} - \left\{1 + \left[\frac{(1+b)\alpha_w}{\alpha_0}\Phi_0 - b\frac{E}{P}\right]^{\omega}\right\}^{\frac{1}{\omega}} . \qquad \text{(S4.7)}$$

Eq. (S4.7) can be rewritten as:

$$\left[1 + (1+b)\left(\frac{\alpha_w}{\alpha_0}\Phi_0 - \frac{E}{P}\right)\right]^{\omega} = 1 + \left[(1+b)\frac{\alpha_w}{\alpha_0}\Phi_0 - b\frac{E}{P}\right]^{\omega} \ . \qquad \text{(S4.8)}$$

An equation similar to Eq. (21) can be obtained expressing E/E_0 as a function of $\Phi_0^{-1} = P/E_0$:

$$\left[1 + \frac{(1+b)}{\Phi_0^{-1}}\left(\frac{\alpha_w}{\alpha_0} - \frac{E}{E_0}\right)\right]^{\omega} = 1 + \left(\frac{1}{\Phi_0^{-1}}\right)^{\omega}\left[\frac{(1+b)\alpha_w}{\alpha_0} - b\frac{E}{E_0}\right]^{\omega} . \qquad \text{(S4.9)}$$

Eqs. (S4.8) and (S4.9) obtained from the Fu-Zhang formulation correspond respectively to $E/P = B_1'(\Phi_0)$ (Eq. 20) and $E/E_0 = B_2'(\Phi_0^{-1})$ (Eq. 21) obtained with the Turc-Mezentsev equation.

Using a similar reasoning as in Eq. (22), the expression of α_0 can be inferred by matching Eqs. (S4.8) and (S4.1): for a given value of the aridity index Φ, we have the same value of E/P. This leads to:

$$\left\{(1+b)\left[\left(\frac{\alpha_w}{\alpha_0} - 1\right)\Phi + (1 + \Phi^{\omega})^{\frac{1}{\omega}}\right] - b\right\}^{\omega} = 1 + \left\{\left[\frac{(1+b)\alpha_w}{\alpha_0} - b\right]\Phi + b(1 + \Phi^{\omega})^{\frac{1}{\omega}} - b\right\}^{\omega} . \qquad \text{(S4.10)}$$

Eq. (S4.10) is equivalent to Eq. (22), but with a transcendental form. It can be resolved numerically and Fig. (S1) shows the variation of the Priestley-Taylor coefficient α_0 as a function of the aridity index Φ for different values of the ω parameter. The shape of the curves is very similar to those of Fig. (5a) obtained with the parameter λ of the Turc-Mezentsev function.

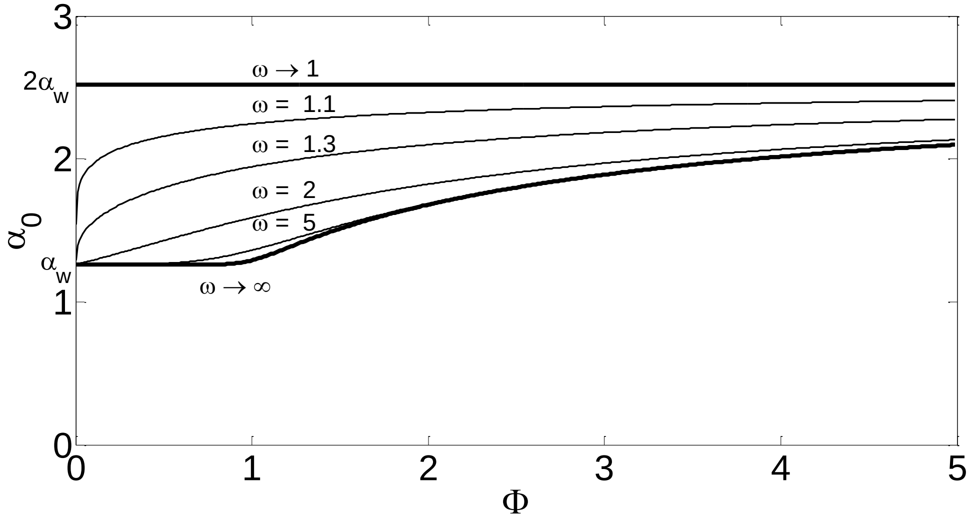

Figure S1: Variation of the Priestley-Taylor coefficient α_0 with b = 1 as a function of the aridity index Φ for different values of the shape parameter ω of the Fu-Zhang function. The bold lines indicate the limits of the feasible domain.

S.5 Results obtained with the Turc-Mezentsev function making b = 4.5 instead of b = 1

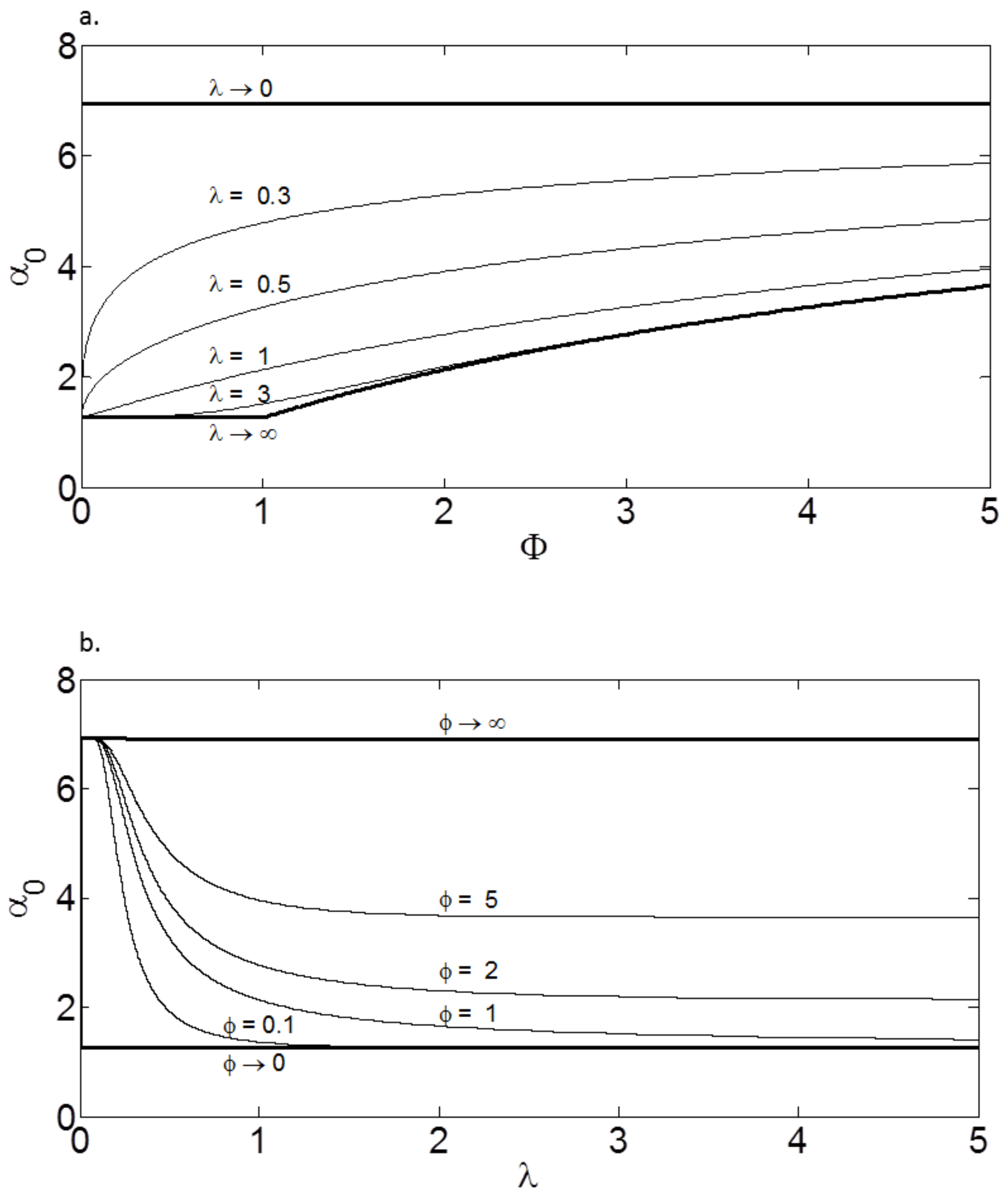

Figure S2: Variation of the Priestley-Taylor coefficient α_0 (Eq. (22) with $b = 4.5$ and $\alpha_w = 1.26$): (a) as a function of the aridity index Φ for different values of the shape parameter λ of the Turc-Mezentsev function; (b) as a function of λ for different values of the aridity index Φ. The bold lines indicate the limits of the feasible domain.